Construction of totally non-negative pfaffian

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Outline

BKP Equation

We start with Clifford algebra for BKP equation. We consider neutral fermions $\{\phi_n, n \in Z\}$, obeying the following canonical anti-communitation relations [Date, Jimbo, Kashiwara and Miwa, 1982]

$$[\phi_m, \phi_n]_+ = \phi_m\phi_n + \phi_n\phi_m = (-)^m\delta_{m,-n}. \tag{1}$$

In particular, $\phi_0^2 = 1/2$.
There are the right and the left vacuum vectors $|0>$ and $<0|$ respectively, having the properties

$$\phi_m|0>= 0, (m<0), \qquad <0|\phi_m = 0, (m>0),$$

and

$$\sqrt{2}\phi_0 = |1>, \sqrt{2}\phi_0|1>= 0|>, <0|\sqrt{2}\phi_0 =<1|, <1|\sqrt{2}\phi_0 =<0|.$$

We have a right and left Fock spaces spanned, respectively, by the right and the left vacuum vectors

$$\begin{aligned} &\phi_{n_1}\phi_{n_2}\phi_{n_3}\cdots\phi_{n_k}|0>, \\ &<0|\phi_{-n_1}\phi_{-n_2}\phi_{-n_3}\cdots\phi_{-n_k}, \end{aligned} \tag{2}$$

where $k = 1, 2, 3, \cdots$ and we assume

$$n_1 > n_2 > n_3 \cdots > n_k \geq 0$$

due to the anti-communitation relations (1). The vacuum expectation values of quadratic elements are given by

$$<0|\phi_i\phi_j|0> = <\phi_i\phi_j> = \begin{cases} (-1)^i\delta_{i,-j}, i < 0, \\ \frac{1}{2}\delta_{j,0}, i = 0, \\ 0, i > 0. \end{cases}$$

Then one introduces the following Hamiltonian operators lableed by odd numbers $n \in 2Z + 1$:

$$H_n = \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^{k+1} \phi_k \phi_{-k-n}. \tag{3}$$

One can check that H_n obey the following Heisenberg algebra relations

$$[H_n, H_m] = H_n H_m - H_m H_n = \frac{n}{2} \delta_{n+m,0}. \tag{4}$$

The anti-communitation relations (1) imply

$$[H_n, \phi_m] = \phi_{m-n}. \tag{5}$$

We also notice that by the definition of vacuum

$$H_n|0>= H_n \phi_0 |0>= 0.$$

For the independent time variables of BKP equation, one sets

$$H(t_1, t_3, t_5, \cdots,) = \sum_{n=1,3,5,\cdots}^{\infty} t_n H_n.$$

It's suitable to introduce the following fermionic field, which depends on a complex parameter p :

$$\phi(p) = \sum_{k=-\infty}^{\infty} p^k \phi_k.$$

For fermionic fields, the anti-communitation relations (1) reads as

$$[\phi(p), \phi(p')]_+ = \sum_{k=-\infty}^{\infty} (\frac{-p}{p'})^k = \delta(\frac{-p}{p'}),$$

and when $|p| \neq |p'|$, one has

$$< 0|\phi(p)\phi(p')|0 >=< \phi(p)\phi(p') >= \frac{(p - p')}{2(p + p')}.$$

By the Wick theorem and the Schur identity, one gets

$$
\begin{aligned}
<\phi(p_1)\phi(p_2)\phi(p_3)\cdots\phi(p_N)> \quad &= \begin{cases} Pf(<\phi(p_i)\phi(p_j)>, \text{N even} \\ 0, \text{N odd} \end{cases} \\
&= \begin{cases} 2^{-N/2}\prod_{i<j}\frac{p_i-p_j}{p_i+p_j}, \text{N even} \\ 0, \text{N odd} \end{cases}
\end{aligned} \tag{6}
$$

where Pf is the pfaffian defined by

$$
Pf(\phi_1, \phi_2, \phi_3, \cdots, \phi_{2N}) \quad = \quad \sum_{\sigma}\epsilon(\sigma)W_{\sigma_1\sigma_2}W_{\sigma_3\sigma_4}\cdots W_{\sigma_{2N-1}\sigma_{2N}}. \tag{7}
$$

Here $\epsilon(\sigma)$ is the sign of the permutation σ and $W_{\sigma_i\sigma_j} = \frac{(p_i-p_j)}{2(p_i+p_j)}$; moreover $\sigma_1 < \sigma_3 < \sigma_5 < \cdots < \sigma_{2N-1}$ and $\sigma_1 < \sigma_2, \sigma_3 < \sigma_4, \cdots, \sigma_{2N-1} < \sigma_{2N}$.

The relation (5) yields

$$[H_n, \phi(p)] = p^n\phi(p),$$

which in turn results in

$$\phi(p)(t) = e^{H(t)}\phi(p)e^{-H(t)} = \phi(p)e^{\xi(t,p)}, \quad \xi(t,p) = \sum_{k=1,3,5,\cdots}^{\infty} t_k p^k.$$

To introduce the τ-function, we define the Clifford group

$$G = \{g | g^{-1} \quad \text{exists and } g\phi_k g^{-1} = \sum_{j \in Z} a_{jk}\phi_j\}. \tag{8}$$

A typical element of G is

$$g = e^{\frac{1}{2}\sum_{r,s\geq 0}\alpha_{rs}\phi_r\phi_s}, \alpha_{rs} = -\alpha_{sr},$$

of a fermion bilinear form. Such an fermion operator defines a τ-function of the BKP equation

$$\tau_g(t) =< 0|e^{H(t)}g|0 > . \tag{9}$$

The space of the BKP τ-function is the G-orbit of the vacuum vector $|0>$.

From the consequence of the algebraic relation

$$\sum_{j\in Z}(-1)^j\phi_j g\otimes\phi_{-j}g=\sum_{j\in Z}(-1)^j g\phi_j\otimes g\phi_{-j}$$

and the formula below called the Boson-Fermion Correspodence :

$$\sqrt{2}<1|e^{H(t)}\phi(p)|V>=X(t,z)<0|e^{H(t)}|V> \tag{10}$$

that holds for arbiartry Fock space V defined in (2), where $X(t,z)$ is the vertex operator

$$X(t,z)=e^{\xi(t,z)}e^{-2D(t,1/z)},\quad D(t,1/z)=\sum_{n=0}^{\infty}\frac{z^{-2n-1}}{2n+1}\partial_{2n+1}.$$

The Hirota equation is

$$\oint \frac{dz}{2\pi i z} e^{\xi(t'-t,z)} \tau(t' - 2[\frac{1}{z}]) \tau(t + 2[\frac{1}{z}]) dz = \tau(t)\tau(t'), \qquad (11)$$

where

$$[\frac{1}{z}] = (\frac{1}{z}, \frac{1}{3z^3}, \frac{1}{5z^5}, \cdots].$$

Thus, the Hirota equation of the BKP equation (13) is obtained from (11) : $(t_1 = x, t_3 = y, t_5 = t)$

$$(D_x^6 - 5D_x^3 D_y - 5D_y^2 + 9D_x D_t)\tau \circ \tau = 0, \qquad (12)$$

where the Hirota derivative is defined as

$$D_t^m D_x^n f(t,x) \circ g(t,x) = \frac{\partial^m}{\partial a^m} \frac{\partial^n}{\partial b^n} f(t+a, x+b) g(t-a, x-b)|_{a=0, b=0}.$$

Letting $\phi = 2(\ln \tau(x,y,t))_x$, one obtains the BKP equation [E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, 1981] or the (2+1) Sawada-Kotera equation in potential form

$$(9\phi_t - 5\phi_{xxy} + \phi_{xxxxx} - 15\phi_x\phi_y + 15\phi_x\phi_{xxx} + 15\phi_x^3)_x - 5\phi_{yy} = 0 \tag{13}$$

When ϕ is independent of y (or $\phi_y = 0$), one has the Sawada-Kotera equation [1974] .

The Schur Q polynomials are defined by , $c = (c_1, c_3, c_5, c_7, \cdots)$

$$e^{2\xi(c,p)} = \sum_{k=0} q_k(c)p^k.$$

For example,

$$q_0(c) = 1, \quad q_1(c) = 2c_1, \quad q_2(c) = (4c_1^3)/3 + 2c_3, \quad q_4(c) = 4c_1c_3 + 2/3c_1^4,$$
$$q_5(c) = 2c_5 + 4c_1^2c_3 + 4/15c_1^5, \quad q_6(c) = 4c_1c_5 + 8/3c_3c_1^3 + 2c_3^2 + 4/45c_1^6,$$
$$q_7(c) = 2c_7 + 4c_1^2c_5 + 4c_1c_3^2 + 4/3c_3c_1^4 + 8/315c_1^7,$$
$$q_8(c) = 4c_1c_7 + 8/3c_5c_1^3 + 4c_5c_3 + 4c_1^2c_3^2 + 8/15c_3c_1^5 + 2/315c_1^8.$$

Thus

$$\phi_i(c) = e^{H(c)}\phi_i|0> = \sum_{k=0}^{\infty} q_k(c/2)\phi_{i-k},$$

and

$$< \phi_i(c)\phi_j(c) > = \frac{1}{2}q_i(c/2)q_j(c/2) + \sum_{k=1}^{j}(-1)^k q_{i+k}(c/2)q_{j-k}(c/2). \quad (14)$$

Since, $q_k(c/2) = 0$ if $k < 0$, it's known that

$$1 = e^{2\xi(p,c)}e^{-2\xi(p,c)} = \sum_{i,j} q_i(c)q_{j-i}(-c) = \sum_{i,j}(-1)^{i-j}q_i(c)q_{j-i}(c)p^j,$$

we have the orthogonal condition for all $n > 0$

$$\sum_{i=0}^{n}(-1)^i q_i(c)q_{n-i}(c) = 0. \tag{15}$$

This is trivial if n is odd and if $n = 2m$ is even, then it gives

$$q_m(c)^2 + 2\sum_{k=1}^{m}(-1)^k q_{m+k}(c)q_{m-k}(c) = 0.$$

Then one can define

$$q_{a,b}(c) = q_a(c)q_b(c) + 2\sum_{k=1}^{b}(-1)^k q_{a+k}(c)q_{b-k}(c). \tag{16}$$

Here we notice that $q_{a,0}(c) = q_a(c)$. It follows from the orthogonal condition (15) that

$$q_{a,b}(c) = -q_{b,a}(c),$$

and in particular, $q_{a,a}(c) = 0$.

Comparing (14) and (16), one has

$$q_{i,j}(c/2) = 2 < \phi_i(c)\phi_j(c) > . \tag{17}$$

Now, consider $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{2n})$, where $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_{2n} \geq 0$. The set of such distinct partition is denoted DP . For $\lambda \in DP$, we define

$$Q_\lambda(c/2) = Pf(q_{\lambda_i,\lambda_j}(c/2)). \tag{18}$$

This is the Schur's Q function. Here we notice that $Q_{\lambda_i,\lambda_j} = q_{\lambda_i,\lambda_j}$ and $Q_{\lambda,0} = Q_\lambda = q_\lambda$. By the Wick Theorem,

$$\begin{aligned} Q_\lambda(c/2) &= Pf(2 < \phi_{\lambda_i}(c)\phi_{\lambda_j}(c) >) \\ &= 2^n < \phi_{\lambda_1}(c)\phi_{\lambda_2}(c)\cdots\phi_{\lambda_{2n}}(c) > . \end{aligned}$$

The τ-function of soliton solution of BKP is [Nimmo and Orlov, 2011]

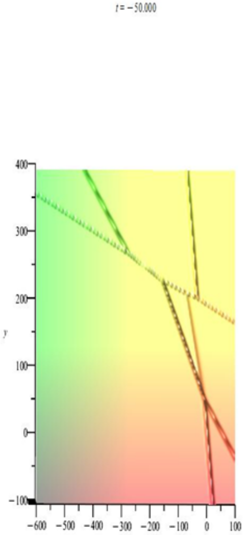
$$
\begin{aligned}
\tau(c,t) &= < e^{H(t)} e^{\sum_{0\le i<j} q_{i,j}(c/2)\phi(p_i)\phi(p_j)} > \\
&= 1 + \sum_{0\le i<j} Q_{i,j}(c/2)\frac{1}{2}\frac{p_i-p_j}{p_i+p_j} e^{\xi(p_i,t)+\xi(p_j,t)} \\
&+ \sum_{0\le i<j<k<l} Q_{i,j,k,l}(c/2)\frac{1}{2^2}\frac{p_j-p_i}{p_j+p_i}\frac{p_k-p_i}{p_k+p_i}\frac{p_l-p_i}{p_l+p_i}\frac{p_k-p_j}{p_k+p_j}\frac{p_l-p_j}{p_l+p_j}\frac{p_l-p_k}{p_l+p_k} \\
&\quad e^{\xi(p_i,t)+\xi(p_j,t)+\xi(p_k,t)+\xi(p_l,t)} \\
&+ \sum_{0\le i<j<k<l<m<n} Q_{i,j,k,l,m,n}(c/2)\frac{1}{2^3}\prod_{0\le\alpha<\beta\le n}\frac{p_\alpha-p_\beta}{p_\alpha+p_\beta} \qquad (19) \\
&\quad e^{\xi(p_i,t)+\xi(p_j,t)+\xi(p_k,t)+\xi(p_l,t)+\xi(p_m,t)+\xi(p_n,t)} \\
&+ \cdots \qquad (20)
\end{aligned}
$$

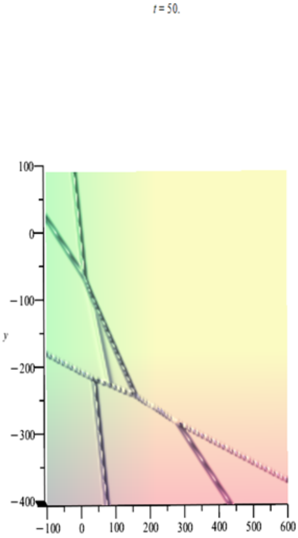
Example:

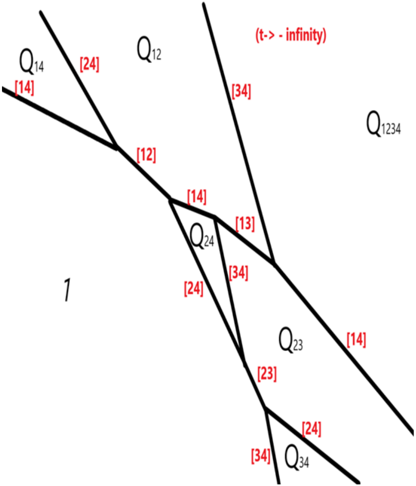
$$B=\begin{bmatrix} 0 & Q_{12} & Q_{13} & Q_{14} \\ -Q_{12} & 0 & Q_{23} & Q_{24} \\ -Q_{13} & -Q_{23} & 0 & Q_{34} \\ -Q_{14} & -Q_{24} & -Q_{34} & 0 \end{bmatrix}, \tag{21}$$

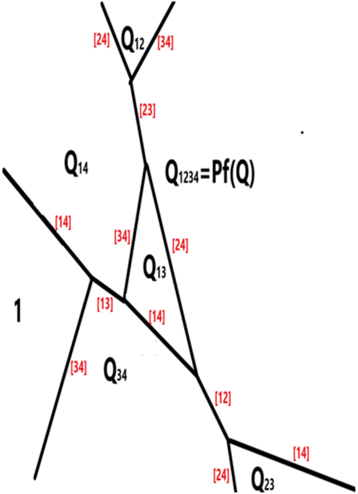
$$\begin{aligned}\tau &= 1+\frac{1}{2}\Big(Q_{12}\frac{p_1-p_2}{p_1+p_2}e^{\xi(p_1,t)+\xi(p_2,t)}+Q_{13}\frac{p_1-p_3}{p_1+p_3}e^{\xi(p_1,t)+\xi(p_3,t)}\\ &+ Q_{14}\frac{p_1-p_4}{p_1+p_4}e^{\xi(p_1,t)+\xi(p_4,t)}+Q_{23}\frac{p_2-p_3}{p_2+p_3}e^{\xi(p_2,t)+\xi(p_3,t)}\\ &+ Q_{24}\frac{p_2-p_4}{p_2+p_4}e^{\xi(p_2,t)+\xi(p_4,t)}+Q_{34}\frac{p_3-p_4}{p_3+p_4}e^{\xi(p_3,t)+\xi(p_4,t)}\Big)\\ &+ \frac{1}{4}(Q_{12}Q_{34}-Q_{13}Q_{24}+Q_{14}Q_{23})\\ &\quad \frac{p_1-p_2}{p_1+p_2}\frac{p_1-p_3}{p_1+p_3}\frac{p_1-p_4}{p_1+p_4}\frac{p_2-p_3}{p_2+p_3}\frac{p_2-p_4}{p_2+p_4}\frac{p_3-p_4}{p_3+p_4}e^{\xi(p_1,t)+\xi(p_2,t)+\xi(p_3,t)+\xi(p_4,t)},\end{aligned}$$

where $p_1^2>p_2^2>p_3^2>p_4^2$.


t = −50.000
y
400
300
200
100
0
−100
−600 −500 −400 −300 −200 −100 0 100


t = 50.
y
100
0
−100
−200
−300
−400
−100 0 100 200 300 400 500 600


(t-> - infinity)
Q14
[14]
[24]
Q12
[34]
Q1234
[12]
[14]
[13]
Q24
1
[24]
[34]
Q23
[14]
[23]
[24]
[34]
Q34


[24]
Q12
[34]
[23]
Q14
Q1234=Pf(Q)
[14]
[34]
[24]
Q13
1
[13]
[14]
[34]
Q34
[12]
[14]
[24]
Q23

Given a Young digram of DP $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_{2n} \geq 0)$, there are

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} = 2^{2n-1}$$

terms in the expansion (20). The sum in (20) is over all the DP of λ and this corresponds to a $2n-1$ soliton.

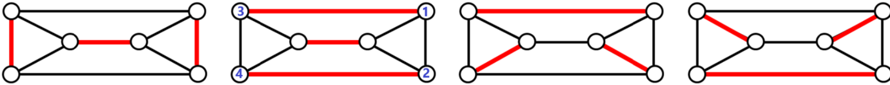
To obtain non-singular resonant-solitons solutions of the BKP equation (13), one has to assume $Q_{DP\in\lambda}(c) \geq 0$ for all DP of λ. Then one has the following

Definition

Let A be a $2n \times 2n$ skew-symmetric matrix. Then A is a totally positive (non-negative) pfaffian (TNNP)if every $2m \times 2m (m \leq n)$ principal sub skew-symmetric matrix of A has a positive (non-negative) pfaffian.

Definition

A planar graph is drawn on the plane in such a way that its edges intersect only at their endpoints. Given a graph G with edges E and vertices V, a perfect matching in G is a subset M of E, such that every vertex in V is adjacent to exactly one edge in M. The number of perfect matchings is M(G). If U is a subset of vertices in V(G), then $G\backslash U$ is the subgraph of G induced by the vertices of $V\backslash U$.



For example, this plane graph G has six vertices, nine edges and M(G)=4. There are four boundary vertices.

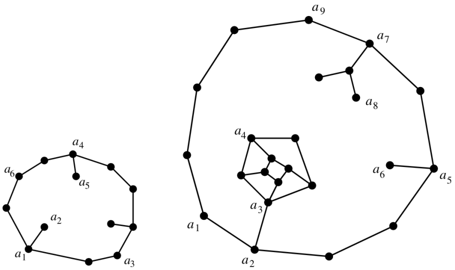
Theorem

[E.H. Kuo (2004) and M. Ciucu(2015)]
Let G be a planar graph with the vertices $a_1, a_2, ..., a_{2n}$ *(boundary vertices) appearing in that cyclic order among the vertices of some face of G. Consider the skew-symmetric matrix* $A = (a_{ij})_{1\le i,j\le 2n}$ *with nonzero entries given by*

$$a_{ij} = \begin{cases} M(G\backslash\{a_i, a_j\}), & \text{if } i < j \\ -M(G\backslash\{a_i, a_j\}), & \text{if } i > j \end{cases}$$

Then we have that

$$[M(G)]^{n-1} M(G\backslash\{a_1, a_2 ..., a_{2n}\}) = Pf(A).$$



Then it can be seen that for any index $I = \{a_{i_1}, a_{i_2}, \cdots, a_{i_{2k}}\} \subseteq \{a_1, a_2 ..., a_{2n}\}$ in the graph G, we obtain

$$[M(G)]^{k-1} M(G \backslash \{a_{i_1}, a_{i_2}, \cdots, a_{i_{2k}}\}) = Pf(A_I).$$

Then one has

Theorem

Given any planar graph G with boundary vertices, we can contruct a totally non-negative pfaffian.

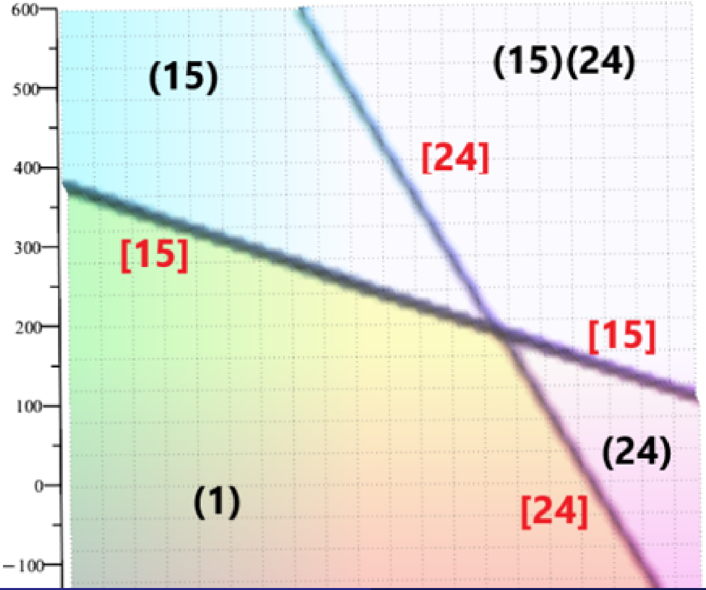
Example 1

$$Pf(\begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 2 & 1 \\ -1 & -2 & 0 & 1 \\ -2 & -1 & -1 & 0 \end{bmatrix}) = 4 \times 1.$$

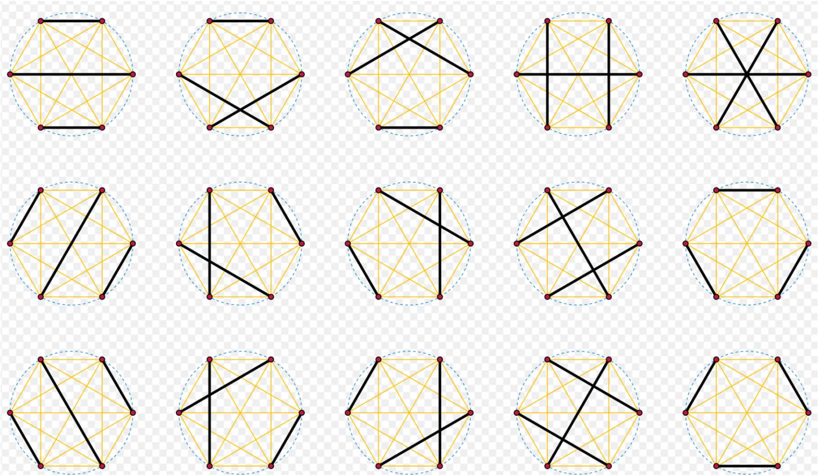
Example 2

$$Pf\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) = 0.$$

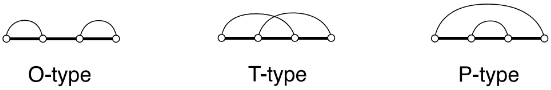
$$\begin{aligned} \tau &= 1 + \frac{1}{2}\left(\frac{p_1 - p_5}{p_1 + p_5} e^{\xi(p_1,t)+\xi(p_5,t)} + \frac{p_2 - p_4}{p_2 + p_4} e^{\xi(p_2,t)+\xi(p_4,t)}\right) \\ &+ \frac{1}{4}\frac{p_1 - p_5}{p_1 + p_5}\frac{p_2 - p_4}{p_2 + p_4} e^{\xi(p_1,t)+\xi(p_2,t)+\xi(p_4,t)+\xi(p_5,t)}, \end{aligned}$$


600
500
400
300
200
100
0
−100
(15)
(15)(24)
[24]
[15]
[15]
(24)
(1)
[24]

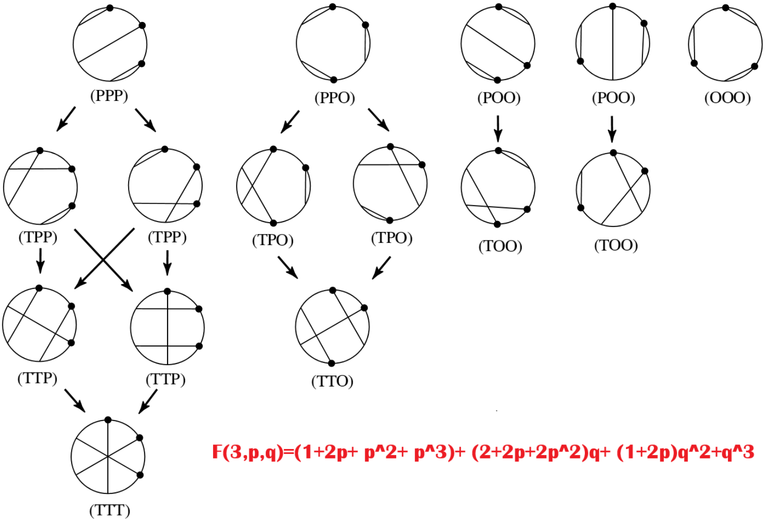
A chord diagram is a diagram of N chords joining $2N$ points on a circle in disjoint pairs.



Remarks:
(1) There are the number $1 \times 3 \times 5 \times 7 \cdots \times (2N-1) = \frac{(2N)!}{2^N(N!)}$ of all chord diagrams.
(2) There are three types of chord diagrams: P-type (nesting), O-type (alignment) and T-type (crossing). They are used to describe resonant solitons of water waves in the KP theory[Sarbarish Chakravarty and Yuji Kodama, 2008].



(3) The generating function of
$F(N,p,q) = \sum_{r,s=0}^{\frac{N(N-1)}{2}} c_{sr} p^s q^r, c_{rs} = c_{sr},$ where r is the number of nestings and s is the number of crossings, can be expressed by Stieltjes-type continued fraction. In particular, $F(1,0)$=the total number of non-crossing= $\frac{1}{N+1}\binom{2N}{N}$, the Catalan number.



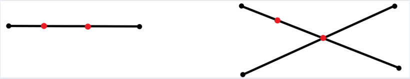
F(3,p,q)=(1+2p+ p^2+ p^3)+ (2+2p+2p^2)q+ (1+2p)q^2+q^3

Theorem

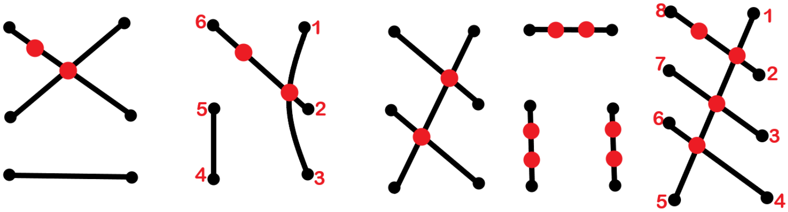
Given a chord diagram, one can construct a totally non-negative pfaffian.

Proof.

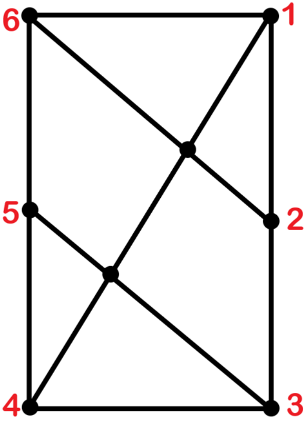
The proof is simple. One considers the following figures:



If there is no intersection, we add two points to the lines. If there are even intersection points, one will add one point to the intersection. If there are odd intersection points, one will add one extra point to a line such that the interior points are even. Then the Kuo-Ciucu Theorem is applied. □


6
1
5
2
4
3
8
1
2
7
3
6
5
4

Remark: Some pfaffians are singular ($M(G) = 0$) . One can construct more non-singular TNNPs from a chord diagram by adding more edges to the vertices.



$$A = \begin{pmatrix} 0 & 3 & 1 & 3 & 1 & 3 \\ -3 & 0 & 2 & 1 & 1 & 2 \\ -1 & -2 & 0 & 3 & 2 & 1 \\ -3 & -1 & -3 & 0 & 3 & 1 \\ -1 & -1 & -2 & -3 & 0 & 2 \\ -3 & -2 & -1 & -1 & -2 & 0 \end{pmatrix}$$

Pf(A)=49=7^2 X1

Pf(1234)=14=7 X 2

Thanks for your attention.