Relativistic cosmology from F to Sz

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Contents

17. Spherically symmetric inhomogeneous models

A spherically symmetric metric, by the Killing equations discussed in Lecture 1, can be put in the form:

$$\mathrm{d}s^2 = \mathrm{e}^{C(t,r)}\mathrm{d}t^2 - \mathrm{e}^{A(t,r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \qquad (17.1)$$

where C, A and R are functions to be determined from the Einstein equations (EEs).

We wish to apply this metric to the post-recombination epoch, so we assume the source in the EEs is dust, $p = 0$.

Then a simple calculation shows that $C = 0$, and the (geodesic!) dust velocity field is

$$u^\alpha = (1, 0, 0, 0) \qquad (17.2)$$

so the coordinates of (17.1) are comoving.

The function R is the *angular diameter distance* between the observer at $R = 0$ and the light source at arbitrary (t, r).

$$ds^2 = dt^2 - e^{A(t,r)} dr^2 - R^2(t,r)\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \quad (17.1)$$

For show, let us display the Einstein equations for (17.1), with Λ included:

$$\frac{{R_{,t}}^2}{R^2} + \frac{A_{,t}\, R_{,t}}{R} - e^{-A}\left(2\frac{R_{,rr}}{R} + \frac{{R_{,r}}^2}{R^2} - \frac{A_{,r}\, R_{,r}}{R}\right) + \frac{1}{R^2} = 8\pi G\rho/c^2 - \Lambda,$$

$$\underline{e^{-A}\left(2\frac{R_{,tr}}{R} - \frac{A_{,t}\, R_{,r}}{R}\right) = 0,}$$

$$2\frac{R_{,tt}}{R} + \frac{{R_{,t}}^2}{R^2} - e^{-A}\,\frac{{R_{,r}}^2}{R^2} + \frac{1}{R^2} = -\Lambda,$$

$$\frac{1}{4}\left(4\frac{R_{,tt}}{R} + 2\frac{A_{,t}\, R_{,t}}{R} + 2A_{,tt} + {A_{,t}}^2\right) - \frac{1}{4}e^{-A}\left(4\frac{R_{,rr}}{R} - 2\frac{A_{,r}\, R_{,r}}{R}\right) = -\Lambda.$$

The first one just defines ρ. The other are 3 equations for 2 functions, but they are not independent.

One solution of the underlined equation is $R_{,r} = 0$, which leads to a cosmologically irrelevant (but interesting and well investigated! [35-36]) metric . The other one is

$$e^A = \frac{{R_{,r}}^2}{1 + 2E(r)} \quad (17.2)$$

where E(r) is an arbitrary function. This way of writing the integration ``constant'' will help in understanding the physical meaning of E(r).

[35] V. A. Ruban, Spherically symmetric T-models in the general theory of relativity. *Zh. Eksper. Teor. Fiz.* **56**, 1914 (1969); English translation with comments: *Gen. Relativ. Gravit.* **33**, 375 (2001).

[36] A. Krasiński and G. Giono, The charged dust solution of Ruban -- matching to Reissner--Nordström and shell crossings. *Gen. Relativ. Gravit*. **44** ,239 (2012).

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right)\right]$$

18. The Lemaître – Tolman (L-T) model

$$S_{,t}{}^2 = \frac{2GM}{c^2 S} - k - \frac{1}{3}\Lambda S^2$$

$\rho S^3 = 3G\mathcal{M}/(4\pi c^2)$

This model is the final solution of the equations from the previous slide with $R_{,r} \neq 0$.

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{A(t,r)}\mathrm{d}r^2 - R^2(t,r)(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \qquad (18.1)$$

$$\mathrm{e}^A = \frac{R_{,r}{}^2}{1+2E(r)}\ , \qquad (18.2)$$

R(t,r) is determined by the equation:

$$R_{,t}{}^2 = 2E(r) + \frac{2M(r)}{R} - \frac{1}{3}\Lambda R^2, \qquad (18.3)$$

M(r) is a second arbitrary function, and the mass density ρ is

$$\frac{8\pi G}{c^2}\rho = \frac{2M_{,r}}{R^2 R_{,r}} \qquad (18.4)$$

The Friedmann models are the special case of (18.1) – (18.4) defined by

$R(t,r) = rS(t)$, $E(r) = -kr^2/2$, $M = \mathcal{M} r^3$, $\mathcal{M}$ and k being constant (18.5)

The solution (18.1) – (18.4) was found by Georges Lemaître [37] in 1933, and its meaning was clarified by Richard Tolman [38] in 1934 and Hermann Bondi [39] in 1947.

The geometrical properties and astronomical applications of this model were investigated in over 100 published papers. This number keeps increasing.

[37] G. Lemaître, L'Univers en expansion [The expanding Universe], *Ann. Soc. Sci. Bruxelles* **A53**, 51 (1933); *Gen. Rel. Grav.* **29**, 641 (1997).
[38] R. C. Tolman, Effect of inhomogeneity on cosmological models, *Proc. Nat. Acad. Sci. USA* **20**, 169 (1934); *Gen. Rel. Grav.* **29**, 935 (1997).
[39] H. Bondi, Spherically symmetrical models in general relativity. *Mon. Not. Roy. Astr. Soc.* **107**, 410 (1947; *Gen. Rel. Grav.* **31**, 1783 (1999).

$$S_{,t}{}^2 = \frac{2GM}{c^2 S} - k - \frac{1}{3}\Lambda S^2 \qquad (9.2)$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R_{,r}{}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \qquad (18.1)$$

$$R_{,t}{}^2 = 2E(r) + \frac{2M(r)}{R} - \frac{1}{3}\Lambda R^2, \qquad (18.3)$$

The priority of G. Lemaître in discovering this model is beyond any doubt.

But, to avoid confusion with the Friedmann – Lemaître models, this class is usually called Lemaître – Tolman (L-T) or Lemaître – Tolman – Bondi (LTB).

Since eqs. (9.2) and (18.3) are so similar, also their solutions are similar.

For example, when E(r) > 0 and Λ = 0, the parametric solution of (18.3) is

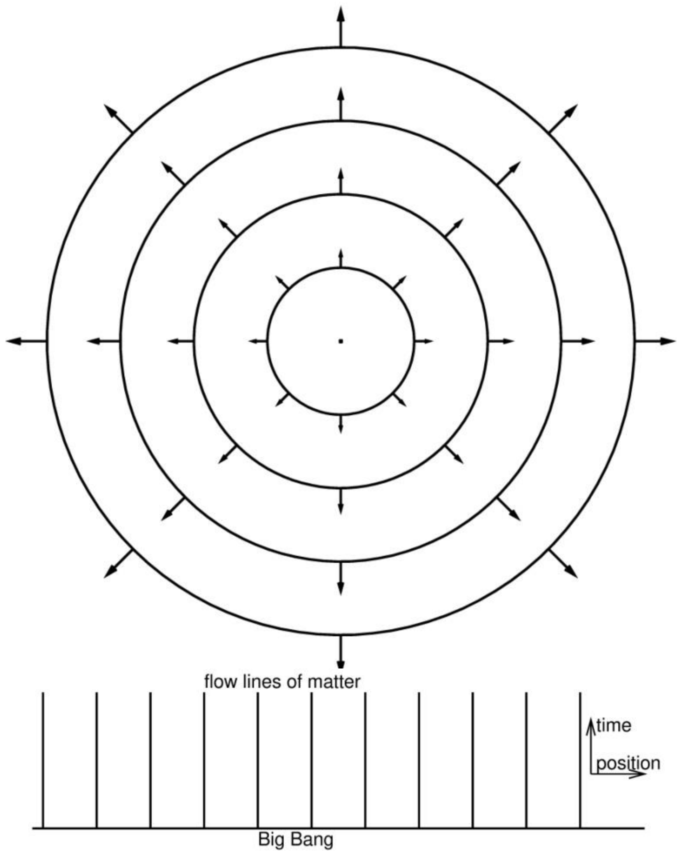
$$\begin{aligned} R(t,r) &= \frac{M}{2E}(\cosh\eta - 1), \\ \sinh\eta - \eta &= \frac{(2E)^{3/2}}{M}\left[t - t_B(r)\right]. \end{aligned} \qquad (18.6)$$

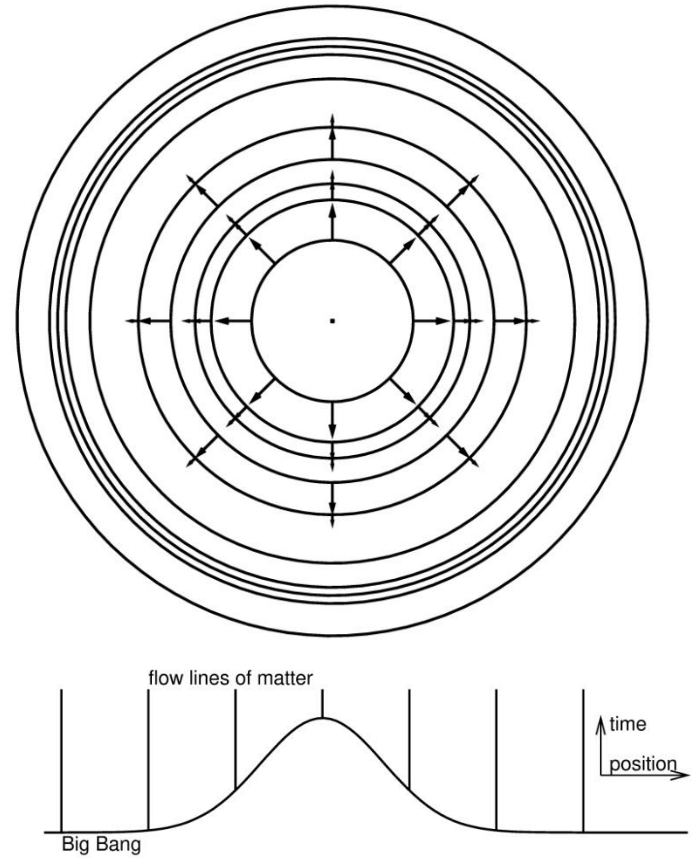
The BB singularity at $t = t_B(r)$ occurs in general at ***different t for each r = constant shell***!

<u>(18.1) does not change when r is transformed by r = f(r') with arbitrary f.</u>

The L-T model can be imagined as a family of expanding (or contracting) spherical shells whose velocities are independent.

Unlike in Friedmann models, where the shell velocities are rigidly connected to their radii by the Hubble law.





Expansion in Friedmann models.

Velocity of expansion of each matter shell **is proportional to** its **distance** from the observer. The BB occurs simultaneously in the coordinates of (18.1).

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R,_r}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \qquad (18.1)$$

Expansion in L-T models.

Velocity of expansion is uncorrelated with the radius of a matter shell.

The BB is non-simultaneous

→ the age of matter particles depends on r.

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$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R_{,r}}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \qquad (18.1)$$

$$\frac{8\pi G}{c^2}\rho = \frac{2M_{,r}}{R^2 R_{,r}} \qquad (18.4)$$

$${R_{,t}}^2 = 2E(r) + \frac{2M(r)}{R} - \frac{1}{3}\Lambda R^2, \qquad (18.3)$$

The quantity c^2M/G in (18.3) and (18.4) is the ***active gravitational mass*** that generates the gravitational field.

It is not coincident with c^2N/G – the sum of masses of particles in the gravitating body.

Suppose that matter fills the interior V of a sphere of coordinate radius $r = r_S$ centered at the centre of symmetry $r = r_c$. Then, from (18.1):

$$c^2 N/G = \int_V \rho\sqrt{-g}\, \mathrm{d}_3 V \equiv 4\pi \int_{r_c}^{r_S} \frac{\rho R^2 R_{,r}}{\sqrt{1+2E(r)}}\, \mathrm{d}r \qquad (18.7)$$

while the active gravitational mass M is, from (18.4)

$$c^2 M/G = 4\pi \int_{r_c}^{r_S} \rho R^2 R_{,r}\, \mathrm{d}r \qquad (18.8)$$

$$c^2 N/G = \int_V \rho\sqrt{-g}\, \mathrm{d}_3 V \equiv 4\pi \int_{r_c}^{r_S} \frac{\rho R^2 R_{,r}}{\sqrt{1+2E(r)}}\, \mathrm{d}r \qquad (18.7)$$

$$c^2 M/G = 4\pi \int_{r_c}^{r_S} \rho R^2 R_{,r}\, \mathrm{d}r \qquad (18.8)$$

M > N when E > 0; M < N when E < 0, and M = N when E = 0.

If M < N, then (N - M) is called the ***relativistic mass defect***. It is analogous to the mass defect known from nuclear physics.

When M < N (E < 0), part of the energy contained in the component particles was shed in creating the body. Such a body is *bound*.

When E > 0, the body's (excess energy)/c^2 adds to the sum of masses of components. The body is unbound and can disperse without any energy input.

When E = 0, the system is ``marginally bound'' – no energy has been shed to form it, and there is no excess energy.

This interpretation of E was one of the results of Bondi [39].

[39] H. Bondi, Spherically symmetrical models in general relativity. *Mon. Not. Roy. Astr. Soc.* **107**, 410 (1947; *Gen. Rel. Grav.* **31**, 1783 (1999).

$$ds^2 = dt^2 - \frac{{R,_r}^2}{1+2E(r)}dr^2 - R^2(t,r)\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \qquad (18.1)$$

The function E(r) has one more interpretation.

In a subspace $t = t_0$ = constant of (18.1), R depends only on r, so can be used as the radial coordinate:

$r = R(t_0)$ and $R,_r(t_0) = 1$.

In such coordinates with E = 0 the metric of the 3-space t = constant in (18.1) is the Euclidean metric in spherical coordinates.

So, $E \neq 0$ is a measure of curvature of the subspaces of constant t.

But this curvature is *local* – it depends on r. It may be positive in one region, but negative elsewhere.

→ The prominent role of the curvature index k is a peculiarity of the R-W class of *models*, and not a general property of our Universe.

The same spacetime may be approximately like the k > 0 Friedmann model in one neighbourhood and like the k < 0 model in another.

→ Metrics (18.1) with different signs of E can be different regions of the same spacetime.

$$ds^2 = dt^2 - \frac{R_{,r}{}^2}{1+2E(r)}dr^2 - R^2(t,r)\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \qquad (18.1)$$

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

$$S_{,t}{}^2 = \frac{2GM}{c^2 S} - k - \frac{1}{3}\Lambda S^2 \qquad (9.2)$$

A few facts about the L-T metric (18.1) are often misunderstood and must be emphasised:

1. Each shell of constant r in the L-T metric evolves exactly as a shell in a Friedmann model, but each one has in general a different M and k in (9.2).

2. An L-T model is not an alternative to F, but a generalisation that includes F as a special case.

3. An L-T metric is not meant to be the model of the whole Universe but an exact local perturbation of a F model. Several small L-T regions may be matched into the same F background and used to study the evolution of independent (spherically symmetric) structures within the exact Einstein theory.

4. All physical processes studied in F models also occur in an L-T model. Along any single observer world line O they can be made exactly the same as in F. Small variations of physical parameters occur in the neighbourhood of O, e.g. neighbouring galaxies may be younger or older than O's galaxy.

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right] \quad (7.1)$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R,_r}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \quad (18.1)$$

$$ {R,_t}^2 = 2E(r) + \frac{2M(r)}{R} - \frac{1}{3}\Lambda R^2, \quad (18.3)$$

19. Formation of voids in the Universe

The discovery of voids in 1978 [9] was a surprise; until then astronomers believed that galaxies are distributed uniformly in space.

In fact, the first papers indicating that voids and condensations should be ubiquitous were published in 1934 by Tolman [37] and Sen [38], but had not been understood.

Their main result was that the Friedmann models are unstable against the growth of inhomogeneities. This is proved as follows.

We compare an L-T model (18.1) with a Friedmann model (7.1).

Let the initial conditions at $t = t_1$ be such that $R(t_1,r) = rS(t_1)$ and $R_{,t}(t_1,r) = rS_{,t}(t_1)$.

The first is just a choice of the coordinate r: whatever $r = r'$ was initially, we choose the new r by $r = R(t_1,r')/S(t_1)$.

With $R(t_1,r)$ and $R_{,t}(t_1,r)$ defined as above, (18.3) imposes a relation between M(r) and E(r) → we still have one free function at our disposal.

→ The mass density ρ is not yet determined and $\rho(t_1,r)$ may be assumed any.

[9] S. A. Gregory and L. A. Thompson, The Coma/A1367 supercluster and its environs. *Astrophys. J.* **222**, 784 (1978).
[37] R. C. Tolman, Effect of inhomogeneity on cosmological models, *Proc. Nat. Acad. Sci. USA* **20**, 169 (1934); *Gen. Rel. Grav.* **29**, 935 (1997).
[38] N. R. Sen, *Z. Astrophysik* **9**, 215 (1934); *Gen. Relativ. Gravit.* **29**, 1477 (1997).

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right] \quad (7.1)$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R,_r}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \quad (18.1)$$

$${R,_t}^2 = 2E(r) + \frac{2M(r)}{R} - \frac{1}{3}\Lambda R^2, \quad (18.3)$$

$$\frac{8\pi G}{c^2}\rho = \frac{2M,_r}{R^2 R,_r} \quad (18.4)$$

$$\rho S^3 = 3G\mathcal{M}/(4\pi c^2) \quad (9.1)$$

From the assumed equations $R(t_1,r) = rS(t_1)$ and $R_{,t}(t_1,r) = rS,_t(t_1)$ it follows that

$(R,_{tr}/R,_r)(t_1) = (S,_t/S)(t_1)$. (19.1)

$R,_t$ is a measure of the velocity of expansion, so with $R_{,t}(t_1,r) = rS,_t(t_1)$ this perturbation of the Friedmann model leaves the initial velocity distribution unperturbed.

The conditions $R(t_1,r) = rS(t_1)$, $R_{,t}(t_1,r) = rS,_t(t_1)$ and (19.1) imply one more thing:

$(\partial/\partial t)(\ln \rho_{LT} - \ln \rho_F)|_{t=t1} = 0$, (19.2)

where ρ_{LT} and ln ρ_F are the mass densities given by (18.4) and (9.1), respectively.

By manipulating (18.4) and (9.1) further, Tolman obtained the equation

$$\frac{\partial^2}{\partial t^2}\left(\ln\rho_{LT} - \ln\rho_F\right) = \frac{1}{2}\kappa\left(\rho_{LT} - \rho_F\right). \quad (19.3)$$

Together with (19.2) this means: wherever $\rho_{LT} - \rho_F \neq 0$, the difference in densities will be increasing in time and the sign of $\rho_{LT} - \rho_F$ will be preserved.

→ An L-T model with spatial fluctuations of mass density will evolve away from the initial nearly-Friedmannian state.

In Tolman's own words [37]:

``… at those values of r where the density in the distorted model is different from that in the Friedmann model, there is at least an initial tendency for the differences to be emphasised … in cases where condensation is taking place … the discrepancies will continue until we reach a singular state involving infinite density or reach a breakdown in the simplified equations.''

Sen [38] assumed the initial density to be unperturbed, with the initial velocity distribution being non-Friedmannian. By a similar method as Tolman, he concluded that ``the models are unstable for initial rarefaction''.

This was a clear prediction that the Universe should evolve away from a spatially homogeneous mass distribution, and in particular, that *voids should form*.

Unfortunately, no-one noticed this prediction until voids were observed [9].

[37] R. C. Tolman, Effect of inhomogeneity on cosmological models, *Proc. Nat. Acad. Sci. USA* **20**, 169 (1934); *Gen. Rel. Grav.* **29**, 935 (1997).
[38] N. R. Sen, *Z. Astrophysik* **9**, 215 (1934); *Gen. Relativ. Gravit.* **29**, 1477 (1997).
[9] S. A. Gregory and L. A. Thompson, The Coma/A1367 supercluster and its environs. *Astrophys. J.* **222**, 784 (1978).

20. The redshift

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R_{,r}}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \quad (18.1)$$

$$1+z = \frac{\nu_e}{\nu_o} = \frac{(k_\alpha u^\alpha)_e}{(k_\alpha u^\alpha)_o} \quad (8.3)$$

In (8.3) the field of tangent vectors to the light ray k^α is affinely parametrised.

Finding this parametrisation may be not easy.

The following method to avoid this difficulty was introduced by Bondi [39].

From (18.1), a radial null geodesic proceeding toward the observer obeys

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{R_{,r}(t,r)}{\sqrt{1+2E(r)}} \quad (20.1)$$

Let two light rays be emitted in the same direction, the second one later by a small τ.

Let their equations be t = T(r) and t = T(r) + τ(r). Both rays must obey (20.1), so

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{R_{,r}(T(r),r)}{\sqrt{1+2E(r)}}, \qquad \frac{\mathrm{d}(T+\tau)}{\mathrm{d}r} = -\frac{R_{,r}(T(r)+\tau(r),r)}{\sqrt{1+2E(r)}} \quad (20.2)$$

Since τ(r) was assumed small, we have, to first order in τ

$$R_{,r}(T(r)+\tau(r),r) = R_{,r}(T(r),r) + \tau(r)R_{,tr}(T(r),r) \quad (20.3)$$

Using (20.3) and the first of (20.2) in the second of (20.2) we get

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\tau(r)\frac{R_{,tr}(T(r),r)}{\sqrt{1+2E(r)}} \quad (20.4)$$

[39] H. Bondi, Spherically symmetrical models in general relativity. *Mon. Not. Roy. Astr. Soc.* **107**, 410 (1947; *Gen. Rel. Grav.* **31**, 1783 (1999).

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\tau(r)\frac{R_{,tr}\left(T(r),r\right)}{\sqrt{1+2E(r)}} \qquad (20.4)$$

$$1+z = \frac{\nu_e}{\nu_o} = \frac{\left(k_\alpha u^\alpha\right)_e}{\left(k_\alpha u^\alpha\right)_o} \qquad (8.3)$$

If τ is the period of the light wave, then $\tau(r_{obs})/\tau(r_{em}) = 1 + z(r_{em})$.

Keeping the observer at fixed r_{obs} and taking two sources at r_{em} and $r_{em}+ dr$, we find

$(d\tau/dr)/\tau = - (dz/dr)/(1 + z)$.

Using this in (20.4):

$$\frac{1}{1+z}\,\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{R_{,tr}\left(T(r),r\right)}{\sqrt{1+2E(r)}} \qquad (20.5)$$

Hence, the redshift may be calculated numerically from:

$$\ln(1+z(r)) = \int_{r_{\rm obs}}^{r_{\rm em}} \frac{R_{,tr}\left(T(r),r\right)}{\sqrt{1+2E(r)}}\mathrm{d}r \qquad (20.6)$$

This formula is equivalent to (8.3), but applies only on radial rays.

On nonradial rays there is no way to avoid calculating the affinely parametrised k^α from the full set of geodesic equations, and then using (8.3).

21. The blueshift

$$1 + z = \frac{\nu_e}{\nu_o} = \frac{(k_\alpha u^\alpha)_e}{(k_\alpha u^\alpha)_o} \qquad (8.3)$$

$$\begin{aligned} R(t,r) &= \frac{M}{2E}(\cosh\eta - 1), \\ \sinh\eta - \eta &= \frac{(2E)^{3/2}}{M}\left[t - t_B(r)\right]. \end{aligned} \qquad (18.6)$$

Note first what blueshift means.

The blueshifted frequency is $\nu_o > \nu_e$, so $z < 0$.

The blueshift is infinite when $\nu_o \to \infty$.

By (8.3), for the infinitely blueshifted ray $z = -1$.

The existence of infinite blueshifts in L-T models on rays emitted from the BB was casually mentioned without proof by P. Szekeres in 1980 [40].

The necessary conditions for infinite blueshift are [41]:

1) The ray is emitted from the BB radially.

2) It is emitted at such a point of the BB where $dt_B/dr \neq 0$.

Whether these are also sufficient conditions has not so far been proved analytically, there exist only perturbative [41] and numerical [42] calculations confirming it.

We verify the first condition.

[40] P. Szekeres, Naked singularities. In: *Gravitational Radiation, Collapsed Objects and Exact Solutions*. Edited by C. Edwards. Springer (Lecture notes in physics, vol. **124**), New York, p. 477 (1980).
[41] Hellaby, C. and Lake, K. (1984). The redshift structure of the Big Bang in inhomogeneous cosmological models. I. Spherical dust solutions, *Astrophys. J.* **282**, 1; + erratum *Astrophys. J.* **294**, 702 (1985).
[42] A. Krasiński, Blueshifts in the Lemaître -- Tolman models. *Phys. Rev*. **D90**, 103525 (2014).

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R_{,r}}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \quad (18.1)$$

$$1+z = \frac{\nu_e}{\nu_o} = \frac{(k_\alpha u^\alpha)_e}{(k_\alpha u^\alpha)_o} \quad (8.3)$$

A geodesic in an L-T model (18.1) is null when its tangent vector k^α obeys

$$\left(k^t\right)^2 - \frac{{R_{,r}}^2\left(k^r\right)^2}{1+2E} - R^2\left[\left(k^\vartheta\right)^2 + \sin^2\vartheta\left(k^\varphi\right)^2\right] = 0\,. \quad (21.1)$$

The geodesic equations for the L-T metric have a first integral

$$\left(k^\vartheta\right)^2 + \sin^2\vartheta\left(k^\varphi\right)^2 = C^2/R^4 \quad (21.2)$$

With C = 0 the geodesic is radial because $k^\vartheta = k^\varphi = 0$. Using (21.2), eq. (21.1) becomes

$$\left(k^t\right)^2 = \frac{{R_{,r}}^2\left(k^r\right)^2}{1+2E} + \frac{C^2}{R^2} \quad (21.3)$$

One can rescale the affine parameter to obtain on past-directed rays

$k^t(t_o) = -1.$ (21.4)

In comoving coordinates $u^\alpha = \delta^\alpha{}_0$, so (8.3) implies via (21.4)

$1 + z = - k^t(t_e)$ (21.5)

On nonradial rays, on which $C \neq 0$, the last term in (21.3) → ∞ when R → 0.

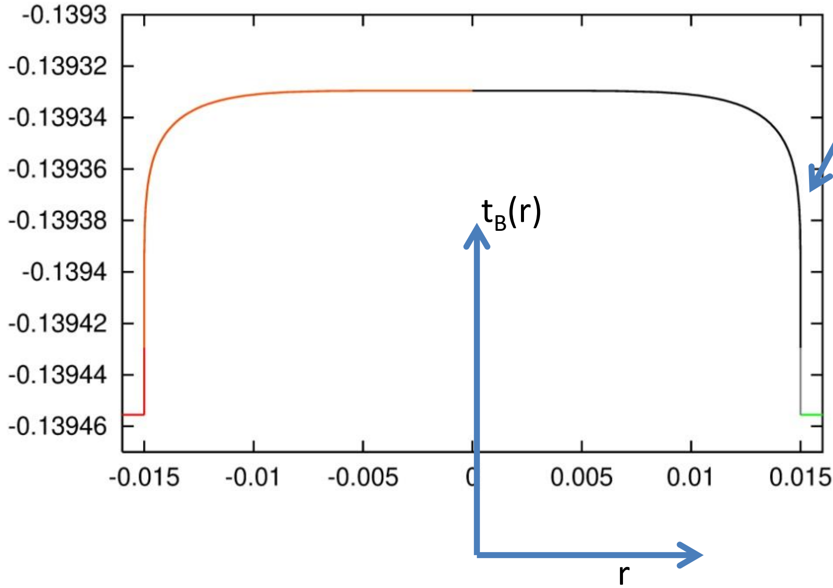
Thus, at the BB,

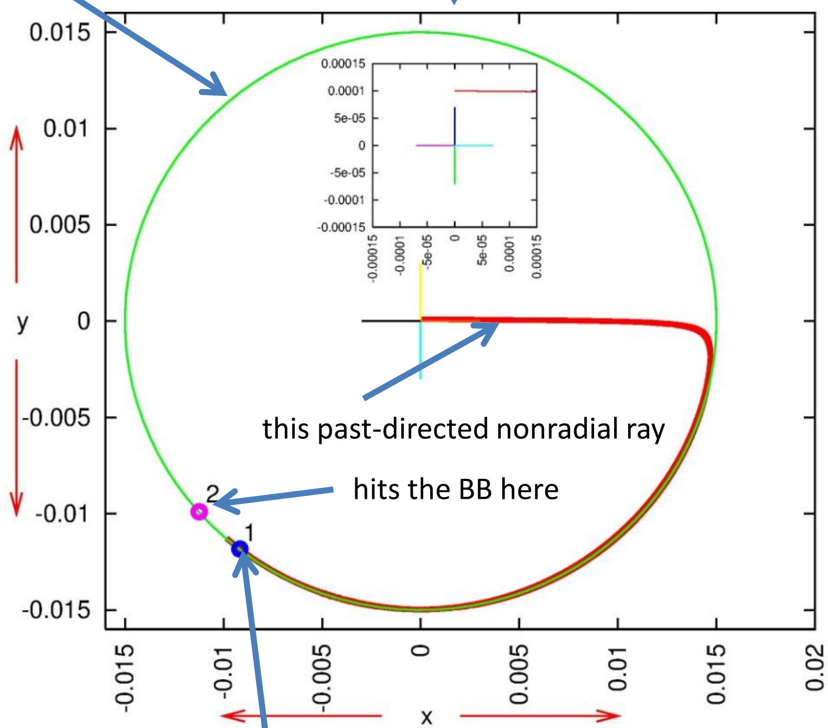
$-k^t(t_e) \to \infty$, so $1 + z \to \infty$.

→ On nonradial rays coming from the BB, z → ∞ just as in the Friedmann models.

t-axis perpendicular to the figure plane

edge of the hump in the BB profile





Example of behaviour of a light ray near a non-constant BB:

A past-directed nonradial ray approaching the BB with the $t_B(r)$ profile shown at left bends around and hits the BB tangentially.

Here the ray crosses the last scattering hypersurface.
At earlier times p ≈ 0 is not a realistic approximation,
so the Sz model cannot apply.

Figures adapted from
[43] A. Krasiński, Cosmological blueshifting may explain the gamma ray bursts. *Phys. Rev.* **D93**, 043525 (2016).

22. ``Accelerated expansion'' without ``dark energy''

The accelerated expansion of the Universe was inferred from observations of type Ia supernovae [17-18].

Their maximal absolute luminosity was ***assumed*** the same for all ("standard candles").

Their observed luminosities were inconsistent with the Friedmann $\Lambda = 0$ model.

For ***other Friedmann models***, the best consistency with observations resulted when [15]

$$S_{,t}{}^2 = \frac{2GM}{c^2 S} - k - \frac{1}{3}\Lambda S^2$$

- $k = 0$,
- 32% of the energy density comes from matter (visible or dark)
- 68% of the energy density comes from ``dark energy" (for example, from Λ).

→ ***The accelerated expansion resulted from the assumption that the Universe has the Friedmann geometry.***

→ ***It follows from interpreting the observations via a model; it is NOT an objective fact.***

[17] S.Perlmutter *et al.*, Measurements of Omega and Lambda from 42 High-Redshift Supernovae. *Astrophys. J.* **517**, 565 (1999).
[18] A. G. Riess *et al.*, Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, *Astron. J.* **116**, 1009 (1998).
[15] Planck collaboration, Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys*. **571**, A16 (2014).

The example given in the next slides shows how an L-T model can imitate the accelerated expansion without introducing the ``dark energy''.

The first idea came from Marie-Noëlle Cèlèrier [44] who proved this by a perturbative method assuming small z.

Iguchi, Nakamura and Nakao [45] generalised her result to arbitrary z by numerical calculations based on exact formulae.

Following Ref. [45] I provided [46] an illustration showing how the light cones of the ΛCDM and the imitating L-T model are related.

[44] M.-N. Cèlèrier, Do we really see a cosmological constant in the supernovae data? *Astronomy and Astrophysics* **353**, 63 (2000).
[45] H. Iguchi, T. Nakamura and K. Nakao, Is Dark Energy the Only Solution to the Apparent Acceleration of the Present Universe? *Progr. Theor. Phys*. **108**, 809 (2002).
[46] A. Krasiński, Accelerating expansion or inhomogeneity? A comparison of the ΛCDM and Lemaître -- Tolman models. *Phys. Rev.* **D89**, 023520 (2014); erratum: *Phys. Rev.* **D89**, 089901(E) (2014).

$$\ln(1+z(r)) = \int_{r_{\rm em}}^{r_{\rm obs}} \frac{R_{,tr}\,(t(r),r)}{\sqrt{1+2E(r)}}\mathrm{d}r \qquad (20.6)$$

$$R(t,r) = \frac{M}{2E}(\cosh\eta - 1),$$
$$\sinh\eta - \eta = \frac{(2E)^{3/2}}{M}\left[t - t_B(r)\right]. \qquad (18.6)$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R_{,r}}^2}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \qquad (18.1)$$

When we know t(r) along a radial ray (by solving the null geodesic equations) and z(r) along the same ray (from (20.6)),

then the (uncorrected) *luminosity distance* $D_L(z)$ from the central observer to the light source at redshift z in the L-T model is [15, 47]

$D_L(z) = (1 + z)^2 R(t(z),r(z))$. (22.1)

In the ΛCDM model the same quantity is, as stated in Lecture II

$$D_L(z) = \frac{1+z}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}\ . \qquad (22.2)$$

Assume $E/r^2 = C$ = constant (as in a Friedmann model; the solution will exist for any C).

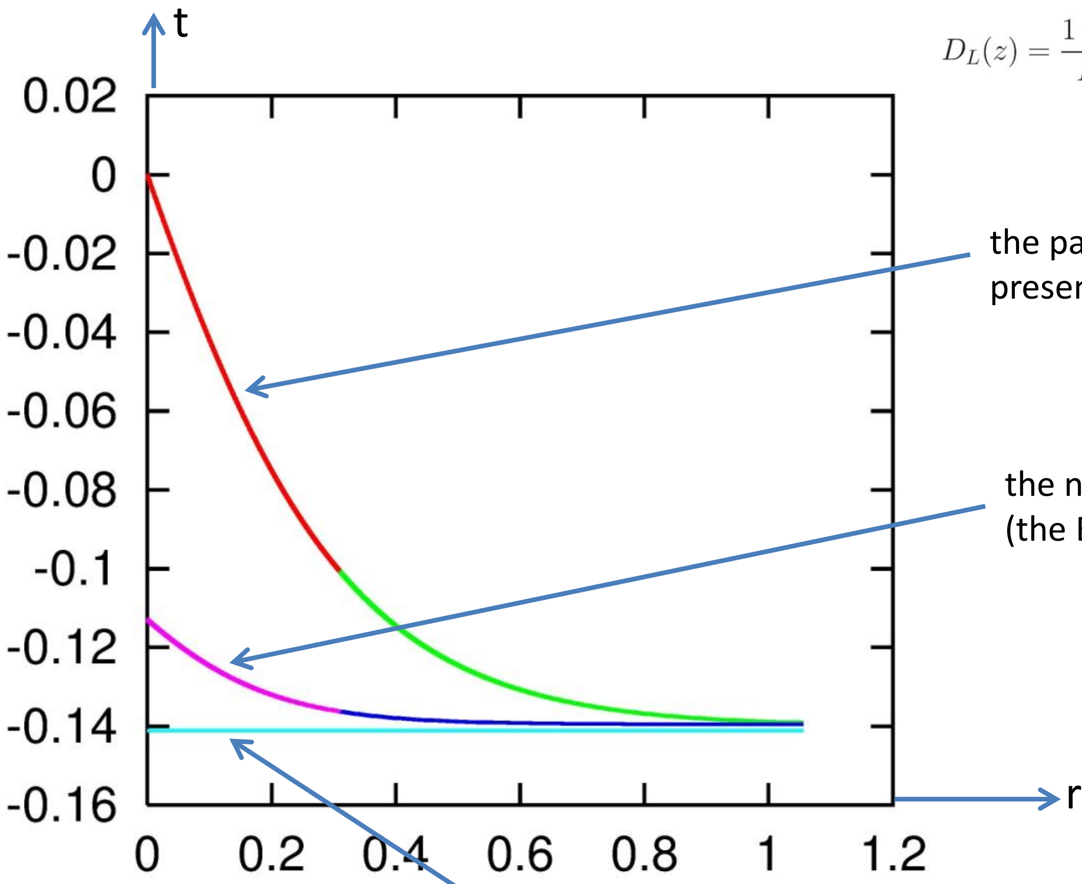
Now take H_0 = 67.1 km/(s×Mpc), $\Omega_m = 0.32$ and $\Omega_\Lambda = 0.68$ from observations [15], but put the $D_L(z)$ from (22.1) on the l.h.s. in (22.2).

Then (22.2) determines $t_B(r)$ via (18.6).

This $t_B(r)$ and the assumed E(r) define an L-T model with the same $D_L(z)$ as in (22.2).

[15] Planck collaboration, Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys*. **571**, A16 (2014).
[47] K. Bolejko, A. Krasiński, C. Hellaby and M.-N. Cèlèrier, Structures in the Universe by exact methods -- formation, evolution, interactions. Cambridge University Press 2010, p. 107.

$$D_L(z) = \frac{1+z}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_m(1+z')^3+\Omega_\Lambda}} \qquad (22.2)$$

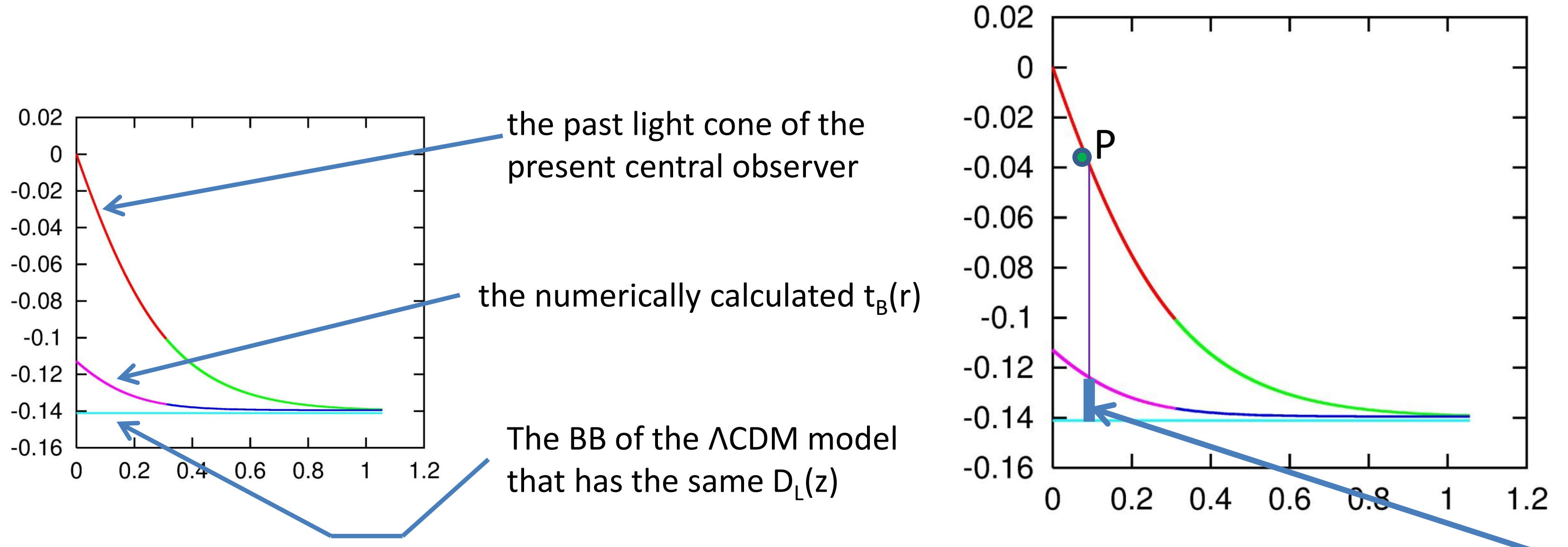


the past light cone of the present central observer

the numerically calculated $t_B(r)$ (the BB of the L-T model)

The Big Bang of the ΛCDM model that has the same $D_L(z)$

The past light cone of the present central observer in the L-T model that reproduces the $D_L(z)$ from (22.2) <u>using only $t_B(r)$</u>.



In the L-T model the BB occurs progressively later when we approach the observer.

→ At point P● the particle in the L-T model is ``younger'' than in a Friedmann model and the ***age difference*** increases toward the observer.

→ The velocity of expansion at P is greater in L-T than in the Friedmann model with $\Lambda = 0 = k$, and the velocity difference increases toward the observer.

→ The expansion velocity increases on approaching the observer.

When this effect is interpreted against an R-W background, the observer has the illusion that the expansion velocity increases with time.

→ ***If an L-T model were used to interpret the observations, there would be no ``accelerated expansion'' and no need to introduce the ``dark energy''***.

In the preceding example, the $D_L(z)$ of the ΛCDM model was reproduced using just $t_B(r)$, with E(r) being the same as in a Friedmann model.

The converse is also possible: we may take t_B = constant as in a Friedmann model and construct numerically such a function E(r) that the same $D_L(z)$ is reproduced [48].

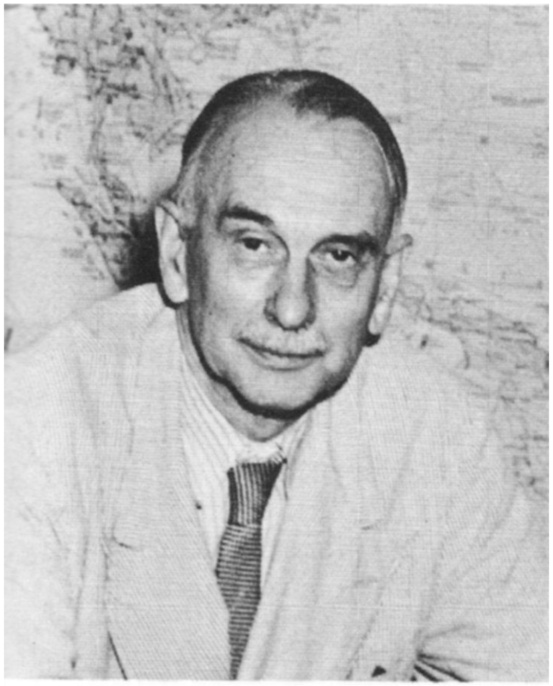
→ If both t_B and E(r) are non-Friedmannian, then we can reproduce two sets of cosmological observations using an L-T model.

Such a result was also published [49].

[48 A. Krasiński, Accelerating expansion or inhomogeneity? Part 2: Mimicking acceleration with the energy function in the Lemaître -- Tolman model. *Phys. Rev.* **D90**, 023524 (2014).
[49] M.-N. Cèlèrier, K. Bolejko and A. Krasiński, A (giant) void is not mandatory to explain away dark energy with a Lemaître -- Tolman model. *Astronomy and Astrophysics* **518**, A21 (2010).

23. Conclusion

Slavish sticking to the R-W class of models makes us blind to several phenomena.



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