Stratification of the Jeffrey-Weitsman half-density quantization

of the Moduli Space of flat connections
and Witten-Chern-Simons invariants

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The Challenge

Fix N, a 3-manifold. Witten's conjecture:

The Chern Simons Partition function

$$Z(N,k) = \int_M e^{ikCS(A)} d\mathcal{A}$$

- M is the space of connections over the trivial SU(2) principal bundle over N modulo Gauge transforms.
- This space is $\Omega^1(N, \mathfrak{g})$: the infinite dimensional "Lie algebra valued one forms" modulo gauge.
- CS(A) is the Chern Simons functional: $CS(A) = 1/4\pi \int A \wedge dA + 2/3A^3$.

Stationary Phase approximation

Witten's conjecture, with refinement by Freed and Gompf:

The expansion of stationary phase

$$Z(N,k) \sim \frac{1}{2} e^{-3\pi i(1+b^1(N))/4} \sum_{A_{flat}} \sqrt{\tau_N(A_i)} e^{-2\pi i(I_{A_i/4}+dim(H^0))}$$
$$\times e^{2\pi i(k+2)CS(A)}(k+2)^{(dim H^1 - dim H^0)/2}$$

- where τ is the Reidemeister-Ray-Singer torsion of the connection, I_A is the spectral flow.
- The sum is over the critical points of the CS functional, precisely the flat connections.

Geometric Quantization

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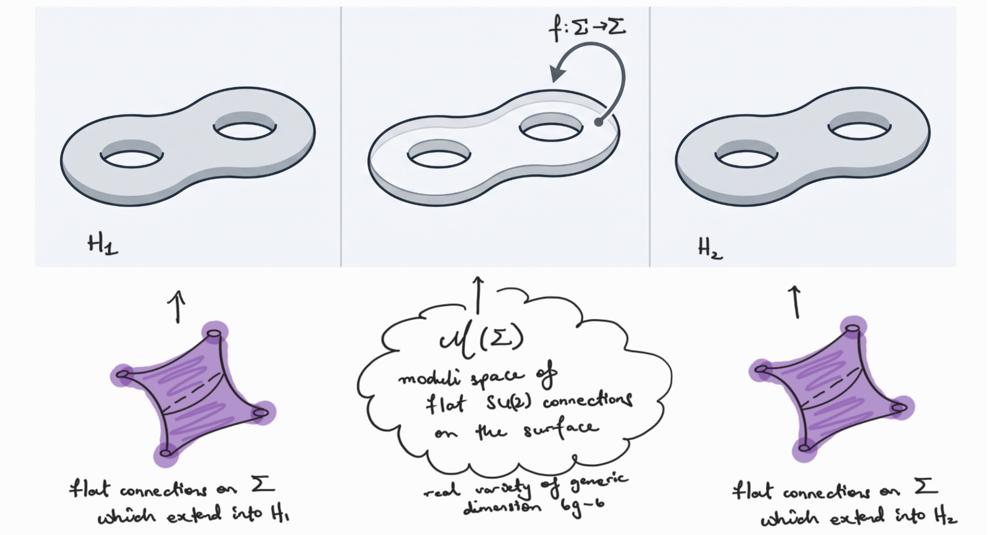
Inspiration from TQFT: genus 2 example of splitting the closed 3-manifold

Geometric Quantization



Z(N,k) from Geometric Quantization

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Witten's conjecture consolidated:

The expansion of stationary phase

$$Z(N,k) \sim e^{i\cdot global} \sum_{A_{flat}} \left(e^{i\cdot n\pi/2}\right)\left(\sqrt{\tau_N(A_i)}\right)\left(e^{2\pi i(k+2)CS(A)}\right)\left((k+2)^{(dimH^1 - dimH^0)/2}\right)$$

- The prequantum line bundle is the Chern-Simons line bundle with covariant constant section contributing $e^{ikCS(A)}$.
- the proposed torsion has a generalization as a volume form over the moduli spaces. For the handlebody, this is an element in:

$$\tau(A_{handlebody}) \in \det(H^1_A) \otimes \det(H^0_A)^*$$

The Moduli Space of flat SU(2) connections

- Famously symplectic with the Atiyah-Bott form

Fundamental group of the surface

$$\pi_1 =< a_1, \ldots, a_g, b_1 \ldots, b_g : \Pi a_i b_i a_i^{-1} b_i^{-1} >$$

Can be viewed as the SU(2) character variety, where due to Goldman, the symplectic form is given by the cup product in group cohomology:

$SU(2)$-Character variety of the surface

$$Hom(\pi_1, SU(2))/SU(2)$$

and we can view this space as:

$$\{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in SU(2)^{2g} : \Pi A_i B_i A_i^{-1} B_i^{-1} = Id\}_{/SU(2)}$$

Geometric Quantization

Irreducibility vs reducibility

- Structure of SU(2): maximal torii are conjugate circles, intersecting in Z(SU(2))
- One special leaf of the polarization of the moduli space:

$$\{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in SU(2)^{2g} : \Pi A_i B_i A_i^{-1} B_i^{-1} = Id\}_{/SU(2)}$$

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Flat connections which extend into the handlebody

$$L_H=\{(A_1,\dots,A_g)\in SU(2)^g\}/SU(2)$$

These are the connections which are trivial on the B_i's and thus extend as flat connections into the surfaces bounding handlebody.

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The reducibles

- Under simultaneous conjugation the stabilizer options are $Z(SU(2)) = \pm Id$, some maximal torus, T or all of SU(2)
- For the connections stabilized by T, this space is of the form:

$$\bigcup h\Big((T^*)^g\Big)h^{-1}$$

for $h \in SU(2)$ simultaneously conjugating a tuple into a different maximal torus.

Geometric Quantization

The space of reducibles can thus be viewed as:

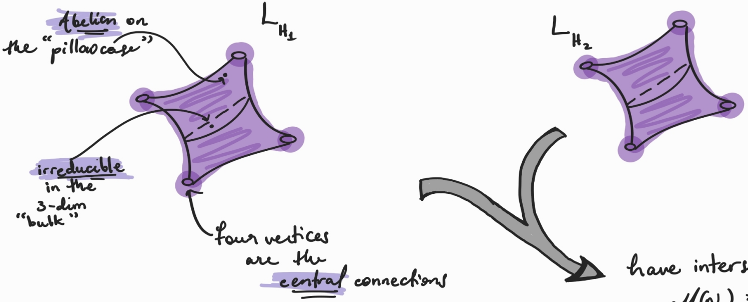
$$Hom(\pi_1, T)/W(T)$$

where $W(T)$ is the Weyl group (order 2) of the fixed maximal torus.

- Another moduli space of flat connections! The abelian moduli space, modulo an order 2 gluing.
- The dimension of this stratum is g.
- Locally the tangent space is modelled by $\dim H^1(M; \mathfrak{t}_{\tilde{x}})$
- (technically: the 2^g-many central connections are deleted from this space)

A Stratified BKS pairing

for genus $g=2$, moduli of flats for each handlebody H_1 and H_2



have intersection in $\mathcal{M}(\Sigma^g)$

$\mathcal{M}(N)$: flat connections on the 3-mfld N

First Abelian Cohomology Dimensions

Manifold (M)	$\pi_1(M)$	**Structure Group:** $SU(2)$ $\dim H^1(M; \mathfrak{g}_{\tilde{x}})$	**Structure Group:** $U(1)$ $\dim H^1(M; \mathfrak{l}_{\tilde{x}})$
$\mathbb{S}^3$	$\{1\}$	0	0
$L(p,q)$	$\mathbb{Z}_p$	0	0
$\mathbb{T}^3$	$\mathbb{Z}^3$	3	3
$\mathbb{S}^1 \times \Sigma_g$	$\mathbb{Z} \times \pi_1(\Sigma_g)$	Grows with Zariski T_x	$2g+1$

- For $L(p,q)$, the first cohomology vanishes for non-trivial reducible connections.
- For $\mathbb{S}^1 \times \Sigma_g$, the $SU(2)$ dimension sits in the singular locus (exceeding the irreducible $6g-2$), while the $U(1)$ untwisted cohomology yields exactly $2g+1$.

A stratified invariant

Theorem (Chitan, arXiv:2509.17656)

Let N be a 3-manifold such that $M(N)$ has stratified clean intersection, then $Z(N,k)$ is a topological invariant given by:

$$Z(N,k) = \sum_{i=0,1,3} \frac{k^{(dim(H^1)-dim(H^0))/2}}{Vol(Stab)} \int_{M^i(N)} e^{ikCS(x)} \tau(N,x)^{1/2} \otimes \mathbf{1}_x^{-2}.$$

where $CS(x)$ is the Chern-Simons invariant of a connection determined by x, $\tau(N,x)$ is the torsion of the connection.

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- Clean intersection of tangent spaces mean the tangent space to the intersection is the intersection of the two tangent spaces before intersection.
- Stratification thus leads to the only precondition of clean intersection (per stratum).
- Each non-empty stratum necessarily contributes to the invariant.
- 1_x is the unit volume element of the trivial stabilizer bundle of a given stratum.

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