Relativistic cosmology from F to Sz

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Contents

Lecture 1: Fundamentals of general relativity

1. Gravitation as geometry

22.06.1864 Aleksota (now Kaunas, Lithuania)
--12.01.1909 Göttingen, Germany

General relativity is an extension of special relativity (SR).

The basic postulate of SR is that each observer moving with zero acceleration will measure the same velocity of light c:

$$(\Delta s)^2 := c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0, \qquad (1.1)$$

Hermann Minkowski noticed in 1908 that the ***Lorentz transformations*** that preserve the equation $(\Delta s)^2 = 0$, also preserve the ***value of*** $(\Delta s)^2$.

In Euclidean geometry the distance between points of coordinates (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$ is

$$L^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (1.2)$$

and is preserved by rotations that are linear in (x, y, z), like the Lorentz transformations.

From the similarity between (1.1) and (1.2) Minkowski concluded that special relativity is the geometry of a space which is today called the ***Minkowski spacetime***.

$$(\Delta s)^2 := c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0. \quad (1.1)$$

Einstein found out some time before 1908 that ***gravitational field can be simulated by acceleration*** of the observer.

The Lorentz transformations preserving the ***metric form*** (1.1) describe the transition to another reference system moving without acceleration (with constant velocity).

In a system moving *with* acceleration the new variables (t', x', y', z') can be *arbitrary* functions of the old (t, x, y, z).

The coefficients of the transformed form (1.1) will be no longer constant.

Example:

$x = x' + t'^2 \quad \rightarrow \quad \Delta x = \Delta x' + 2t' \Delta t' \quad \rightarrow$

$$(\Delta s)^2 = (c^2 - 4t'^2) (\Delta t')^2 - 4t' \Delta x' \Delta t' - (\Delta x')^2 - (\Delta y)^2 - (\Delta z)^2.$$

→ In a spacetime with gravitational field, the coefficients of squares and products of Δt, Δx, Δy and Δz in (1.1) should also be functions of the coordinates.

Such a geometry was introduced by ***Bernhard Riemann*** in his ***habilitation lecture*** in 1854.

Bernhard Riemann, Über Die Hypothesen, Welche Der Geometrie Zu Grunde Liegen. Springer 2013, with comments by J. Jost.

$$L^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (1.2)$$

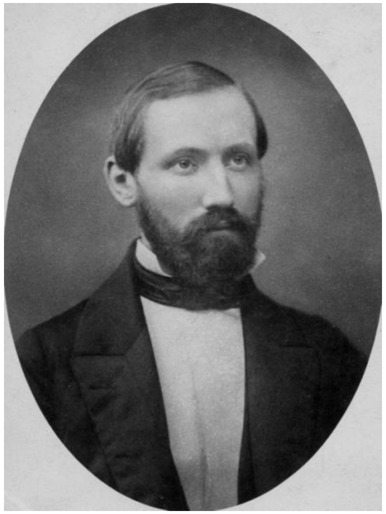
Riemann's idea was to generalise the Pythagoras formula (1.2) to a symmetric quadratic form in an n-dimensional space:

$$ds^2 = g_{11}(x_1, ..., x_n)(dx_1)^2 + 2\, g_{12}(x_1, ..., x_n)\, dx_1\, dx_2 + ... + g_{nn}(x_1, ..., x_n)(dx_n)^2, \qquad (1.3)$$

whose coefficients are functions of the coordinates. The ds is a distance between points of coordinates $(x_1, x_2, ..., x_n)$ and $(x_1 + dx_1, x_2 + dx_2, ..., x_n + dx_n)$.

in ≈1850

https://upload.wikimedia.org/wikipedia/commons/9/9e/Bernhard_Riemann_2.jpg



in 1863

https://www.livescience.com/65577-riemann-hypothesis-big-step-math.html

Georg Friedrich Bernhard Riemann
17.09.1826, Breselenz (Hannover, Germany) – 16.06.1866, Selasca (Lago Maggiore, Italy)

2. A quick introduction to general relativity

Following Riemann, Einstein assumed that the geometry of spacetime is described by the ***fundamental form*** (usually called ***metric form***, although it is not positive-definite):

$$\Phi = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \equiv g_{00}\,(dx^0)^2 + 2\, g_{01}\, dx^0\, dx^1 + 2\, g_{02}\, dx^0\, dx^2 + 2\, g_{03}\, dx^0\, dx^3$$
$$+ g_{11}\,(dx^1)^2 + 2\, g_{12}\, dx^1\, dx^2 + 2\, g_{13}\, dx^1\, dx^3$$
$$+ g_{22}\,(dx^2)^2 + 2\, g_{23}\, dx^2\, dx^3 + 2\, g_{33}\,(dx^3)^2, \qquad (2.1)$$

where dx^0, dx^1, dx^2, dx^3 are coordinate differences between two neighbouring points A and B, and the collection of coefficients $g_{\alpha\beta}(x)$ is called the ***metric tensor*** (or just ***metric***).

If points A and B can be connected by an arc $\mathscr{I}$ on which $\Phi > 0$ everywhere, then A and B are said to be in a ***timelike relation***. Between such points one can travel with a velocity smaller than the light velocity, and the quantity

$$\int_{\mathscr{I}\ A}^{B} \Phi^{1/2}\, d\lambda := T_{AB}$$

is the travel time along $\mathscr{I}$.

Among such arcs there is one on which T_{AB} is maximum, called a ***timelike geodesic.*** Timelike geodesics are paths of free motion of massive bodies.

$$\begin{aligned}\Phi = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} \equiv\ & g_{00}\,(dx^0)^2 + 2\, g_{01}\, dx^0\, dx^1 + 2\, g_{02}\, dx^0\, dx^2 + 2\, g_{03}\, dx^0\, dx^3 \\ & + g_{11}\,(dx^1)^2 + 2\, g_{12}\, dx^1\, dx^2 + 2\, g_{13}\, dx^1\, dx^3 \\ & + g_{22}\,(dx^2)^2 + 2\, g_{23}\, dx^2\, dx^3 + 2\, g_{33}\,(dx^3)^2, \qquad (2.1)\end{aligned}$$

If points A and B can be connected by an arc $\mathcal{N}$ on which $\Phi = 0$ everywhere, but cannot be connected by an arc on which $\Phi > 0$ everywhere, then A and B are said to be in a ***null relation*** (sometimes, ***light-like relation***).

The arc $\mathcal{N}$ is then called a ***null curve.***

Null curves are paths of electromagnetic signals (e.g., light rays).

The set of points in null relation to A has the equation $\Phi = 0$ (see (2.1)).

It is called a ***null cone*** because $\Phi = 0$ at a fixed point can be transformed to

$$c^2\,(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = 0, \qquad (2.2)$$

which looks like the equation of a cone in a 4-dimensional Euclidean space.

Each spacetime point is the vertex of a null cone.

$$\Phi = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \equiv g_{00}\,(dx^0)^2 + 2\, g_{01}\, dx^0\, dx^1 + 2\, g_{02}\, dx^0\, dx^2 + 2\, g_{03}\, dx^0\, dx^3 + g_{11}\,(dx^1)^2 + 2\, g_{12}\, dx^1\, dx^2 + 2\, g_{13}\, dx^1\, dx^3 + g_{22}\,(dx^2)^2 + 2\, g_{23}\, dx^2\, dx^3 + 2\, g_{33}\,(dx^3)^2, \qquad (2.1)$$

If points A and B cannot be connected by an arc on which $\Phi > 0$ everywhere or $\Phi = 0$ everywhere, then A and B are said to be in a ***spacelike relation***.

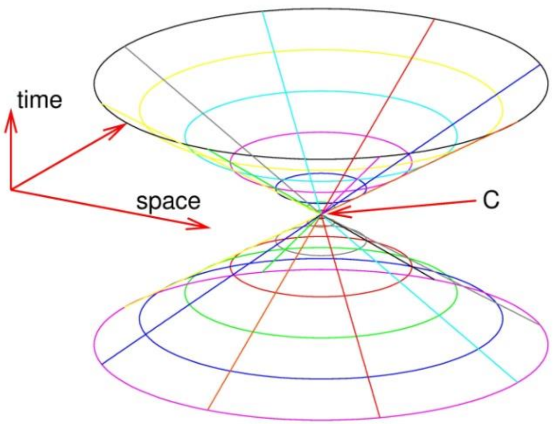
The quantity

$$\int_{\mathfrak{S}\,A}^{B} (-\Phi)^{1/2}\, d\lambda := \mathcal{L}_{AB}$$

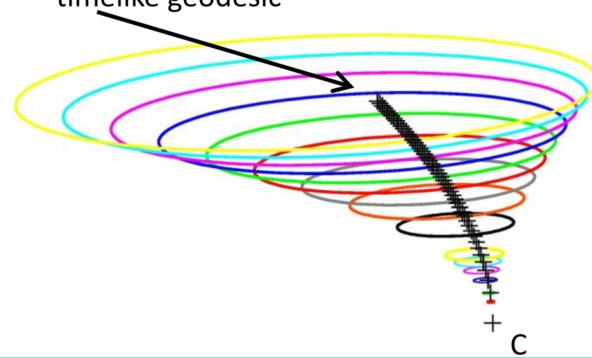
is the length of the arc $\mathfrak{S}$ between A and B.

Among those arcs there is one on which $\mathcal{L}_{AB}$ is ***minimum***, it is called a ***spacelike geodesic.***

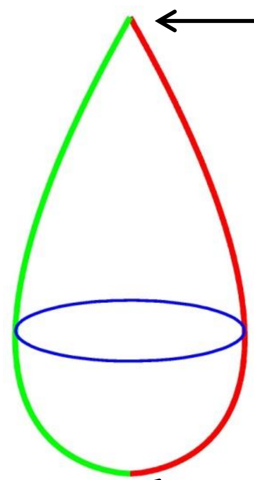
Examples of light cones



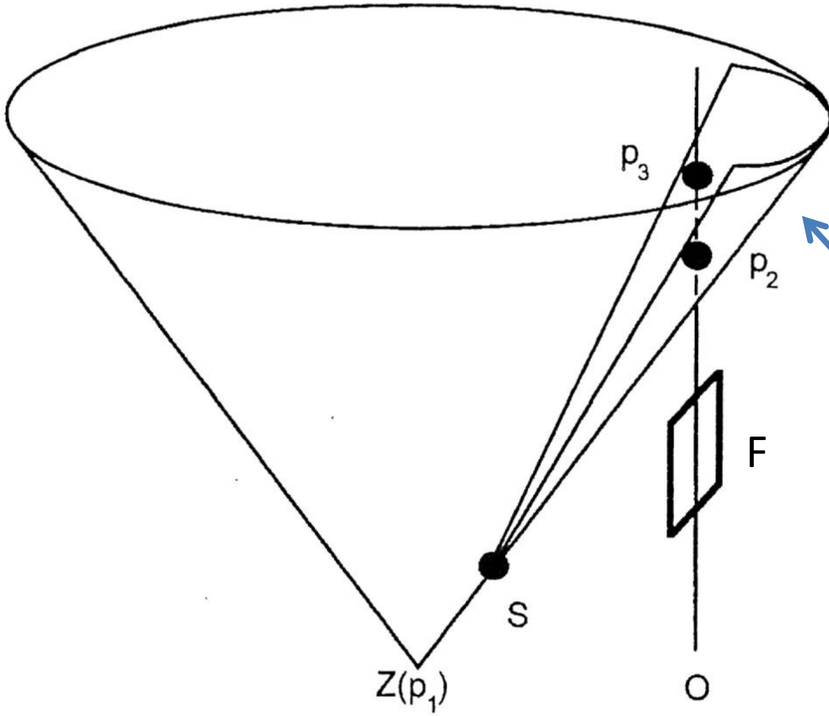
The light cone of event C in the Minkowski spacetime

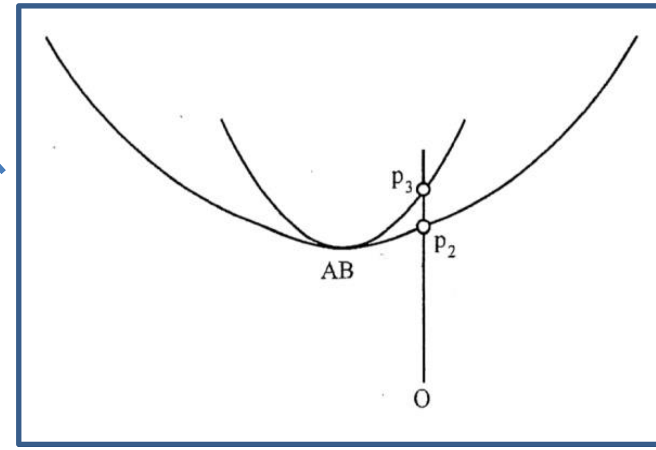


The future light cone of event C in a spacetime with gravitational field (a fictitious illustration)



The past light cone of the present observer in the now-standard model of the Universe





The intersection of the light cone
with plane F.

A light cone with a self-intersection.
This situation is typical in a gravitational lens.
The spherical lens is at S, the light source is at $Z(p_1)$.

In general relativity, the geometry of spacetime is connected with matter distribution via the ***Einstein equations***. This is the recipe for writing them out:

Step 1 – calculate the ***Christoffel symbols***:

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \frac{1}{2}\, g^{\alpha\rho} \left(g_{\beta\rho,\gamma} + g_{\gamma\rho,\beta} - g_{\beta\gamma,\rho} \right)$$

$_{,\beta}$ means $\partial/\partial x^{\beta}$; $g^{\alpha\varrho}$ is the inverse matrix to $g_{\alpha\varrho}$, repeated index means we sum over all its values: 0, 1, 2, 3.

Step 2 – calculate the ***curvature tensor*** (often called the Riemann tensor):

$$R^{\alpha}{}_{\beta\gamma\delta} = -\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_{,\delta} + \left\{ \begin{matrix} \alpha \\ \beta\delta \end{matrix} \right\}_{,\gamma} + \left\{ \begin{matrix} \alpha \\ \rho\gamma \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \beta\delta \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \rho\delta \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \beta\gamma \end{matrix} \right\}$$

Step 3 – calculate the ***Ricci tensor*** $R_{\alpha\beta}$ and its trace R:

$R_{\alpha\beta} = R^{\varrho}{}_{\alpha\varrho\beta}$, $R = R^{\varrho}{}_{\varrho} \equiv g^{\varrho\sigma} R_{\varrho\sigma}$

Step 4 – calculate the ***Einstein tensor*** and equate it to the ***energy-momentum tensor***:

$$G_{\alpha\beta} \stackrel{\text{def}}{=} R_{\alpha\beta} - g_{\alpha\beta} R^{\rho}{}_{\rho} = \frac{8\pi G}{c^2}\, T_{\alpha\beta} \qquad (2.3)$$

← these are the Einstein equations

where:

$$T_{\alpha\beta} = \begin{bmatrix} \text{energy density (a scalar)} & \text{momentum density (a matrix } 1\times 3) \\ \text{momentum density (a matrix } 3\times 1) & \text{the stress tensor (a matrix } 3\times 3) \end{bmatrix}$$

In general, the Einstein eqs. are a set of 10 partial differential equations in 4 variables. With symmetries present, both these numbers decrease (see further).

Sometimes the Einstein equations are considered in a slightly generalised form:

$R_{\alpha\beta} - (1/2)\, R^{\rho}{}_{\rho}\, g_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = (8\pi G/c^2)\, T_{\alpha\beta}$ (2.3')

where Λ is the ***cosmological constant*** (its value is to be determined from observations).

We shall mostly assume $\Lambda = 0$.

The r.h.s of (2.3) most frequently used in cosmology is ***perfect fluid***

$T_{\alpha\beta} = (\rho c^2 + p)\, u_\alpha u_\beta - p g_{\alpha\beta}$,

where ρ is the mass density, p is the pressure, u_α is the ***four-velocity*** field of the fluid.
u_α always obeys $u_\alpha u^\alpha = 1$.

The subcase p = 0 is called ***dust***. In this case u^α is necessarily geodesic (see below).

In ***comoving coordinates*** $u^\alpha = (u^t, u^x, u^y, u^z) = (1, 0, 0, 0)$, and then

$$T^{\alpha}{}_{\beta} = \begin{bmatrix} \rho c^2 + p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{bmatrix} \Longleftrightarrow T_{\alpha\beta} = (\rho c^2 + p)\, u_\alpha u_\beta - p g_{\alpha\beta}$$

All geodesics obey the equation

$$\frac{\mathrm{d}k^\alpha}{\mathrm{d}s} + \begin{Bmatrix} \alpha \\ \rho\sigma \end{Bmatrix} k^\rho k^\sigma = 0 \qquad (2.4)$$

where s is the ***affine parameter*** and $k^\alpha = dx^\alpha/ds$ is the tangent vector to the geodesic.

In a non-affine parametrisation, the 0 on the r.h.s. of (2.4) changes to λk^α. s is determined up to s = as'.

Equation (2.4) can be derived from the variational principle:

$$\delta\left[\int_A^B g_{\alpha\beta}\,(dx^\alpha/ds)\,(dx^\beta/ds)\,ds\right] = 0, \qquad (2.5)$$

while the Newtonian equations of motion can be derived from

$$\delta\left[\int_A^B (m\mathbf{v}^2/2 - m\varphi)\,dt\right] = 0. \qquad (2.6)$$

By requiring that (2.5) goes over into (2.6) in the limit $c \to \infty$ one obtains [1]:

$$g_{00} = 1 + \frac{2\varphi}{c^2} + O\left(\frac{1}{c^3}\right), \qquad g_{0I} = O\left(\frac{1}{c^2}\right), \qquad g_{IJ} = -\delta_{IJ} + O\left(\frac{1}{c}\right) \qquad (2.7)$$

where $I, J = 1, 2, 3$, and $O(1/c^n)$ means

$$\lim_{c \to \infty} c^{n-1} O(1/c^n) = 0.$$

[1] J. Plebański and A. Krasiński, An introduction to general relativity and cosmology. Cambridge University Press 2006.

$$g_{00} = 1 + \frac{2\varphi}{c^2} + O\left(\frac{1}{c^3}\right), \qquad g_{0I} = O\left(\frac{1}{c^2}\right), \qquad g_{IJ} = -\delta_{IJ} + O\left(\frac{1}{c}\right) \quad (2.7)$$

$$G_{\alpha\beta} \stackrel{\text{def}}{=} R_{\alpha\beta} - g_{\alpha\beta} R^{\rho}{}_{\rho} = \frac{8\pi G}{c^2} T_{\alpha\beta} \quad (2.3)$$

$$\frac{\mathrm{d}k^{\alpha}}{\mathrm{d}s} + \left\{ \begin{matrix} \alpha \\ \rho\sigma \end{matrix} \right\} k^{\rho} k^{\sigma} = 0 \quad (2.4)$$

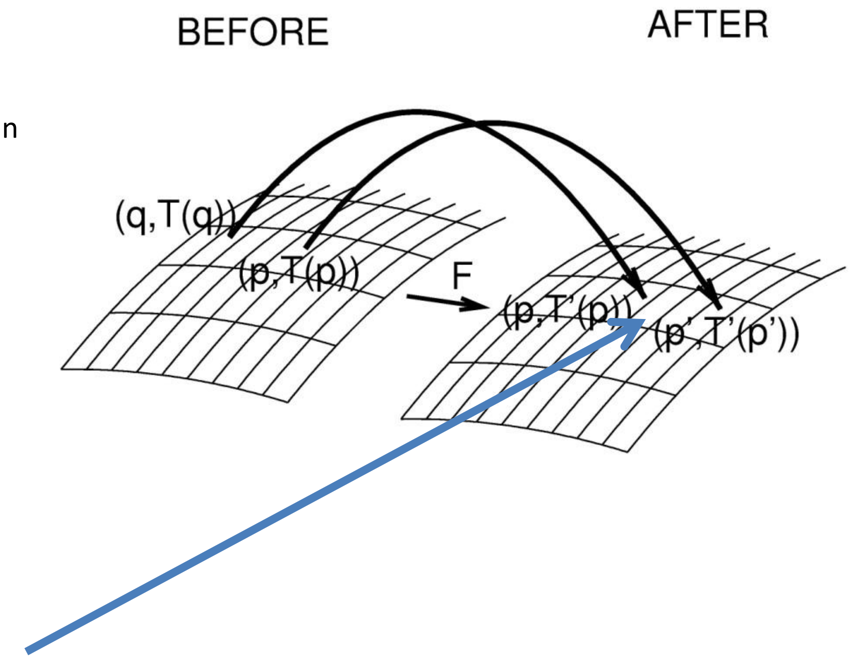
Substituting the approximation (2.7) into the Einstein equations (2.3) we conclude that in the limit c →∞ the equation α = β = 0 goes over into the Poisson equation

$\Delta\varphi = 4\pi G \times$(mass density)

while the remaining Einstein equations become identities 0 = 0.

Going to higher orders in 1/c one can discuss *post-Newtonian corrections* to Newton's theory of gravitation.

3. Symmetries



Let $F:M_n \rightarrow M_n$ be a mapping of the spacetime M_n into itself.

Let p be a point in M_n and p' = F(p) be its image.

Then, tensors attached at p are transformed to tensors attached at p'.

A tensor T at p becomes T' at p'.

But p becomes an image of another point q, p = F(q) and tensors at q are sent to p.

→ We have two tensors at p: T(p) that was there before the transformation and T'(p) that was sent from q to p by F.

If T'(p) = T(p) for all points of the spacetime, then T is ***invariant under the action of F***, and F is an ***invariance transformation of T***.

If M_n is a Riemann space, and the metric tensor of M_n is invariant under F, then F is a ***symmetry*** (also called ***isometry***) of M_n.

Suppose F is a member of a continuous one-parameter group of transformations *G.*

$G = \{F_t : M_n \rightarrow M_n , t \in [t_1, t_2] \}$ with F_{t0} being the identity transformation, $t_1 \leq t_0 \leq t_2$.

Example:

Let $M_n = R^3$ and F_t be the rotation of R^3 around a fixed axis A by the angle t. Then *G* is the set of rotations of R^3 around A by all angles in the range $0 \leq t < 2\pi$ and $t_0 = 0$.

Apply to a point $P \in M_n$ the mappings F_t corresponding to all $t \in [t_1, t_2]$.

The collection of all images of P will then be an arc of a curve in M_n passing through $P = F_{t0}(P)$, and each $P \in M_n$ may be used to generate such an arc ***C***.

Now imagine the derivative by t of any tensor field *T*.

This is a derivative along the tangent vector field *k* to ***C***.

It is called the ***Lie derivative***, denoted $₤_k T$ and the vectors *k* are called ***generators*** of *G*.

$F_t \in G$ are isometries of M_n when $₤_k\, g_{\alpha\beta} = 0$ along ***C***.

$₤_k T$ can be calculated for any tensor (or other) field T, its explicit form depends on the transformation properties of T under coordinate changes. Examples:

$₤_k\Phi = k^\alpha \Phi,_\alpha$ (directional derivative) for a scalar field Φ,

$₤_k V^\alpha = k^\varrho V^\alpha,_\varrho - k^\alpha,_\varrho V^\varrho$ (the commutator of k and V) for a contravariant vector field V^α,

$₤_k W_\alpha = k^\varrho W_{\alpha},_\varrho + k^\varrho,_\alpha W_\varrho$ (the anticommutator of k and W) for a covariant vector field W_α,

$₤_k g_{\alpha\beta} = k^\varrho g_{\alpha\beta},_\varrho + k^\varrho,_\alpha g_{\varrho\beta} + k^\varrho,_\beta g_{\alpha\varrho}$ for any doubly covariant tensor field, in particular, for the metric tensor.

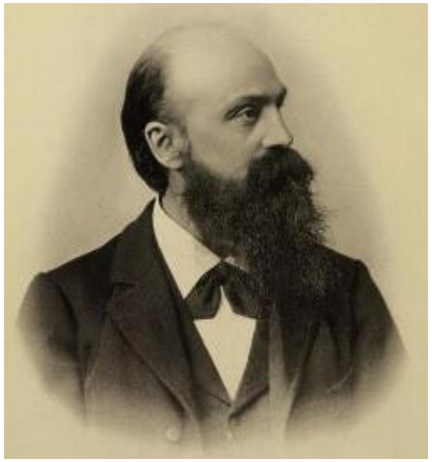
For the metric tensor the equations

$$₤_k g_{\alpha\beta} = k^\varrho g_{\alpha\beta},_\varrho + k^\varrho,_\alpha g_{\varrho\beta} + k^\varrho,_\beta g_{\alpha\varrho} = 0 \qquad (3.1)$$

are called ***Killing equations***.

They allow us to find the generators of symmetries k^α for a given metric $g_{\alpha\beta}$ or the most general metric $g_{\alpha\beta}$ that has a given symmetry group (for example, SO(3)).

With k^α known, the symmetry transformations $x^\alpha(x')$ are found from $dx^\alpha/ds = k^\alpha$ [1].



Wilhelm Karl Joseph Killing
10 May 1847, Burbach near Siegen, Germany
– 11 February 1923, Münster, Germany

[1] J. Plebański and A. Krasiński, An introduction to general relativity and cosmology. Cambridge University Press 2006.

A historical digression:

The notion of the Lie derivative was invented by the Polish mathematician Władysław Ślebodziński [2,3], who did not care to give it a name.

The misleading name "Lie derivative" was later introduced by D. van Dantzig and made popular by J. A. Schouten [4].

Władysław Ślebodziński
6 February 1884, Pysznica near Stalowa Wola, Poland
– 3 January 1972 Wrocław, Poland

[2] Władysław Ślebodziński, Sur les équations canoniques de Hamilton.} *Bulletins de la Classe des Sciences, Acad. Royale de Belg.* (5) **17**, 864 -- 870 (1931). English translations: *Gen. Relativ. Gravit.* **42**, 2525 -- 2535 (2010), and:
[3] Władysław Ślebodziński, in *Golden Oldies in General Relativity. Hidden Gems*. Edited by A. Krasiński, M. A. H. MacCallum and G. F. R. Ellis, Springer, Berlin-Heidelberg 2013, pp. 3 – 14.
[4] A. Trautman, page 4 in Ref. [3].

$£_k g_{\alpha\beta} = k^\varrho g_{\alpha\beta},_\varrho + k^\varrho,_\alpha g_{\varrho\beta} + k^\varrho,_\beta g_{\alpha\varrho} = 0$ (3.1)

4. An example of a solution of Einstein's equations – the Friedmann model [5,6]

Assume, that each 3-space of constant t is homogeneous and isotropic (i.e., is a 3-dimensional sphere, Euclidean space or a space of constant negative curvature).

By imposing the Killing equations (3.1) on the generators of symmetries of these 3-spaces one is led to the following metric [7,8,1]

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right] \qquad (4.1)$$

where k is an arbitrary constant (k > 0 for spherical space, k = 0 for Euclidean space, k < 0 for negative-curvature space),

and S(t) is to be found from the Einstein equations (see next page).

This is how these spacetimes, now called Friedmann–Lemaître–Robertson–Walker (FLRW) first appeared in literature – out of mathematical speculation, with a high symmetry assumed for simplicity.

[5] A. A. Friedmann, Über die Krümmung des Raumes. *Z. Physik* **10**, 377 (1922); *Gen. Relativ. Gravit.* **31**, 1991 (1999) + addendum: **32**, 1937 (2000).
[6] A. A. Friedmann, Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes. *Z. Physik* **21**, 326 (1924); *GRG* **31**, 2001 (1999) ; both reprinted papers with an editorial note by A. Krasiński and G. F. R. Ellis, *Gen. Relativ. Gravit.* **31**, 1985 (1999).
[7] H. P. Robertson, Relativistic cosmology. *Rev. Mod. Phys.* **5**, 62 (1933); *Gen. Relativ. Gravit.* **44**, 2115 (2012) , with an editorial note by G. F. R. Ellis, *Gen. Relativ. Gravit.* **44**, 2099 (2012).
[8] A. G. Walker, On Riemannian spaces with spherical symmetry about a line, and the conditions for isotropy in general relativity. *Quart. J. Math. Oxford*, ser. 6, 81 (1935).
[1] J. Plebański and A. Krasiński, An introduction to general relativity and cosmology. Cambridge University Press 2006.

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right] \quad (4.1)$$

With (4.1), the Einstein equations imply for the velocity $u^\alpha = (1, 0, 0, 0)$. Such coordinates are called ***comoving***.

For dust, the Einstein equations with the metric (4.1) reduce to

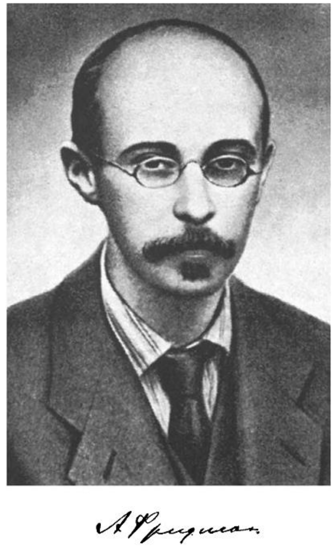
$$S_{,t}{}^2 = \frac{2GM}{c^2 S} - k - \frac{1}{3}\Lambda S^2 \quad (4.2)$$

where $\rho = \frac{3M}{4\pi S^3}$ is the mass density. M and Λ are constant, so ϱ depends only on t.

The ***assumption*** of high symmetry reduced the whole geometrical wealth of the Einstein theory to one equation for a function of one variable.

Equations (4.1) – (4.2) are used until today, with various modifications of the r.h.s. of (4.2), as ``standard'' models of the actual Universe.

Their properties, and some problems connected with them, will be discussed in the next lecture.



https://pl.wikipedia.org/wiki/Aleksandr_Friedman#/media/Plik:Fridman_AA.jpg

https://kierul.files.wordpress.com/2013/06/aleksandr_fridman.png

Aleksandr Aleksandrovich Fridman
[ФРИДМАН Александр Александрович]()
16 July 1888, Sankt Peterburg - 16 September 1925, Leningrad (same city)

Monsignor Georges Lemaître
17 Jul 1894, Charleroi, Belgium
- 20 Jun 1966, Louvain, Belgium
https://todayinsci.com/L/Lemaitre_Georges/
LemaitreGeorges-Quotations.htm

Howard Percy Robertson
27 Jan 1903, Hoquiam, Washington, USA
-- 26 Aug 1961, Pasadena, California, USA
https://history.aip.org/phn/11605024.html

Arthur Geoffrey Walker
17 July 1909, Watford, Hertfordshire, England
- 31 March 2001, Chichester, Sussex,England
https://memim.com/arthur-geoffrey-walker.html