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Topological Feynman integrals and the odd graph complex

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Based on arXiv 2503.09558 (Documenta Mathematica ...) together with Simone Hu,
and arXiv 2408.03192 (JHEP ...) together with Davide Gaiotto.
Slides and links are available from paulbalduf.com/research.

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Introduction

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Theorem **[Balduf and Hu 2025].** The topological form is the Pfaffian form,

$$\alpha_G = \phi_G \qquad \text{(up to constants).}$$

This raises questions, among them:

1. What is the topological form α_G? What does it compute in topological QFT?
2. What is the Pfaffian form ϕ_G? How is it used in the odd graph complex?
3. (We skip the proof of the theorem)

   What can one learn from them being equal?

TQFT Propagator $P_n(\vec{x})$

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- Field theory for field Φ, Lagrangian $\mathcal{L} = \frac{1}{2}\Phi\mathcal{D}\Phi + \ldots$ quadratic part is "free field differential operator" $\mathcal{D}$. E.g. $\mathcal{D} = \partial_\mu \partial^\mu - m^2$.
- Consider n-dimensional topological QFT, position variable $\vec{x} = (x^{(1)}, \ldots, x^{(n)})^\intercal$ with $\mathcal{D}$ = de Rham operator = exterior derivative:

$$\mathcal{D} = \mathrm{d} = \mathrm{d}x^{(1)}\partial_{x^{(1)}} + \mathrm{d}x^{(2)}\partial_{x^{(2)}} + \ldots + \mathrm{d}x^{(n)}\partial_{x^{(n)}}.$$

- Propagator is Green function of $\mathcal{D}$, hence $\mathrm{d}P_n(\vec{x}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta^n(\vec{x})\ \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. It is

$$P_n(\vec{x}) = \frac{\Omega_n}{|\vec{x}|^n} = \frac{\sum_{j=1}^n (-1)^j x^{(j)}\, \mathrm{d}x^{(1)} \wedge \widehat{\mathrm{d}x^{(j)}} \wedge \mathrm{d}x^{(n)}}{\sqrt{\vec{x}\cdot\vec{x}}^{\,n}}.$$

- Ω_n is the projective n-dimensional volume form (= $(n-1)$-dimensional infinitesimal surface element of a sphere in n dimensions). For example:

$$P_1 = \frac{x}{|x|} = \operatorname{sgn}(x), \qquad P_2 = \frac{x^{(2)}\,\mathrm{d}x^{(1)} - x^{(1)}\,\mathrm{d}x^{(2)}}{{x^{(1)}}^2 + {x^{(2)}}^2} = \frac{r^2 \sin^2\phi\,\mathrm{d}\phi + r^2\cos^2\phi\,\mathrm{d}\phi}{r^2} = \mathrm{d}\phi.$$

Parametric representation of the TQFT propagator $P_n(\vec{x})$

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- Recall integral representation of Euler gamma function,

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\left|\vec{x}\right|^n} = \int_0^\infty e^{-\frac{\vec{x}^2}{a}} \frac{\mathrm{d}a}{a^{\frac{n}{2}+1}}.$$

- [Gaiotto, Kulp, and Wu 2025; Budzik et al. 2023] For each component $x^{(j)}$ introduce $s^{(j)} := \frac{x^{(j)}}{\sqrt{a}}$. Then, $\mathrm{d}s^{(j)} = \frac{\mathrm{d}x^{(j)}}{a^{\frac{1}{2}}} - \frac{x^{(j)}}{2a^{\frac{3}{2}}}\mathrm{d}a$. Explicit calculation yields (recall $\mathrm{d}a \wedge \mathrm{d}a = 0$):

$$\mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\mathrm{d}x^{(1)} \wedge \ldots \wedge \mathrm{d}x^{(n)}}{a^{\frac{n}{2}}} + \frac{\mathrm{d}a \wedge \Omega_n}{2a^{\frac{n}{2}+1}}.$$

- If one integrates a, first term vanishes, and

$$\int_0^\infty e^{-\vec{s}^2}\,\mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\Gamma\left(\frac{n}{2}\right)}{2}\frac{\Omega_n}{\left(\vec{x}^2\right)^{\frac{n}{2}}} = \frac{\Gamma\left(\frac{n}{2}\right)}{2}P_n(\vec{x}).$$

- Notice that the integrand factorizes: $e^{-s^{(1)2}}\,\mathrm{d}s^{(1)} \wedge e^{-s^{(2)2}}\,\mathrm{d}s^{(2)} \wedge e^{-s^{(3)2}}\,\mathrm{d}s^{(3)} \wedge \ldots$.

Brackets

- Use BRST formalism: BRST differential Q such that gauge-invariant "physical" observables A are 0^{th} cohomology group. That is,

$$QA = 0 \qquad \text{and} \qquad \nexists B : A = QB.$$

- A classically gauge invariant observable might violate gauge invariance at quantum level ("anomaly"). Work in perturbation theory, let $\mathcal{O}_j$ be local operators. Define *bracket* [Gaiotto, Kulp, and Wu 2025]

$$\{\mathcal{O}_1, \dots, \mathcal{O}_k\} := Q\left(\int_{\mathbb{R}^{n(k-1)}} \mathcal{O}_1 \cdots \mathcal{O}_k\right).$$

- The integral is a sum over Feynman integrals with k vertices in the n-dimensional TQFT,

$$\{\mathcal{O}_1, \mathcal{O}_2, \dots\} = \sum_{\text{Graphs } G} \frac{1}{|\mathrm{Aut}(G)|} I_G \prod_{v \in V_G} \prod_i \varphi_{i,v}.$$

symmetry factor

Feynman integral

External leg structure

The topological form

- Recall that parametric integrand factorizes along dimension ⇒ consider 1-dimensional integrand. Schwinger parameter $a_e \in \mathbb{R}$ for each edge. Then the *topological form* α_G is a differential form of degree ℓ in a_e,

$$I_G = \int \underbrace{\alpha_G \wedge \alpha_G \wedge \dots}_{n \text{ factors}} \qquad \text{where} \quad \alpha_G := \frac{1}{\pi^{\frac{|E_G|}{2}}} \int_{\mathbb{R}^{|V_G|-1}} \cdots \int \bigwedge_{e \in E_G} e^{-s_e^2}\, \mathrm{d}s_e .$$

  The integral in α_G is over vertex positions $x_v \in \mathbb{R}$.

- Key results of [Balduf and Gaiotto 2025]:

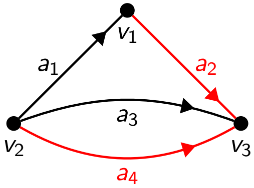
$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^\ell \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{\substack{T \text{ spanning} \\ \text{tree}}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma \in \mathfrak{S}_T} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)} \right) \bigwedge_{f \notin T} \mathrm{d}a_f ,$$

  and $\alpha_G \wedge \alpha_G = 0$ for all graphs (Kontsevich Formality theorem).

  Here $\mathbb{I}$ is the edge-vertex incidence matrix, ψ_G is the Symanzik polynomial, ψ^{e_1,e_2} are edge-induced Dodgson polynomials (See appendix. All of these can be produced easily with a computer).

Topological differential form for the dunce's cap

$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^{\ell} \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{\substack{T \text{ spanning}\\ \text{tree}}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma \in \mathfrak{S}_T} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)} \right) \bigwedge_{f \notin T} \mathrm{d}a_f .$$



G has five spanning trees T. For example, consider $T = \{2, 4\}$. Then $E \setminus T = \{f_1, f_2\} = \{1, 3\}$ and $\mathbb{I}[T] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\psi^{1,3} = -a_4$ (I didn't introduce how to compute this, see appendix). One obtains the contribution

$$\frac{(+1)}{16\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} \cdot (-2a_4)\,\mathrm{d}a_1 \wedge \mathrm{d}a_3 .$$

End result:

$$\alpha_G = \frac{-a_4(\mathrm{d}a_1 \wedge \mathrm{d}a_3 + \mathrm{d}a_2 \wedge \mathrm{d}a_3) + a_3(\mathrm{d}a_1 \wedge \mathrm{d}a_4 + \mathrm{d}a_2 \wedge \mathrm{d}a_4) - (a_1 + a_2)\,\mathrm{d}a_3 \wedge \mathrm{d}a_4}{8\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} .$$

Pfaffians

- Let M be a $2n \times 2n$ skew-symmetric matrix. The *Pfaffian* is

$$\operatorname{Pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot M_{\sigma(1),\sigma(2)} \cdots M_{\sigma(2n-1),\sigma(2n)}.$$

- If a skew-symmetric M has odd dimensions, set $\operatorname{Pf}(M) = 0$.
  Then $\operatorname{Pf}(M)^2 = \det(M)$ for all skew-symmetric matrices.
- This (like the determinant) assumes that the entries of M commute.
- Examples:

$$\operatorname{Pf}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b, \qquad \operatorname{Pf}\begin{pmatrix} 0 & b & c & d \\ -b & 0 & g & h \\ -c & -g & 0 & l \\ -d & -h & -l & 0 \end{pmatrix} = bl - ch + dg.$$

The Pfaffian form

- Consider a graph with even loop number ℓ. Collect Schwinger parameters in diagonal matrix $\mathbb{D}$. Let $\mathcal{C}$ be its edge-cycle incidence matrix, and $\mathbb{\Lambda} = \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$ the cycle Laplacian, and $\mathrm{d}\mathbb{\Lambda}$ its differential w.r.t. Schwinger parameters,

  $$\mathrm{d}\mathbb{\Lambda} = \mathrm{d}\left(\mathcal{C}^\intercal \mathbb{D} \mathcal{C}\right) = \mathcal{C}^\intercal \, \mathrm{d}\mathbb{D} \mathcal{C}.$$

  Then the matrix $\mathrm{d}\mathbb{\Lambda} \cdot \mathbb{\Lambda}^{-1} \cdot \mathrm{d}\mathbb{\Lambda}$ is a $\ell \times \ell$ (=even), skew-symmetric matrix whose entries are 2-forms (hence they commute).
- The *Pfaffian form* is defined as [Brown, Hu, and Panzer 2024]

  $$\phi_G := \frac{1}{(-2\pi)^{\frac{\ell}{2}}} \frac{\operatorname{Pf}\left(\mathrm{d}\mathbb{\Lambda} \cdot \mathbb{\Lambda}^{-1} \cdot \mathrm{d}\mathbb{\Lambda}\right)}{\sqrt{\det \mathbb{\Lambda}}}.$$

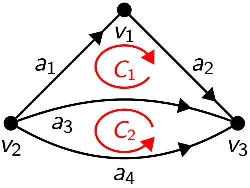
- Change of cycle basis $\mathcal{C}' = A^\intercal \mathcal{C} A$ with constant matrix A leads to

  $$\mathrm{d}\mathbb{\Lambda}' \mathbb{\Lambda}'^\intercal \, \mathrm{d}\mathbb{\Lambda} = A^\intercal \, \mathrm{d}\mathbb{\Lambda} A \; \left(A^\intercal \mathbb{\Lambda} A\right)^{-1} A^\intercal \, \mathrm{d}\mathbb{\Lambda} A = A^\intercal \; \mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1} \mathrm{d}\mathbb{\Lambda} \; A$$

  known: $\operatorname{Pf}(A^\intercal B A) = \det(A) \operatorname{Pf}(B)$.

  $\Rightarrow \phi_G$ changes sign by $\det(A)$ under change of basis (becomes important later!).

Example: Pfaffian form of the dunce's cap



$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{\Lambda}^{-1} = \frac{1}{\psi_G} \begin{pmatrix} a_3 + a_4 & -a_3 \\ -a_3 & a_1 + a_2 + a_3 \end{pmatrix}$$

$$\mathbb{\Lambda} = \begin{pmatrix} a_1 + a_2 + a_3 & a_3 \\ a_3 & a_3 + a_4 \end{pmatrix}, \quad \mathrm{d}\mathbb{\Lambda} = \begin{pmatrix} \mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 & \mathrm{d}a_3 \\ \mathrm{d}a_3 & \mathrm{d}a_3 + \mathrm{d}a_4 \end{pmatrix}.$$

$\mathrm{d}\mathbb{\Lambda}\, \mathbb{\Lambda}^{-1}\, \mathrm{d}\mathbb{\Lambda}$ is a 2×2 matrix. Recall $\operatorname{Pf} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b$.

We only need the top right entry of

$$\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda} = \frac{1}{\psi_G} \begin{pmatrix} \mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 & \mathrm{d}a_3 \\ \mathrm{d}a_3 & \mathrm{d}a_3 + \mathrm{d}a_4 \end{pmatrix} \begin{pmatrix} (a_3 + a_4)(\mathrm{d}a_1 + \mathrm{d}a_2) + a_4\,\mathrm{d}a_3 & a_4\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_4 \\ -a_3(\mathrm{d}a_1 + \mathrm{d}a_2) + (a_1 + a_2)\,\mathrm{d}a_3 & (a_1 + a_2)(\mathrm{d}a_3 + \mathrm{d}a_4) + a_3\,\mathrm{d}a_4 \end{pmatrix}$$

This yields

$$\phi_G = \frac{a_4\,\mathrm{d}a_1\,\mathrm{d}a_3 + a_4\,\mathrm{d}a_2\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_1\,\mathrm{d}a_4 - a_3\,\mathrm{d}a_2\,\mathrm{d}a_4 + (a_1 + a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{-2\pi\psi_G^{\frac{3}{2}}}.$$

The main result

Compare the two example calculations for the dunce's cap:

$$\phi_G = \frac{a_4\,\mathrm{d}a_1\,\mathrm{d}a_3 + a_4\,\mathrm{d}a_2\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_1\,\mathrm{d}a_4 - a_3\,\mathrm{d}a_2\,\mathrm{d}a_4 + (a_1+a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{-2\pi\psi_G^{\frac{3}{2}}},$$

$$\alpha_G = \frac{-a_4(\,\mathrm{d}a_1\,\mathrm{d}a_3 + \,\mathrm{d}a_2\,\mathrm{d}a_3) + a_3(\,\mathrm{d}a_1\,\mathrm{d}a_4 + \,\mathrm{d}a_2\,\mathrm{d}a_4) - (a_1+a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{8\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} = \frac{1}{4}\phi_G.$$

Theorem [Balduf and Hu 2025]. Let $\mathcal{C}$ be any choice of cycle incidence matrix and $\mathcal{P}$ any choice of path matrix, then $\det(\mathcal{C}\,|\,\mathcal{P}) \in \{+1,-1\}$ and for all graphs

$$\alpha_G = \frac{\det(\mathcal{C}\,|\,\mathcal{P})}{2^\ell}\cdot\phi_G.$$

Proof: Linear algebra, expansion formulas for Pfaffians, match the Dodgson polynomial formula for the topological form α_G.

What is the Pfaffian form good for?

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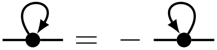
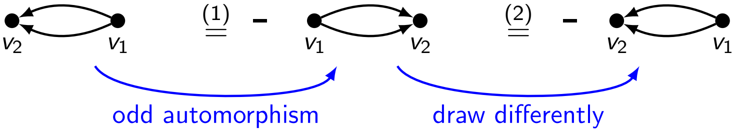
It solves a combinatorics problem on the odd graph complex...

The odd graph complex

- The *odd graph complex* GC_3 is a $\mathbb{Q}$ vector space of “oriented graphs” (G, η),
- G has vertex valence at least 3, modulo graph isomorphism.
  Grading $\deg(G) = |E| - 3\ell$.
- Orientation η is (ordering of vertices and a choice of edge directions).
  orientation is equivalent to (cycle basis and edge order) [Conant and Vogtmann 2003].
- E.g. Tadpoles vanish: 
- More generally, graphs with odd automorphism (=exchange odd number of elements) vanish. E.g. multi edges with even number of edges:

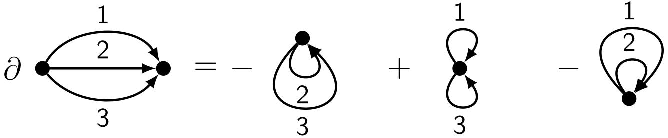


Boundary map of graph complexes

- Let G/γ denote shrinking of subgraph $\gamma \subset G$ to a vertex. Define the boundary operator

$$\partial(G,\eta) = \sum_{j=1}^{n} (-1)^j \left(G/e_j, \eta/e_j\right).$$

Example:



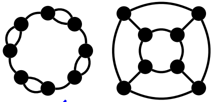
- *Graph homology* is $H_\bullet(\mathrm{GC}_3) = \ker \partial / \operatorname{im} \partial$.
  I.e. we want graphs G such that $\partial G = 0$ and there is no F with $\partial F = G$.
  Homology is graded by degree, H_n where $n = \deg(G) = |E| - 3\ell$, and by loop number .
- Example: The above graph D_3 (=dipole on 3 edges) has $\partial D_3 = 0$ since all resulting graphs contain tadpoles. $\deg(D_3) = 6 - 3 \times 2 = 0$. Turns out it is not exact, $\nexists F : \partial F = D_3$. Hence $D_3 \in H_0(\mathrm{GC}_3)$, at loop number $\ell = 2$.

Homology of the odd graph complex

- *Graph homology* is $H_\bullet(\mathrm{GC}_3) = \ker \partial / \operatorname{im} \partial$. It is graded by (homological) degree, H_n where $n = \deg(G) = |E| - 3\ell$, and by loop number.
- Homologies are known up to $\ell \approx 10$ [Brun and Willwacher 2024]. One finds only few classes, but for $\ell \to \infty$, their dimension grows super-exponentially [Borinsky and Zagier 2024].
- H_{-3} related to “algebra of 3-graphs” [Duzhin, Kaishev, and Chmutov 1998; Vogel 2011].

Homologies of GC_3:

H_{-8}		known to						0	
H_{-7}		vanish					0	0	
H_{-6}						1	1	2	
H_{-5}					0	0	0	0	
H_{-4}				0	0	0	0	0	
H_{-3}		1	1	1	2	2	3	4	
ℓ	1	2	3	4	5	6	7	8	...



Brown’s canonical differential forms

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- Let G be a connected graph with cycle Laplacian $\mathbb{\Lambda} = \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$. Define *canonical form* [Brown 2021]

$$\beta_G^n := \operatorname{tr}\left(\left(\mathbb{\Lambda}^{-1} \mathrm{d}\mathbb{\Lambda} \right)^n \right).$$

(distinct objects are called “canonical forms” in the literature. This one is canonical because it is invariant under multiplying $\mathbb{\Lambda}$ by any invertible matrix A with $\mathrm{d}A = 0$.)

- β has various good properties, for example
  - $\mathrm{d}\beta^{4k+1} = 0$,
  - if $k > 0$, the form is projectively invariant,
  - β_G^n is zero unless $n = 4k+1$ for $k \in \mathbb{N}_0$,
  - have algebra structure, where products might have different degree. E.g. $\beta^5 \wedge \beta^9$ has degree $14 \neq 4k+1$.
  - Have Hopf algebra structure where β_j are primitive (i.e. define a coproduct Δ such that $\Delta \beta^{4k+1} = \mathbb{1} \otimes \beta^{4k+1} + \beta^{4k+1} \otimes \mathbb{1}$).
  - If ω_G is a canonical form of degree n and $|E| = n+1$, then ω is proportional to the projective volume form $\Omega_{|E|}$,

$$\omega_G = \frac{\text{some polynomial}}{\psi^{\text{some integer}}} \Omega_{|E|}.$$

Computing graph homology with canonical integrals

- Canonical forms can be used to find cohomology classes in the graph complex.
  Let G be some linear combination of graphs such that $\partial G = 0$ (this can be checked by explicit computation).
  Hard part: How to establish whether $\exists F$ such that $\partial F = G$?
- As $\mathrm{d}\beta = 0$, also $\int_F \mathrm{d}\beta = 0$ for every graph F,
  where $\int_F = \int_{\sigma_F}$ with $\sigma_F = [a_1 : \ldots : a_{|E|}] \in \mathbb{P}(\mathbb{R}^{|E|})_+$ ("graph simplex").
- Stokes theorem:

$$0 = \int_F \mathrm{d}\beta = \int_{\partial F} \beta = \int_G \beta \qquad (\text{if } \partial F = G).$$

  This integral vanishes for all primitive canonical forms β.
  (There are more terms for a non-primitive $\omega = \beta \wedge \beta \wedge \ldots$, but it still vanishes).
- Conversely: If one finds *any* β such that $\int_G \beta \neq 0$, one knows that $G \neq \partial F$.
  This is a proof that G is not exact, and since $\partial G = 0$, this G defines a cohomology class in the even graph complex.

The role of the Pfaffian form

- The canonical forms β_G^{4k+1} are *invariant* under change of cycle basis (i.e. they operate on the *even* graph complex).
- The odd graph complex requires a form that flips sign in the same way as the graphs do.
- The Pfaffian form ϕ_G has this property [Brown, Hu, and Panzer 2024], it is an "orientation form". Concretely, for a change of cycle basis, $\mathbb{\Lambda} \mapsto A^\intercal \mathbb{\Lambda} A$, we have

$$\beta_G^{4k+1} \mapsto \beta_G^{4k+1}, \qquad \text{but} \qquad \phi_G \mapsto \det(A)\phi_G.$$

  $\Rightarrow \int_G \phi_G \wedge \omega$ is well-defined on the odd graph complex, where ω is any product of β forms.
- Can use $\int_G \phi_G \wedge \omega$ to detect homology: If this integral is $\neq 0$, then $G \neq \partial F$.
- Example from [Brown, Hu, and Panzer 2024]: For $\ell = 6$, the form $\beta^5 \wedge \phi$ is of degree 11. There is a linear combination of graphs with $\ell = 6$ and $|E| = 12$ where the integral is non-vanishing, it spans the homology H_{-6} at $\ell = 6$.

Summary

- There is a certain "topological" differential form, α_G, of degree ℓ in Schwinger parameters which computes BRST anomalies in TQFTs.
- There is another, "Pfaffian", differential form, ϕ_G, of degree ℓ which realizes the combinatorial sign of the odd graph complex GC_3 and therefore makes integrals $\int_G \phi_G \wedge \omega_G$ well-defined. These integrals detect homology classes in GC_3.
- The two forms are the same.

Consequences

A certain physics question (anomalies in TQFT) and a certain pure math problem (homology of GC_3) are answered by the same method in differential geometry. We have ...

- Obtained physical interpretation of the Pfaffian form ϕ_G: It computes BRST anomalies.
- Obtained a nice new representation for the topological α_G in terms of relatively simple matrices. $\Rightarrow$ many of its properties follow easily from linear algebra, or from known properties of ϕ_G
  - $\mathrm{d}\alpha_G = 0$, and $\int \alpha_G$ is finite, projective, well-defined under change of labelings, etc.
  - Much simplified proof of Kontsevich formality theorem $\alpha_G \wedge \alpha_G = 0$ (i.e. there are no anomalies in topological QFT with 2 or more dimensions).
- Shown that their properties match one by one, e.g.
  - L_∞-relations of topological form α_G correspond to Stokes relations of Pfaffian form ϕ_G.
  - The sum of dipole/multi-edge graphs plays a special role on both sides.
  - On both sides, one is interested in products between this form and some other forms.

Open question: Was it clear that they are the same? What is the fundamental relation between graph cohomology and anomalies in QFT?

Thank you!

Background: Deformation quantisation

- Given is a *classical field theory*: Smooth manifold M. Field variable $\phi(t,\mathbf{x})$, canonical conjugate $\pi(t,\mathbf{x})$ are smooth functions on M. Hamilton function $H(\phi(t,\mathbf{x}),\pi(t,\mathbf{x}))$. Skew-symmetric *Poisson bracket* $\{f,g\}\in C^\infty(M)$. Gives equations of motion:

$$\partial_t\phi=\{\phi,H\}\,,\qquad \partial_t\pi=\{\pi,H\}\,,\qquad \{\phi,\pi\}=1.$$

- Naive quantisation: Replace $\{f,g\}$ by $\frac{i}{\hbar}\left[\hat{f},\hat{g}\right]$. Runs into inconsistencies for powers of fields. Deformation quantisation: Find a "star product" $\star$ such that

$$[f,g]_\star := f\star g-g\star f\overset{!}{=}\hbar\{f,g\}+\mathcal{O}\left(\hbar^2\right).$$

- Power series ansatz with (to be determined) differential operators $B_j(f,g)$.

$$f\star g=B_0(f,g)+\hbar B_1(f,g)+\hbar^2B_2(f,g)+\ldots,$$

  Clearly $B_0(f,g)=f\cdot g$ and $B_1(f,g)=\frac{1}{2}\{f,g\}$. What are the higher B_j?
- Two conditions:
  1. Should be associative $f\star(g\star h)=(f\star g)\star h$,
  2. Should be invariant under diffeomorphisms $f\mapsto f+\hbar D_1(f)+\hbar^2D_2(f)+\ldots$.

Background: Deformation quantisation 2

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- Solution in [Kontsevich 2003]: Consider graphs Γ embedded in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ with hyperbolic metric.
- In Γ, each vertex with 2 outgoing edges corresponds to a factor $\omega^{ij}\partial_i\partial_j$. (i.e. a graph Γ encodes a nesting of Poisson brackets, a differential operator B_Γ). Graph has n upper vertices and 2 vertices at bottom line $\mathbb{R}$, corresponding to arguments f, g of $B_n(f, g)$.
- Define angle $\phi(p, q)$ between geodesic $p \longrightarrow q$ and vertical line $p \longrightarrow i\infty$.
- Each graph is weighted by a weight integral $W_\Gamma = \text{const} \times \int \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e$. Star product is (details omitted)

$$\star = \cdot \; + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma} W_\Gamma B_\Gamma .$$

Background: Deformation quantisation 3

- Crucial step: Show that the so-defined $\star$ is associative.
- Associativity condition at order $\hbar^n$,

$$\sum_{k=0}^{n} B_k \left(B_{n-k}(f,g), h \right) = \sum_{k=0}^{n} B_k \left(f, B_{n-k}(g,h) \right),$$

  amounts to insertion of operators B_j, hence nesting/shrinking of graphs.
- Obstructions to associativity are given by certain integrals over the boundary of configuration space,

$$c_\Gamma = \int\limits_{\partial \bar{C}_{n,m}} \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e .$$

  These integrals can be shown to vanish and $\star$ is associative. More general, abstract statement: "Formality theorem".

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

- α_Γ is a differential form in $\mathrm{d}a_e$, to be integrated over Schwinger parameters a_e. Itself, α_Γ is an integral over vertex positions x_v of some integrand W_Γ. Schematically:

$$\mathcal{F}(\Gamma) = \int_{\{a_e\}} \alpha_\Gamma = \int_{\{a_e\}} \int_{\{x_v\}} W_\Gamma, \qquad W_\Gamma = \bigwedge_{e \in E_\Gamma} e^{-s_e^2}\, \mathrm{d}s_e.$$

- There is one Schwinger variable a_e for each edge, but there could be more than one coordinate x_v for each vertex (i.e. the vertex coordinate is a vector $(x_v^{(1)}, x_v^{(2)}, \ldots)$. Consider a 2-dimensional theory

$$\mathcal{F}(\Gamma) = \int_{\{a_e\}} \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma.$$

- Here, $\alpha_\Gamma \wedge \alpha_\Gamma$ is some differential form in the $\mathrm{d}a_e$'s, independent of the x_v. Conversely, we can exchange the order of integration and do the $\mathrm{d}a_e$ integral first. The integrand is

$$W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \exp\left(-\sum_e \left({s_e^{(1)}}^2 + {s_e^{(2)}}^2\right)\right) \bigwedge_e \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)}.$$

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

$$\int_{\{a_e\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \exp\left(- \sum_e \left({s_e^{(1)}}^2 + {s_e^{(2)}}^2 \right) \right) \bigwedge_e \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)}$$

- This expression factorizes for edges. Consider an edge e from point $(0,0)$ to $(x^{(1)}, x^{(2)})$:

$$e^{-{s_e^{(1)}}^2 - {s_e^{(2)}}^2} \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)} = e^{-\frac{\vec{x}^2}{a}} \left(-2a_e^{-2}\,\mathrm{d}a_e \left(x^{(2)}\,\mathrm{d}x^{(1)} - x^{(1)}\,\mathrm{d}x^{(2)} \right) + a_e^{-1}\,\mathrm{d}x^{(1)} \wedge \mathrm{d}x^{(2)} \right).$$

- Only the term $\propto \mathrm{d}a_e$ contributes to integral. Polar coordinates in the plane:

$$\vec{x} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} = r \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}, \qquad \frac{\mathrm{d}\vec{x}}{\mathrm{d}\phi} = r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} -x^{(2)} \\ x^{(1)} \end{pmatrix}.$$

$\Rightarrow \qquad x^{(1)}\,\mathrm{d}x^{(2)} - x^{(2)}\,\mathrm{d}x^{(1)} = \left((-x^{(2)})^2 + (x^{(1)})^2 \right) \mathrm{d}\phi = |\vec{x}|^2\,\mathrm{d}\phi$ is the differential of the 2D angle ϕ of the vector $\vec{x}$.

- Integrate the Schwinger parameter a_e for a single edge:

$$\int_{a_e=0}^{\infty} e^{-{s^{(1)}}^2 - {s^{(2)}}^2} \mathrm{d}s^{(1)} \wedge \mathrm{d}s^{(2)} = \int_{a_e=0}^{\infty} e^{-\frac{|\vec{x}|^2}{a_e}} 2a_e^{-2} |\vec{x}|^2\,\mathrm{d}\phi_e \wedge \mathrm{d}a_e = 2\,\mathrm{d}\phi_e.$$

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

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- We conclude that the 2-dimensional integral is (very schematically)

$$\mathcal{F}(\Gamma) = \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} \int_{\{a_e\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{\text{relative positions } \vec{x}_v\}} \bigwedge_e \mathrm{d}\phi_e .$$

  Closer investigation of the last integral shows: These are the Kontsevich integrals c_Γ which need to vanish in order to make the star product associative and establish the *formality theorem*.

- On the other hand:

$$c_\Gamma = \mathcal{F}(\Gamma) = \int_{\{a_e\}} \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma$$

- Hence $\int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma = 0$ implies the vanishing of Kontsevich integrals.

Formality theorem

- Kontsevich formality theorem [Kontsevich 2003] $\alpha_G \wedge \alpha_G = 0$ (there are no anomalies in TQFTs with $D \geq 2$) proved with some effort in [Balduf and Gaiotto 2025; Wang and Williams 2024].
- Now use that $\mathrm{Pf}(A)^2 = \det(A)$:

$$\phi_G \wedge \phi_G \propto \frac{1}{\det \mathbb{\Lambda}} \left(\mathrm{Pf}\left(\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)\right)^2 = \det\left(\mathbb{\Lambda}^{-1}\right)\det\left(\mathrm{d}\mathbb{\Lambda}\;\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right) = \det\left(\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\;\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)$$

$$= \det\left(\left(\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)^2\right) =: \det\left(M\right) = \frac{1}{(\ell/2)!} B_n\left(s_1, s_2, \ldots\right),$$

where we defined $M := \left(\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)^2$, B_n are Bell polynomials, and

$$s_j = -\frac{(j-1)!}{2}\,\mathrm{tr}\left(M^j\right) = -\frac{(j-1)!}{2}\,\mathrm{tr}\left(\left(\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)^{2j}\right) = -\frac{(j-1)!}{2}\beta_G^{2j} = 0 \quad \forall j.$$

(recall that only $\beta^{4k+1} \neq 0$ due to cyclicity of trace and symmetry of $\mathbb{\Lambda}$).

- Hence $\phi_G \wedge \phi_G = 0$, and therefore $\alpha_G \wedge \alpha_G = 0$.

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$$s_j = -\frac{(j-1)!}{2}\,\mathrm{tr}\left(M^j\right)$$

- Hence $\phi_G \wedge \phi_G = 0$, and therefore $\alpha_G \wedge \alpha_G = 0$.

Graph matrices 1: Incidence matrix and Laplacian

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- Always assume that the graph G is connected. Edge set E, vertex set V.
- $|E| \times (|V|-1)$ *incidence matrix* $\mathbb{I}$ has entry $\mathbb{I}_{e,v} = +1$ if edge e ends at vertex v, and -1 if e starts at v, and 0 else. Column of one vertex $v_\star$ left out.
- $|E| \times |E|$ *edge variable matrix* $\mathbb{D} = \operatorname{diag}\left(a_1, \ldots, a_{|E|}\right)$ contains Schwinger parameters.
- $(|V|-1) \times (|V|-1)$ *vertex Laplacian*

$$\mathbb{L} := \mathbb{I}^\intercal \mathbb{D}^{-1} \mathbb{I}.$$

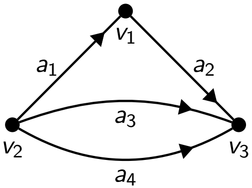
- *First Symanzik polynomial*

$$\psi_G := \det \mathbb{L} \cdot \det \mathbb{D} = \det \mathbb{L} \ \cdot \prod_{e \in E} a_e = \sum_{T \text{ spanning}} \prod_{e \notin T} a_e$$

is homogeneous of degree ℓ in the variables a_e.

Example: The dunce's cap

"Dunce's cap" G is a graph on 3 vertices and 4 edges, with $\ell = 2$ loops. Labels and directions are chosen as:



We further choose $v_3 =: v_\star$ as the vertex to remove from $\vec{x}$.
Remaining: $|V| = 2, |E| = 4$.
$\mathbb{I}$ is 4×2 and $\mathbb{D}$ is 4×4.

With these choices:

$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{D} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

This gives the Laplacian $\mathbb{L} = \mathbb{I}^\intercal \mathbb{D} \mathbb{I}$:

$$\mathbb{L} = \begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2} & -\frac{1}{a_1} \\ -\frac{1}{a_1} & \frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_4} \end{pmatrix}.$$

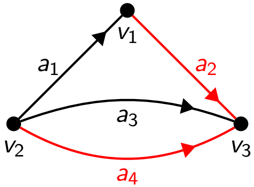
Symanzik polynomial:

$$\psi_G = \det \mathbb{L} \cdot \prod_{e \in E} a_e = a_3 a_4 + a_1(a_3 + a_4) + a_2(a_3 + a_4).$$

(Notice *matrix tree theorem*: The terms of ψ are the complements of spanning trees, $\psi = \sum_T \prod_{e \notin T} a_e$).

Topological differential form for the dunce's cap

$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^{\ell} \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{\substack{T \text{ spanning}\\ \text{tree}}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma \in \mathfrak{S}_T} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)} \right) \bigwedge_{f \notin T} \mathrm{d}a_f .$$



G has five spanning trees T. For example, consider $T = \{2, 4\}$. Then $E \setminus T = \{f_1, f_2\} = \{1, 3\}$ and $\mathbb{I}[T] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\psi^{1,3} = -a_4$ (I didn't introduce how to compute this). One obtains the contribution

$$\frac{(+1)}{16\pi(a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)^{3/2}} \cdot (-2a_4)\, \mathrm{d}a_1 \wedge \mathrm{d}a_3 .$$

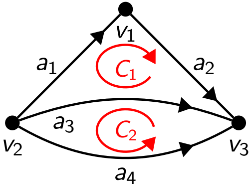
End result:

$$\alpha_G = \frac{-a_4(\mathrm{d}a_1 \wedge \mathrm{d}a_3 + \mathrm{d}a_2 \wedge \mathrm{d}a_3) + a_3(\mathrm{d}a_1 \wedge \mathrm{d}a_4 + \mathrm{d}a_2 \wedge \mathrm{d}a_4) - (a_1 + a_2)\, \mathrm{d}a_3 \wedge \mathrm{d}a_4}{8\pi(a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)^{3/2}} .$$

Graph matrices 2: Cycle incidence matrix

- A *circuit* is a closed path of edges (regardless of edge directions). May visit vertex, but not edge, multiple times.
- Circuits can be added and subtracted, form a vector space over $\mathbb{Z}$ (mod ± 2). *Cycle space*, dimension: $|E| - |V| + 1 = \ell$ is *loop number*.
- A choice of basis for cycle space determines a *cycle incidence matrix* $\mathcal{C}$: Entry $\mathcal{C}_{e,c} = +1$ if edge e is in cycle c in positive direction, -1 if in negative direction.
- Analogously, vertex incidence matrix $\mathbb{I}$ represents a choice of basis in *cut space*.
- The spaces, and hence the matrices $\mathcal{C}$ and $\mathbb{I}$ are orthogonal, $\mathbb{I}^\intercal \mathcal{C} = \mathbb{0}_{(|V|-1)\times \ell}$, $\mathcal{C}^\intercal \mathbb{I} = \mathbb{0}_{\ell \times (|V|-1)}$.

Example: Cycles in the dunce's cap



$\ell = 2 \Rightarrow$ 2 linearly independent circuits to be chosen as basis of cycle space. This choice is not unique.

With C_1 and C_2 as drawn,
$C_1 = \{+a_1, +a_2, -a_3\}$ and $C_2 = \{-a_3, +a_4\}$.

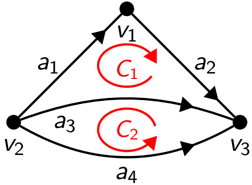
$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \text{recall} \quad \mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

Columns of $\mathcal{C}$ are basis vectors in cycle space, columns of $\mathbb{I}$ are basis vectors in cut space.
Cut space and cycle space are orthogonal, i.e.

$$\mathcal{C}^\intercal \mathbb{I} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Graph matrices 3: Cycle Laplacian

- Recall the vertex Laplacian $\mathbb{L} := \mathbb{I}^\intercal \mathbb{D}^{-1} \mathbb{I}$, is a $(|V|-1)\times(|V|-1)$ sym. matrix.
- Analogously *cycle Laplacian* is the $\ell \times \ell$ symmetric matrix $\mathbb{\Lambda} := \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$.
- Determinant is $\det \mathbb{\Lambda} = \psi_G$ (regardless of the choice of $\mathcal{C}$). Hence, $\mathbb{\Lambda}$ is invertible.



$$C_1 = \{+a_1, +a_2, -a_3\} \text{ and } C_2 = \{-a_3, +a_4\}.$$

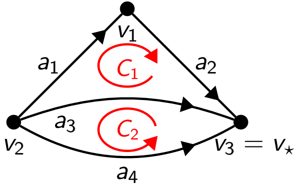
$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{\Lambda} = \begin{pmatrix} a_1+a_2+a_3 & a_3 \\ a_3 & a_3+a_4 \end{pmatrix}.$$

Inverse matrix denominator is Symanzik polynomial $\det \mathbb{\Lambda} = \psi_G$,

$$\mathbb{\Lambda}^{-1} = \frac{1}{\psi_G} \begin{pmatrix} a_3+a_4 & -a_3 \\ -a_3 & a_1+a_2+a_3 \end{pmatrix}.$$

Graph matrices 4: Path matrices

- A *path matrix* $\mathcal{P}$ is a $|E| \times (|V|-1)$–matrix where column j is a directed path of edges from $v_\star$ to v_j.
- $\mathcal{P}$ has the same shape as $\mathbb{I}$, but they are distinct. In fact, $\mathcal{P}^\intercal \mathbb{I} = \mathbb{1}_{(|V|-1)\times(|V|-1)}$.
- One can show that $\det(\mathcal{C} \,|\, \mathcal{P}) \in \{+1,-1\}$. This determinant encodes a (relative) sign ambiguity that arises from the choice of cycle basis in $\mathcal{C}$ [Conant and Vogtmann 2003].



Let $v_\star = v_3$ and paths $P_1 = \{a_1, -a_3\}$ and $P_2 = \{-a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

The concatenation $(\mathcal{C} \,|\, \mathcal{P})$ has full rank and $\det(\mathcal{C} \,|\, \mathcal{P}) = +1$.
One also checks that $\mathcal{P}^\intercal \mathbb{I} = \mathbb{1}_{2\times 2}$.
It is coincidence that all matrices have the same shape.

Dodgson polynomials

- Consider the *expanded Laplacian*, defined as the block matrix

$$\mathbb{M} := \begin{pmatrix} \mathbb{D} & \mathbb{I} \\ -\mathbb{I}^T & 0 \end{pmatrix}.$$

  One can show that $\det(\mathbb{M}) = \psi$.

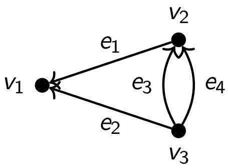
- Let $\mathbb{M}(A,B)$ be $\mathbb{M}$ with rows A and columns B removed. If $|A| = |B|$, this is a square matrix, and its determinant is called *Dodgson polynomial*

$$\psi^{A,B} := \det\left(\mathbb{M}(A,B)\right).$$

- In particular, if $A = \{i\}$ and $B = \{j\}$ each consist of only one index, the Dodgson polynomials $\psi^{i,j}$ are the cofactors of $\mathbb{M}$, i.e. they are entries of the inverse.
- $\mathbb{M}$ has block form, so $\mathbb{M}^{-1}$ has block form. Bottom right block is $\mathbb{L}^{-1}$. $\Rightarrow$ Lemma:

$$\left(\mathbb{L}^{-1}\right)_{i,j} = (-1)^{i+j} \frac{\psi^{i,j}}{\psi} \quad \text{(where } i,j \text{ are indices of vertices).}$$

Example: Dogson polynomials



$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{M} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 1 & -1 \\ 0 & a_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & a_4 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

In $\mathbb{M}$, the first 4 rows and columns refer to edges, the last 2 rows and columns refer to vertices v_1, v_2. Compute vertex-indexed Dodgson polynomials explicitly:

$$\psi^{v_1,v_1} = \det \begin{pmatrix} a_1 & 0 & 0 & 0 & -1 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 1 \\ 0 & 0 & 0 & a_4 & 1 \\ 1 & 0 & -1 & -1 & 0 \end{pmatrix} = a_2\left(a_1 a_3 + a_1 a_4 + a_3 a_4\right)$$

$$\psi^{v_1,v_2} = -a_2 a_3 a_4 = \psi^{v_2,v_1}, \qquad \psi^{v_2,v_2} = (a_1 + a_2) a_3 a_4 .$$

Indeed,

$$\mathbb{L}^{-1} = \frac{1}{a_3 a_4 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4} \begin{pmatrix} a_2(a_3 a_4 + a_1(a_3 + a_4)) & a_2 a_3 a_4 \\ a_2 a_3 a_4 & (a_1 + a_2) a_3 a_4 \end{pmatrix} .$$

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