A multivector generator of time evolution in the phase plane

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The mathematical problem

Consider a 2D linear dynamical system

$$\dot{\mathbf{X}} = A\mathbf{X}, \qquad A \in M_2(\mathbb{R}),$$

where

$$\mathbf{X}(t) \in \mathbb{R}^2,$$

and A is a real 2×2 matrix, the generator of the evolution in the phase plane.

A determines the complete phase-plane dynamics through its exponential map

$$\mathbf{X}(t) = e^{At}\,\mathbf{X}_0.$$

Different structures of A → different behaviors:

- ✓ Rotation (oscillatory dynamics),
- ✓ Reflections and anisotropic deformations,
- ✓ Expansion or contraction of phase-plane area.

Hamiltonian systems:

- ✓ $\mathrm{tr}A = 0$,
- ✓ Phase-plane area is preserved,
- ✓ The dynamics are reversible.

Dissipative systems:

- ✓ $\mathrm{tr}A \neq 0$,
- ✓ Phase-plane area is contracted/expanded,
- ✓ Irreversible dynamics.

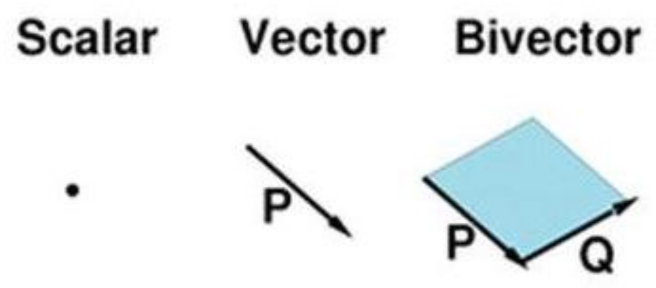
☞ **Question**: Can the generator A represented by a canonical geometric object that reveals its intrinsic geometric structure?

Ref.: V. I. Arnol'd, Mathematical methods of classical mechanics (Springer Science & Business Media, 2013).

The geometric algebra $Cl(2,0)$

Geometric Algebra $Cl(2,0)$ is the associative algebra generated by two orthonormal basis vectors e_1 and e_2 satisfying the Euclidean metric:

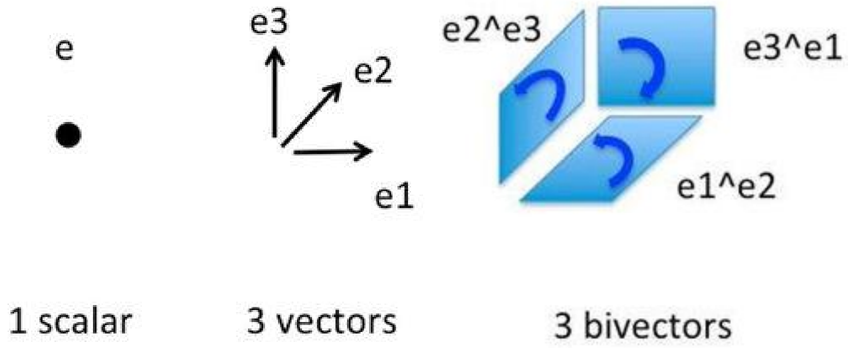
- ✓ $e_1^2 = +1,$
- ✓ $e_2^2 = +1,$
- ✓ $e_1 e_2 = -e_2 e_1$ (anticommutativity).



$Cl(2,0)$ naturally contains:

- ✓ Scalar: 1 → magnitude,
- ✓ Vectors: e_1, e_2 → directions,
- ✓ Bivector: $J = e_1 \wedge e_2 = e_1 e_2$ → oriented area.

- ✓ $J^2 = -1$ → isomorphic to the imaginary unit i.
- ✓ Acts as a geometric generator of 90° rotations in $e_1 - e_2$ plane.



The most general element in $Cl(2,0)$ is the multivector M defined as:

$$M = a1 + be_1 + ce_2 + dJ, \qquad a, b, c, d \in \mathbb{R}.$$

☞ **Goal**: Represent the dynamical generator A by a multivector M whose components have explicit geometric meaning.

Ref.: D. Hestenes, "The Genesis of Geometric Algebra: A Personal Retrospective," *Adv. Appl. Clifford Algebr. 27, 351* (2017).

Matrix representation of $Cl(2,0)$

Let

$$\Phi\colon\ Cl(2,0) \to M_2(\mathbb{R})$$

on the generators of $Cl(2,0)$ by

$$\Phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Phi(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi(J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Theorem 1

The map Φ extends uniquely to an algebra isomorphism

$$\Phi\colon\ Cl(2,0) \cong M_2(\mathbb{R}),$$

satisfying

$$\Phi(MN) = \Phi(M)\Phi(N), \qquad \forall M, N \in Cl(2,0).$$

For a general multivector $M = a1 + be_1 + ce_2 + dJ$, map Φ yields:

$$\Phi(M) = \begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix}.$$

This establishes a one-to-one correspondence between the coefficients a, b, c, d and a unique 2×2 matrix.

Proof

The matrices $\Phi(1)$, $\Phi(e_1)$, $\Phi(e_2)$, $\Phi(J)$ form a basis of $M_2(\mathbb{R})$. They preserve the defining Clifford relations

$$e_1^2 = e_2^2 = +1, \qquad e_1 e_2 = -e_2 e_1.$$

Hence, Φ is bijective and preserves multiplication.

Classical phase-plane dynamics using $Cl(2,0)$

Classical 2D linear system

$$\dot{\mathbf{X}} = A\mathbf{X}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} q \\ p \end{pmatrix}.$$

Embed the phase plane into $\boldsymbol{Cl(2,0)}$

Represent the phase-space state as

$$X = qe_1 + pe_2,$$

so that the Euclidean geometry of the plane is inherited

$$X^2 = q^2 + p^2,$$

The canonical symplectic form becomes

$$\omega = dq \wedge dp \quad \leftrightarrow \quad J.$$

Definition

By Theorem 1

$$M := \Phi^{-1}(A) = a1 + be_1 + ce_2 + dJ$$

with

$$a = \frac{\alpha + \delta}{2}, \qquad b = \frac{\alpha - \delta}{2},$$

$$c = \frac{\beta + \gamma}{2}, \qquad d = \frac{\beta - \gamma}{2}.$$

Hence,

$$\dot{X} = MX.$$

The Multivector Time Generator

We showed that

$$M = \Phi^{-1}(A) = a1 + be_1 + ce_2 + dJ,$$

satisfying

$$\dot{X} = MX.$$

☞ ***M* is called the Multivector Time Generator (MTG).**

Canonical grade decomposition

M admits the unique decomposition

$$M = M^{(0)} + M^{(1)} + M^{(2)}.$$

With respect to the basis $\{1, e_1, e_2, J\}$

$$M = a + (be_1 + ce_2) + dJ.$$

Equivalently,

$$M^{(0)} = a, \qquad M^{(1)} = be_1 + ce_2, \qquad M^{(2)} = dJ.$$

The symplectic condition in $Cl(2,0)$

A matrix A is considered as a Hamiltonian (i.e., it preserves the symplectic structure) if and only if:

$$A^T J + JA = 0.$$

This holds true if and only if:

$$\mathrm{tr}A = 0.$$

Theorem 2

Let

$$M = M^{(0)} + M^{(1)} + M^{(2)}$$

be the canonical grade decomposition of the MTG. Then

$$\widetilde{M}J + JM = 2M^{(0)}J.$$

Equivalently,

$$\widetilde{M}J + JM = 2aJ,$$

thus

$$2aJ = 0 \Leftrightarrow a = 0.$$

Proof

Using

$$\widetilde{e_i} = e_i, \qquad \tilde{J} = -J,$$

we obtain

$$\widetilde{M} = a + be_1 + ce_2 - dJ.$$

Hence,

$$\widetilde{M}J + JM = (a + V - B)J + J(a + V + B),$$

where

$$V = be_1 + ce_2, \qquad B = dJ.$$

Since

$$VJ = -JV, \qquad BJ = -JB,$$

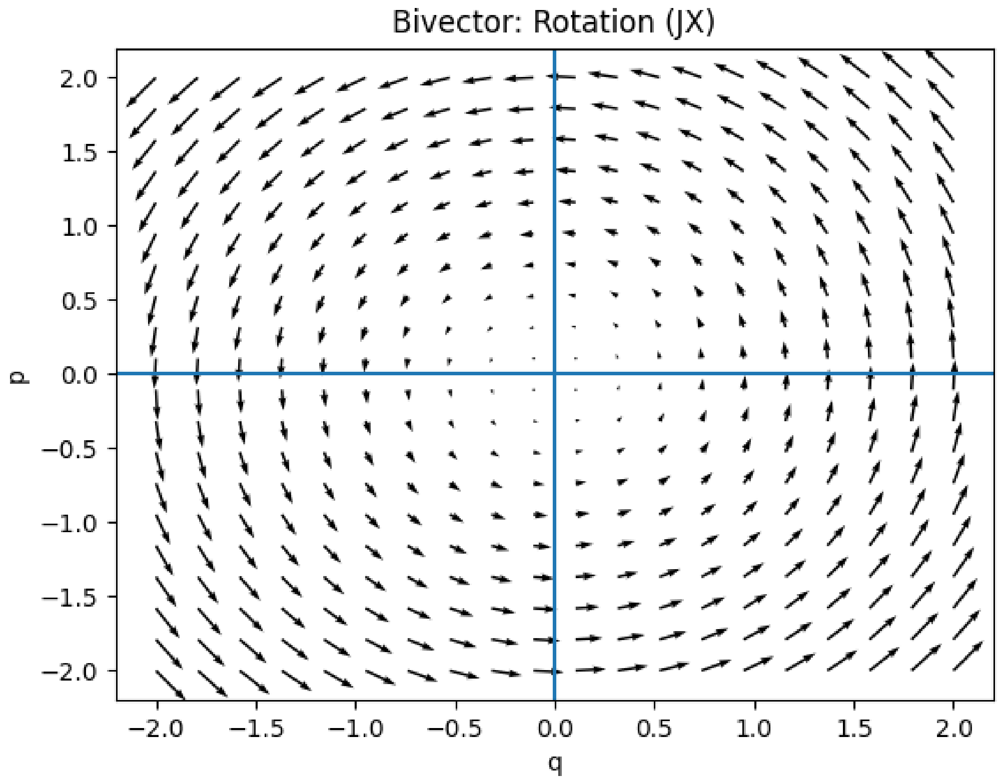
all vector and bivector contributions cancel, leaving

$$2aJ.$$

Bivector component as a generator of rotation

- Bivector part of the multivector M: dJ.
- Already said that $J^2 = -1$ → produces rotation in the phase plane.
- Using the symplectic condition, it is easily shown that $\widetilde{(dJ)}J + J(dJ) = 0$.
- Corresponds to a reversible oscillatory motion.

☞ Preserves area → reversible dynamics (Hamiltonian evolution).

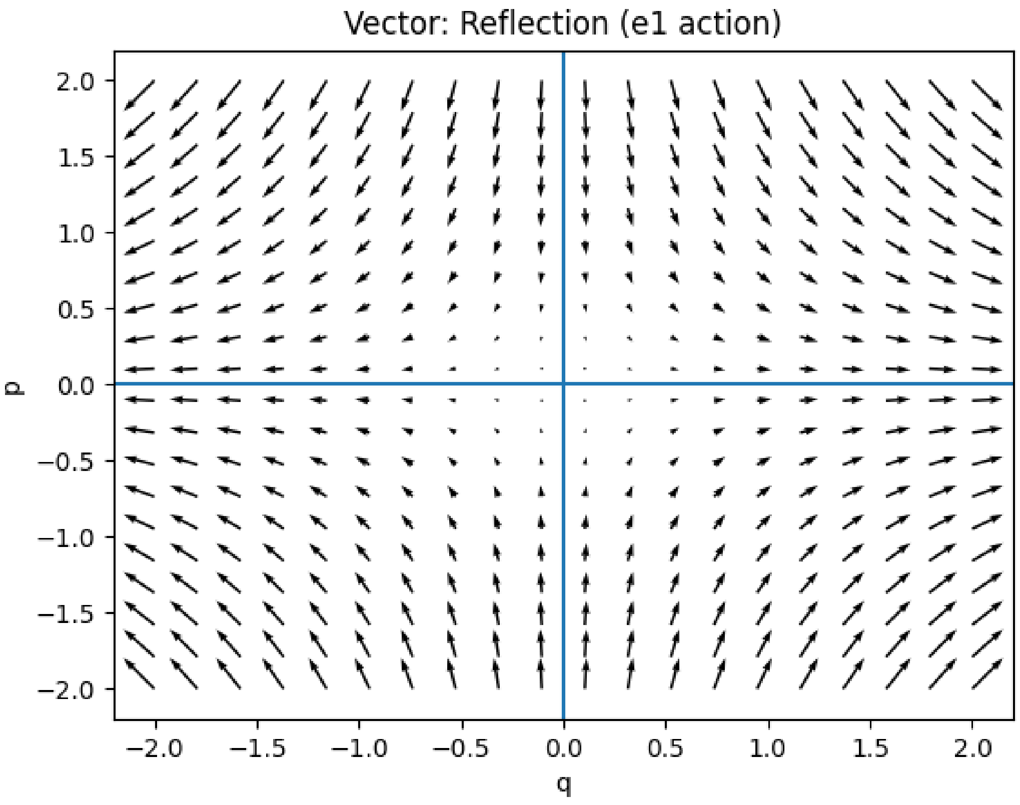


Vector components as generators of reflections

- Vector parts of the multivector M: $be_1 + ce_2$.
- e_1 and e_2 correspond (using the linear map Φ) to the matrices $\Phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\Phi(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\Phi(e_1)$ corresponds to the reflection along the q-axis
- $\Phi(e_2)$ corresponds to the reflection along the line $q = p$ (flipping q and p axes)
- The symplectic condition is satisfied as:

  $\widetilde{e_i}J + Je_i = 0,\ \ i \in \{1, 2\}$.

☞ Preserve distances and phase plane area, reverse the orientation phase plane → reversible, but anti-symplectic.



Scalar component: expansion and contraction

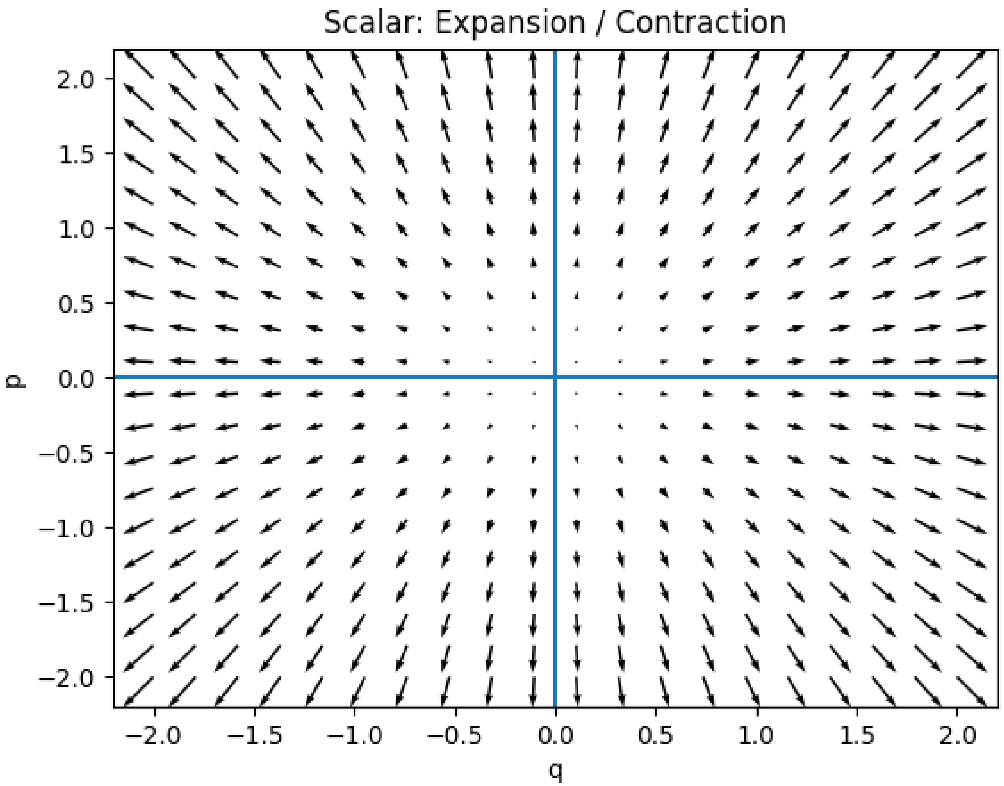
- Scalar part of the multivector M: a.

- Corresponds to uniform expansion or contraction of the phase-plane area.

- The symplectic condition is not satisfied as:

  $\tilde{\alpha}J + J\alpha = 2aJ \neq 0 \rightarrow a \neq 0.$

☞ Unique source of irreversibility in the multivector generator M.



General solution using $Cl(2,0)$

- General solution of the system: $X(t) = e^{Mt}X_0$, where $X_0 = X(0)$ the initial phase-plane vector.
- The multivector time generator be decomposed as $M = a1 + M_{rev}$ with $M_{rev} = be_1 + ce_2 + dJ$.
- The general solution is now written as $X(t) = e^{at}e^{M_{rev}t}X_0 \rightarrow$ irreversible evolution is separated from reversible one, closed form solution are easily derived depending on the form of ${M_{rev}}^2$.
- ${M_{rev}}^2 = b^2 + c^2 - d^2 \rightarrow$ Depending on the sign of ${M_{rev}}^2$, exponential $e^{M_{rev}t}$ takes one of the standard closed forms:
  - ✓ *Elliptic (oscillatory) case* $\left({M_{rev}}^2 < 0\right)$: $e^{M_{rev}t} = \cos(\omega_o t) + \frac{M_{rev}}{\omega_o}\sin(\omega_o t)$, $\omega_o = \sqrt{-{M_{rev}}^2}$
  - ✓ *Hyperbolic case* $\left({M_{rev}}^2 > 0\right)$: $e^{M_{rev}t} = \cosh(M_{rev}t) + \frac{M_{rev}}{\kappa}\sinh(M_{rev}t)$
  - ✓ *Nilpotent case* $\left({M_{rev}}^2 = 0\right)$: $e^{M_{rev}t} = 1 + M_{rev}t$.

Example No 1: Undamped harmonic oscillator

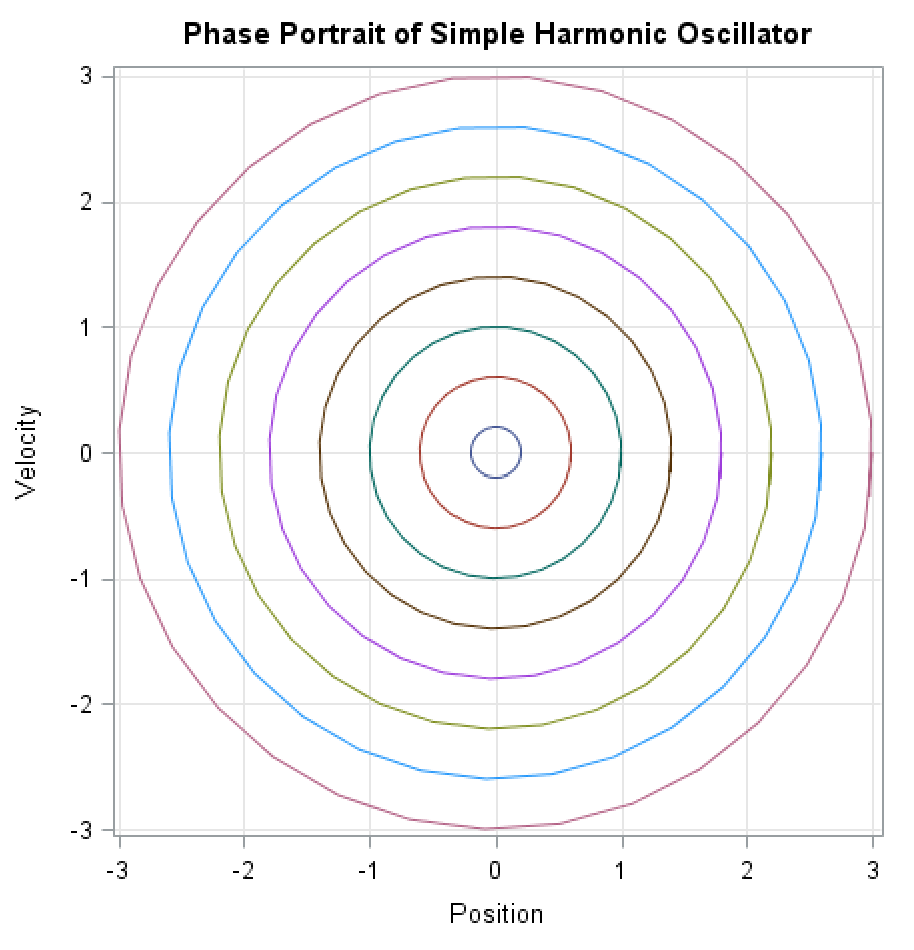
- Equation of motion: $\ddot{q} + {\omega_0}^2 q = 0$, ω_0: angular frequency.

- In matrix form: $\dot{\mathbf{X}} = A\mathbf{X}$, $\quad A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$.

- The multivector from the isomorphism is $M = \omega_0 J \;\rightarrow$ consisting only of the bivector part as A is traceless and antisymmetric.

- General solution: $X(t) = e^{Mt}X_0 = e^{M_{rev}t}X_0 = [\cos(\omega_0 t) + J\sin(\omega_0 t)]X_0$.

☞ Purely Hamiltonian system consisting only by a bivector part corresponding to rotations on the phase plane. Mechanical energy is conserved.



Example No 2: Damped harmonic oscillator

- Equation of motion: $\ddot{q} + 2\mathscr{b}\dot{q} + {\omega_0}^2 q = 0$, $\mathscr{b}$ damping coefficient.

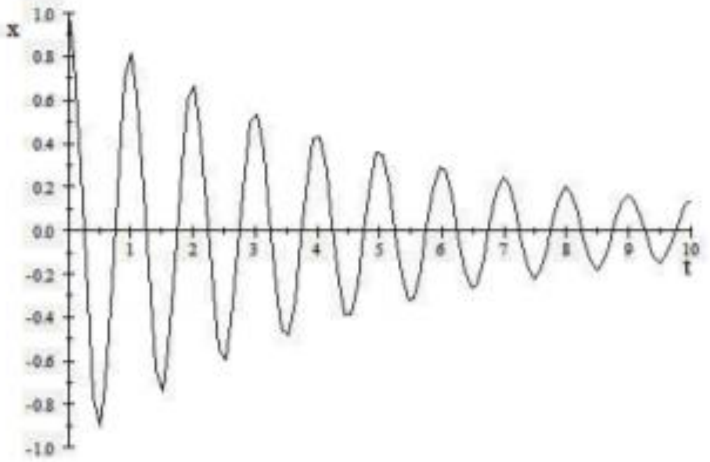
- In matrix form: $\dot{\mathbf{X}} = A\mathbf{X}$, $\quad A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\mathscr{b} \end{pmatrix}$.

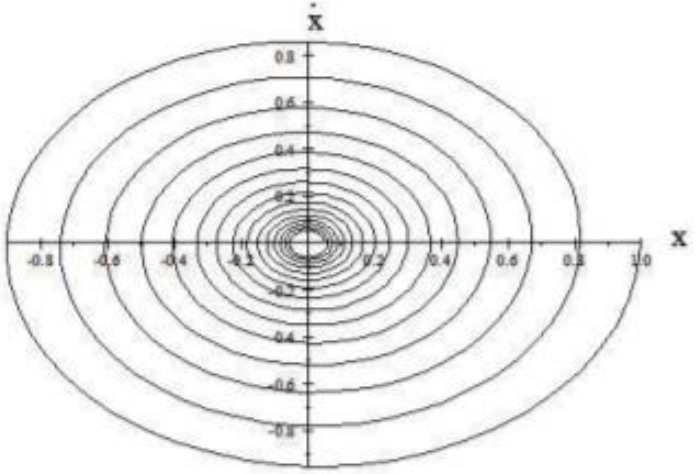
- Multivector is $M = -\mathscr{b} + \mathscr{b}e_1 + \omega_0 J$ → consisting not only by a bivector part, but also by a scalar and a vector part.

- Scalar part corresponds to dissipation, vector part corresponds to anisotropy, bivector part corresponds to oscillation.

- Solution depends on the sign of ${M_{rev}}^2 = \mathscr{b}^2 - {\omega_0}^2$ corresponding to underdamped, critically damped and overdamped cases.

☞ Dissipation is a structured phenomenon, affecting also vector parts in M, not just the scalar part.





Isotropic vs anisotropic dissipation in the phase plane

Consider a 2D linear dynamical system

$$\dot{\mathbf{X}} = A\mathbf{X}, \qquad \mathbf{X}(t) \in \mathbb{R}^2, \;\; A \in M_2(\mathbb{R}).$$

1. Phase-plane area contraction in dissipative systems:

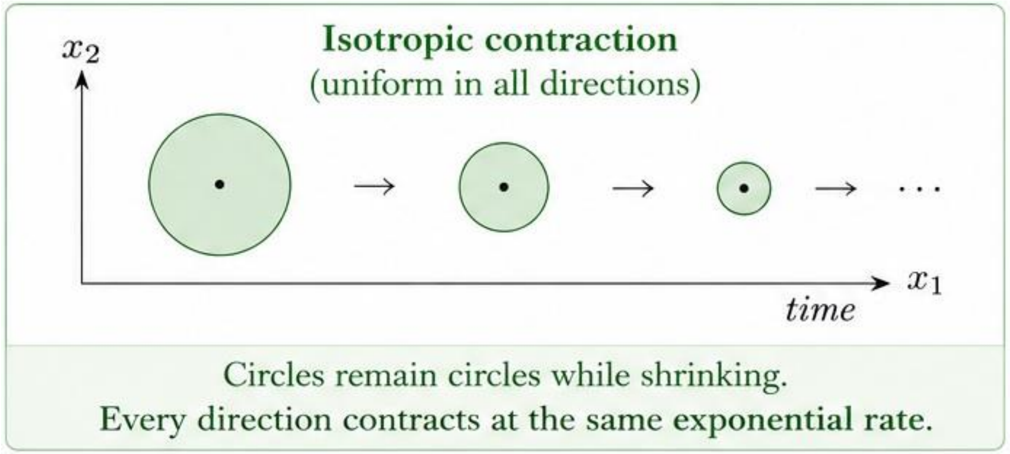
$$\mathrm{tr}A < 0$$

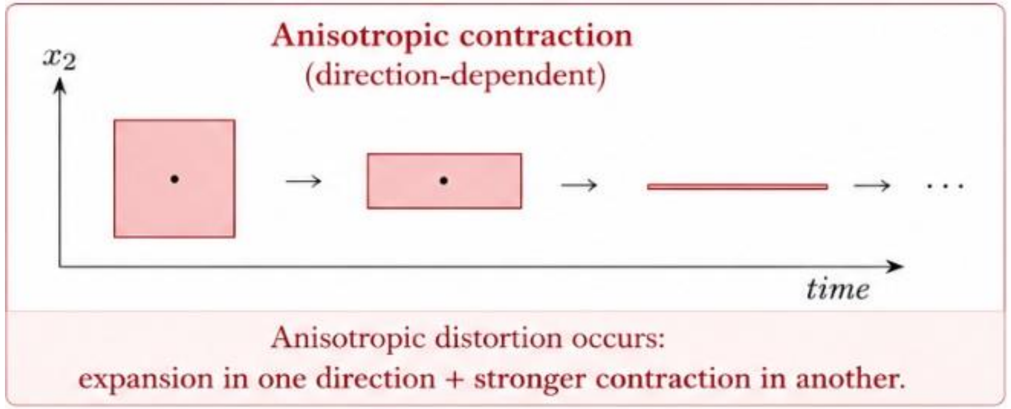
is necessary and sufficient for the phase-plane area to contract exponentially.

2. Isotropic vs anisotropic contraction:

$$\mathrm{tr}A < 0$$

is present when the phase-plane area contracts, but does not change if the shape is distorted. The condition is the same in both isotropic and anisotropic contraction.





☞ **Question**: What is the geometric criterion that controls whether the contraction of the phase-plane area is isotropic or anisotropic?

Geometric criterion for isotropic contraction (1)

Let's decompose the matrix A as

$$A = D + \Omega$$

where

$$D = \frac{A + A^T}{2}, \qquad \Omega = \frac{A - A^T}{2}.$$

D controls stretching/compression, Ω controls rotation.

Proposition

The phase-plane contraction is geometrically isotropic if:

$$D = -aI, \qquad a > 0.$$

The most general isotropically contracting system is

$$A = -aI + \Omega.$$

In 2D,

$$\mathrm{A} = \begin{pmatrix} -a & d \\ -d & -a \end{pmatrix}.$$

All directions contract equally

$$r(t) = r_0 e^{-at},$$

while rotation may still occur through Ω.

Geometric criterion for isotropic contraction (2)

Proof

Let

$$r^2 = \mathbf{X}^T\mathbf{X}.$$

Differentiating,

$$\frac{d}{dt}r^2 = \mathbf{X}^T(A + A^T)\mathbf{X} = 2\mathbf{X}^T D\mathbf{X}.$$

Therefore,

$$\frac{1}{r}\frac{dr}{dt} = \frac{\mathbf{X}^T D\mathbf{X}}{\mathbf{X}^T\mathbf{X}}.$$

For isotropic contraction, the rate must be independent of direction.

$$\frac{\mathbf{X}^T D\mathbf{X}}{\mathbf{X}^T\mathbf{X}} = -a,$$

for every nonzero $\mathbf{X}$. This implies that

$$D = -aI.$$

Thus, the geometric criterion is

$$\frac{A + A^T}{2} = -aI,$$

with

$$a > 0.$$

Translation into the MTG

Recall the MTG decomposition

$$M = a + be_1 + ce_2 + dJ.$$

Also recall that the most general isotropically contracting matrix A in 2D is

$$\mathrm{A} = \begin{pmatrix} -a & d \\ -d & -a \end{pmatrix}.$$

Using the isomorphism Φ: $Cl(2,0) \rightarrow M_2(\mathbb{R})$, the matrix A is isomorphic to

$$M = a + dJ,$$

which means that the vector parts vanish, i.e., $b = c = 0$, and M is mathematically equivalent to a complex number.

Component	**Geometric Action**	**Dynamical Role**
Scalar a	Isotropic contraction	Irreversibility
Vectors be_1, ce_2	Anisotropic deformation	Anti-symplectic structure
Bivector dJ	Hamiltonian rotation	Hamiltonian evolution

Conclusions (1)

- Every 2×2 linear generator A admits a unique multivector representation $A \leftrightarrow M$ through the isomorphism $M_2(\mathbb{R}) \cong Cl(2,0)$.
- The MTG $M = a + be_1 + ce_2 + dJ$ decomposes the dynamics into three independent geometric components:
  - ✓ Scalar part → isotropic contraction/expansion (irreversibility).
  - ✓ Vector parts → reflections and anisotropy.
  - ✓ Bivector part → Hamiltonian rotations.
- The classical symplectic condition acquires a simple geometric interpretation: Hamiltonian systems contain only vector and bivector parts, while the scalar part is responsible for the loss (or gain) of phase-plane area.
- The MTG exponential provides a unified description of elliptic, hyperbolic, and nilpotent evolution.
- Application to the harmonic oscillator shows that the MTG naturally distinguishes conservative and dissipative dynamics:
  - ✓ Undamped oscillator: purely bivector generator (Hamiltonian rotation).
  - ✓ Damped oscillator: scalar, vector and bivector components coexist.
- Dissipation is not described solely by the scalar contraction of phase-plane area; may also contain vector contributions that encode anisotropic contraction.

Conclusions (2)

- The geometric criterion

$$\frac{A + A^T}{2} = -aI$$

shows that isotropic dissipation is precisely characterized by the disappearance of the vector components,

$$b = c = 0$$

so that the MTG reduces to a complex multivector consisting only of scalar and bivector parts.

Future directions

- Applications to higher-dimensional nonlinear dynamical systems.
- Geometric characterization of entropy production and irreversible dynamics.
- Connections with nonequilibrium statistical physics.

Thank you for your attention!

For more information about the study, see the Preprint:
Aslani, K. E. (2026). Multivector Time Generator of Evolution in Phase Plane. *Preprints*.
DOI: 10.20944/preprints202602.0263.v1.

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