Mechanics on algebroids

Janusz Grabowski

(Polish Academy of Sciences)

May 30, 2021

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Plan of the talk

- The classical Tulczyjew triple
- Euler-Lagrange equations
- Tulczyjew triples for algebroids
- Digression on field theories
- Tulczyjew triples for strings
- Lagrangian and Hamiltonian formalism for strings
- Example: Plateau problem
- Some references
- Home work

Dynamics

- By (usually implicit) first-order dynamics on a manifold N we will understand a submanifold (or even subset) D in $\mathsf{T}N$.
- A curve $\gamma : \mathbb{R} \to N$ satisfies this dynamics (is a solution), if its tangent prolongation belongs to D, $\mathrm{t}(\gamma) : \mathbb{R} \to D \subset \mathsf{T}N$.

Example

A vector field X on N, i.e. a section of the tangent bundle $X : N \to \mathsf{T}N$, defines the dynamics $D = X(N) \subset \mathsf{T}N$.

In local coordinates, for the vector field $X = f_a(q)\frac{\partial}{\partial q^a}$, we have

$$D = \{(q^a, \dot{q}^b) \in \mathsf{T}N : \dot{q}^b = f_b(q)\}$$

and the explicit dynamical equations $\frac{\mathrm{d}q^a}{\mathrm{d}t}(t) = f_a(q(t))$ are the equations for trajectories of this vector field.

Dynamics

- By (usually implicit) first-order dynamics on a manifold N we will understand a submanifold (or even subset) D in $\mathsf{T}N$.
- A curve $\gamma : \mathbb{R} \to N$ satisfies this dynamics (is a solution), if its tangent prolongation belongs to D, $\mathrm{t}(\gamma) : \mathbb{R} \to D \subset \mathsf{T}N$.

Example

A vector field X on N, i.e. a section of the tangent bundle $X : N \to \mathsf{T}N$, defines the dynamics $D = X(N) \subset \mathsf{T}N$.

- In local coordinates, for the vector field $X = f_a(q)\frac{\partial}{\partial q^a}$, we have

$$D = \{(q^a, \dot{q}^b) \in \mathsf{T}N : \dot{q}^b = f_b(q)\}$$

and the explicit dynamical equations $\frac{\mathrm{d}q^a}{\mathrm{d}t}(t) = f_a(q(t))$ are the equations for trajectories of this vector field.

Dynamics

- By (usually implicit) first-order dynamics on a manifold N we will understand a submanifold (or even subset) D in $\mathsf{T}N$.
- A curve $\gamma : \mathbb{R} \to N$ satisfies this dynamics (is a solution), if its tangent prolongation belongs to D, $\mathrm{t}(\gamma) : \mathbb{R} \to D \subset \mathsf{T}N$.

Example

A vector field X on N, i.e. a section of the tangent bundle $X : N \to \mathsf{T}N$, defines the dynamics $D = X(N) \subset \mathsf{T}N$.

In local coordinates, for the vector field $X = f_a(q)\frac{\partial}{\partial q^a}$, we have

$$D = \{(q^a, \dot{q}^b) \in \mathsf{T}N : \dot{q}^b = f_b(q)\}$$

and the explicit dynamical equations $\frac{\mathrm{d}q^a}{\mathrm{d}t}(t) = f_a(q(t))$ are the equations for trajectories of this vector field.

Dynamics

- By (usually implicit) first-order dynamics on a manifold N we will understand a submanifold (or even subset) D in $\mathsf{T}N$.
- A curve $\gamma : \mathbb{R} \to N$ satisfies this dynamics (is a solution), if its tangent prolongation belongs to D, $\mathrm{t}(\gamma) : \mathbb{R} \to D \subset \mathsf{T}N$.

Example

A vector field X on N, i.e. a section of the tangent bundle $X : N \to \mathsf{T}N$, defines the dynamics $D = X(N) \subset \mathsf{T}N$.

- In local coordinates, for the vector field $X = f_a(q)\frac{\partial}{\partial q^a}$, we have

$$D = \{(q^a, \dot{q}^b) \in \mathsf{T}N : \dot{q}^b = f_b(q)\}$$

and the explicit dynamical equations $\frac{\mathrm{d}q^a}{\mathrm{d}t}(t) = f_a(q(t))$ are the equations for trajectories of this vector field.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

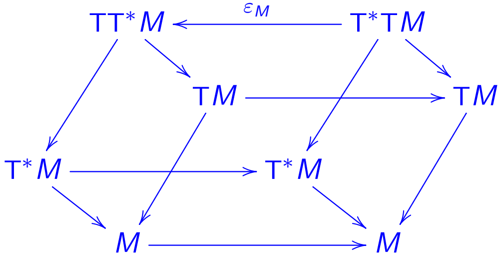
$$\mathcal{D} = \left\{(x,p,\dot{x},\dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot\gamma \in \mathcal{D}$. For the Lagrangian phase equations:

- M - positions,
- $\mathsf{T}M$ - (kinematic) configurations,
- $L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
- T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

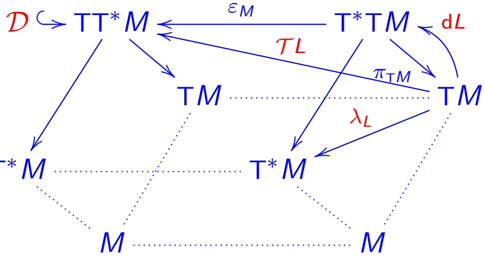
$$\mathcal{D} = \left\{ (x, p, \dot x, \dot p) : \quad p = \frac{\partial L}{\partial \dot x}, \quad \dot p = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot x}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot\gamma \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

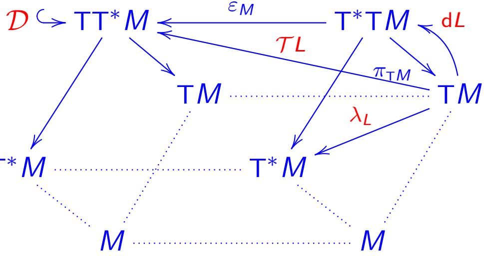
$$\mathcal{D} = \left\{(x, p, \dot x, \dot p) : \quad p = \frac{\partial L}{\partial \dot x}, \quad \dot p = \frac{\partial L}{\partial x}\right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot x}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot\gamma \in \mathcal{D}$. For the Lagrangian phase equations:

- M - positions,
- $\mathsf{T}M$ - (kinematic) configurations,
- $L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
- T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

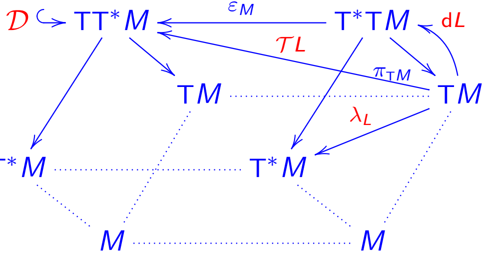
$$\mathcal{D} = \left\{(x, p, \dot x, \dot p) : \quad p = \frac{\partial L}{\partial \dot x}, \quad \dot p = \frac{\partial L}{\partial x}\right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot x}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

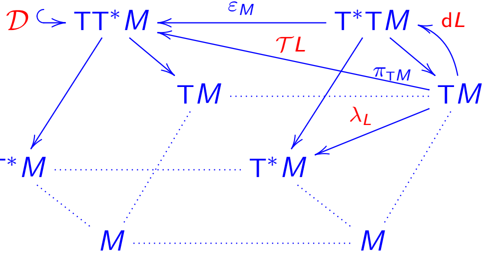
$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

- M - positions,
- $\mathsf{T}M$ - (kinematic) configurations,
- $L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
- T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Hamiltonian side

canonical isomorphism
$\mathsf{T}^*\mathsf{T}M \simeq \mathsf{T}^*\mathsf{T}^*M$,

$E : \mathsf{T}^*M \times_M \mathsf{T}M \to \mathbb{R}$
$\widetilde{H}(p, v) = \langle p, v \rangle - L(v)$
$H : \mathsf{T}^*M \to \mathbb{R}$

$$\mathcal{D} = \Pi_M^{\#}(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \right\},$$

whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

canonical isomorphism
$\mathsf{T}^*\mathsf{T}M \simeq \mathsf{T}^*\mathsf{T}^*M$,

$E : \mathsf{T}^*M \times_M \mathsf{T}M \to \mathbb{R}$
$\widetilde{H}(p,v) = \langle\, p,\ v\,\rangle - L(v)$
$H : \mathsf{T}^*M \to \mathbb{R}$

$$\mathcal{D} = \Pi^{\#}_M(\mathrm{d}H(\mathsf{T}^*M))$$

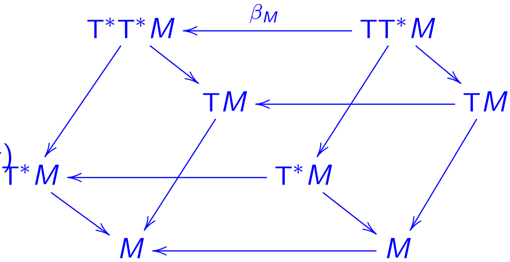
$$\mathcal{D} = \left\{(x,p,\dot{x},\dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

Hamiltonian side of the triple

canonical isomorphism
$\mathrm{T}^*\mathrm{T}M \simeq \mathrm{T}^*\mathrm{T}^*M$,

$E : \mathrm{T}^*M \times_M \mathrm{T}M \to \mathbb{R}$
$\widetilde{H}(p, v) = \langle\, p,\ v\,\rangle - L(v)$
$H : \mathrm{T}^*M \to \mathbb{R}$



$$\mathcal{D} = \Pi_M^{\#}(\mathrm{d}H(\mathrm{T}^*M))$$

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

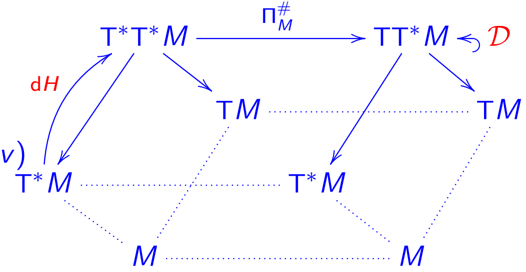
whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

Hamiltonian side of the triple

canonical isomorphism
$\mathsf{T}^*\mathsf{T}M \simeq \mathsf{T}^*\mathsf{T}^*M$,

$E : \mathsf{T}^*M \times_M \mathsf{T}M \to \mathbb{R}$
$\widetilde{H}(p, v) = \langle\, p,\ v\,\rangle - L(v)$
$H : \mathsf{T}^*M \to \mathbb{R}$



$$\mathcal{D} = \Pi^\#_M(\mathrm{d}H(\mathsf{T}^*M))$$

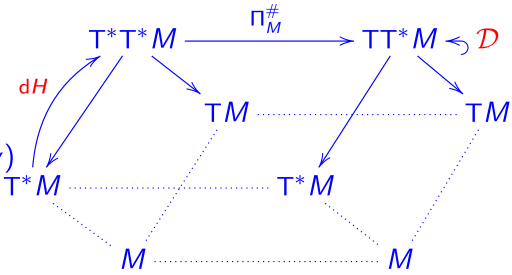
$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

Hamiltonian side of the triple

canonical isomorphism
$\mathsf{T}^*\mathsf{T}M \simeq \mathsf{T}^*\mathsf{T}^*M$,

$E : \mathsf{T}^*M \times_M \mathsf{T}M \to \mathbb{R}$
$\widetilde{H}(p, v) = \langle\, p,\ v\,\rangle - L(v)$
$H : \mathsf{T}^*M \to \mathbb{R}$



$$\mathcal{D} = \Pi_M^\#(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) :\quad \dot{p} = -\frac{\partial H}{\partial x},\quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

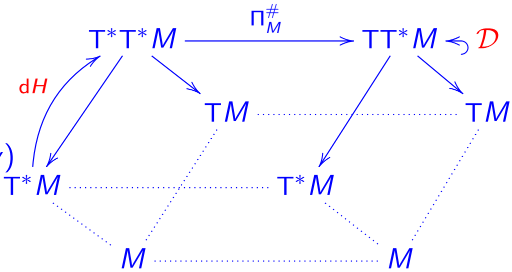
whence the Hamilton equations.

The Tulczyjew triple - Hamiltonian side

Hamiltonian side of the triple

canonical isomorphism
$\mathsf{T}^*\mathsf{T}M \simeq \mathsf{T}^*\mathsf{T}^*M$,

$E : \mathsf{T}^*M \times_M \mathsf{T}M \to \mathbb{R}$
$\widetilde{H}(p, v) = \langle\, p,\ v\,\rangle - L(v)$
$H : \mathsf{T}^*M \to \mathbb{R}$

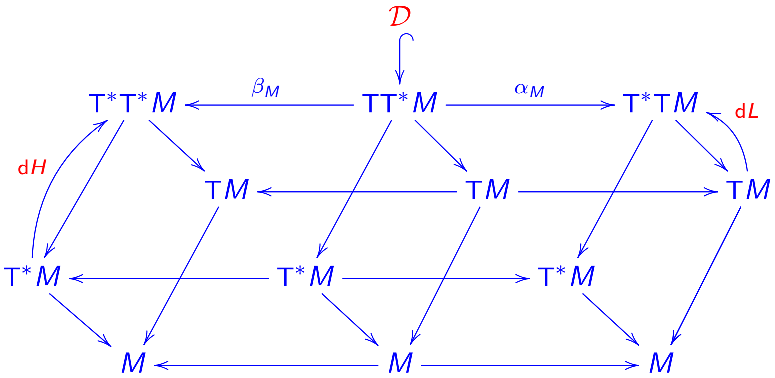


$$\mathcal{D} = \Pi^{\#}_M(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \right\},$$

whence the Hamilton equations.

Tulczyjew triple in mechanics



The dynamics is in the middle, the right-hand side is Lagrangian, the left-hand side – Hamiltonian.

The Legendre transform

- The Legendre transform is a pass from the Lagrange to the Hamilton description of the dynamics:
  we try to describe the Lagrangian phase dynamics as a Hamiltonian phase dynamics.
- It is easy in the case of hyperregular Lagrangians (the Legendre map $(q,p) \mapsto \lambda_L(q,\dot{q}) = (q,p)$ is a diffeomorhism).
- In this case the Lagrangian phase dynamics D_L is simultaneously Hamiltonian with the Hamiltonian function

$$\begin{aligned} H(q,p) &= \dot{q}^a p_a - L(q,\dot{q})\,, \\ (q,\dot{q}) &= \lambda_L^{-1}(q,p)\,. \end{aligned}$$

- In other words, the Lagrangian submanifolds $\mathrm{d}L(\mathsf{T}M) \subset \mathsf{T}^*\mathsf{T}M$ and $\mathrm{d}H(\mathsf{T}^*M) \subset \mathsf{T}^*\mathsf{T}^*M$ are related by the canonical isomorphism $\mathcal{R}_{\tau_M}$.

The Legendre transform

- The Legendre transform is a pass from the Lagrange to the Hamilton description of the dynamics:
  we try to describe the Lagrangian phase dynamics as a Hamiltonian phase dynamics.
- It is easy in the case of hyperregular Lagrangians (the Legendre map $(q,p) \mapsto \lambda_L(q,\dot{q}) = (q,p)$ is a diffeomorhism).
- In this case the Lagrangian phase dynamics D_L is simultaneously Hamiltonian with the Hamiltonian function

$$
\begin{aligned}
H(q,p) &= \dot{q}^a p_a - L(q,\dot{q}), \\
(q,\dot{q}) &= \lambda_L^{-1}(q,p).
\end{aligned}
$$

- In other words, the Lagrangian submanifolds $\mathrm{d}L(\mathrm{T}M) \subset \mathrm{T}^*\mathrm{T}M$ and $\mathrm{d}H(\mathrm{T}^*M) \subset \mathrm{T}^*\mathrm{T}^*M$ are related by the canonical isomorphism $\mathcal{R}_{\tau_M}$.

The Legendre transform

- The Legendre transform is a pass from the Lagrange to the Hamilton description of the dynamics:
  we try to describe the Lagrangian phase dynamics as a Hamiltonian phase dynamics.
- It is easy in the case of hyperregular Lagrangians (the Legendre map $(q,p) \mapsto \lambda_L(q,\dot q) = (q,p)$ is a diffeomorhism).
- In this case the Lagrangian phase dynamics D_L is simultaneously Hamiltonian with the Hamiltonian function

$$
\begin{aligned}
H(q,p) &= \dot q^a p_a - L(q,\dot q)\,,\\
(q,\dot q) &= \lambda_L^{-1}(q,p)\,.
\end{aligned}
$$

- In other words, the Lagrangian submanifolds $\mathrm{d}L(\mathrm{T}M) \subset \mathrm{T}^*\mathrm{T}M$ and $\mathrm{d}H(\mathrm{T}^*M) \subset \mathrm{T}^*\mathrm{T}^*M$ are related by the canonical isomorphism $\mathcal{R}_{\tau_M}$.

The Legendre transform

- The Legendre transform is a pass from the Lagrange to the Hamilton description of the dynamics:
  we try to describe the Lagrangian phase dynamics as a Hamiltonian phase dynamics.
- It is easy in the case of hyperregular Lagrangians (the Legendre map $(q,p)\mapsto\lambda_L(q,\dot q)=(q,p)$ is a diffeomorhism).
- In this case the Lagrangian phase dynamics D_L is simultaneously Hamiltonian with the Hamiltonian function

$$\begin{aligned} H(q,p) &= \dot q^a p_a - L(q,\dot q)\,, \\ (q,\dot q) &= \lambda_L^{-1}(q,p)\,. \end{aligned}$$

- In other words, the Lagrangian submanifolds $\mathrm{d}L(\mathsf{T}M)\subset\mathsf{T}^*\mathsf{T}M$ and $\mathrm{d}H(\mathsf{T}^*M)\subset\mathsf{T}^*\mathsf{T}^*M$ are related by the canonical isomorphism $\mathcal{R}_{\tau_M}$.

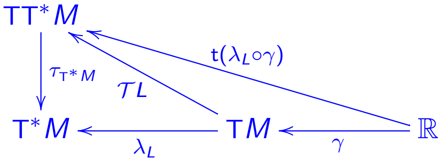
Euler-Lagrange equations

- The Euler-Lagrange equation for a curve $\underline{\gamma}:\mathbb{R}\to M$ takes in this model the form
$$\mathrm{t}(\lambda_L\circ\gamma)=\mathcal{T}L\circ\gamma\,,$$
where $\mathcal{T}L=\varepsilon\circ\mathrm{d}L$ is the Tulczyjew differential and $\gamma=\mathrm{t}(\underline{\gamma})$ is the tangent prolongation of $\underline{\gamma}$.
- In this sense, the Euler-Lagrange equation can be viewed as a first-order differential equation on curves γ in $\mathrm{T}M$:

- The equation just tells that the curve $\mathcal{T}L\circ\gamma$ is admissible, i.e. that it is a tangent prolongation of a curve (it must be $\lambda_L\circ\gamma$) on the phase space, $\mathcal{T}L\circ\gamma=\mathrm{t}(\lambda_L\circ\gamma)$.

Euler-Lagrange equations

- The Euler-Lagrange equation for a curve $\underline{\gamma}:\mathbb{R}\to M$ takes in this model the form
$$\mathrm{t}(\lambda_L\circ\gamma)=\mathcal{T}L\circ\gamma\,,$$
where $\mathcal{T}L=\varepsilon\circ\mathrm{d}L$ is the Tulczyjew differential and $\gamma=\mathrm{t}(\underline{\gamma})$ is the tangent prolongation of $\underline{\gamma}$.
- In this sense, the Euler-Lagrange equation can be viewed as a first-order differential equation on curves γ in $\mathsf{T}M$:



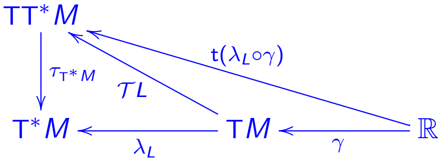
- The equation just tells that the curve $\mathcal{T}L\circ\gamma$ is admissible, i.e. that it is a tangent prolongation of a curve (it must be $\lambda_L\circ\gamma$) on the phase space, $\mathcal{T}L\circ\gamma=\mathrm{t}(\lambda_L\circ\gamma)$.

Euler-Lagrange equations

- The Euler-Lagrange equation for a curve $\underline{\gamma} : \mathbb{R} \to M$ takes in this model the form

$$\mathrm{t}(\lambda_L \circ \gamma) = \mathcal{T}L \circ \gamma\,,$$

  where $\mathcal{T}L = \varepsilon \circ \mathrm{d}L$ is the Tulczyjew differential and $\gamma = \mathrm{t}(\underline{\gamma})$ is the tangent prolongation of $\underline{\gamma}$.
- In this sense, the Euler-Lagrange equation can be viewed as a first-order differential equation on curves γ in $\mathrm{T}M$:



- The equation just tells that the curve $\mathcal{T}L \circ \gamma$ is admissible, i.e. that it is a tangent prolongation of a curve (it must be $\lambda_L \circ \gamma$) on the phase space, $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Euler-Lagrange equations (continued)

- In local coordinates,

$$\mathcal{T}L(q,\dot{q}) = (q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q}), \dot{q}, \frac{\partial L}{\partial q}(q,\dot{q}))\,.$$

For $\gamma(t) = (q(t), \dot{q}(t))$ this implies the equations

$$\dot{q}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}(t)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}(q(t),\dot{q}(t)) = \frac{\partial L}{\partial q}(q(t),\dot{q}(t))\,.$$

- These equations are second-order equations for curves $q = q(t)$ in M.
- Regularity of the Lagrangian is completely irrelevant for this formalism. Singular Lagrangians just produce complicated and implicit dynamics, but the geometric model is the same.

Euler-Lagrange equations (continued)

- In local coordinates,

$$\mathcal{T}L(q,\dot{q}) = (q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q}), \dot{q}, \frac{\partial L}{\partial q}(q,\dot{q}))\,.$$

  For $\gamma(t) = (q(t), \dot{q}(t))$ this implies the equations

$$\dot{q}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}(t)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}(q(t),\dot{q}(t)) = \frac{\partial L}{\partial q}(q(t),\dot{q}(t))\,.$$

- These equations are second-order equations for curves $q = q(t)$ in M.
- Regularity of the Lagrangian is completely irrelevant for this formalism. Singular Lagrangians just produce complicated and implicit dynamics, but the geometric model is the same.

Euler-Lagrange equations (continued)

- In local coordinates,

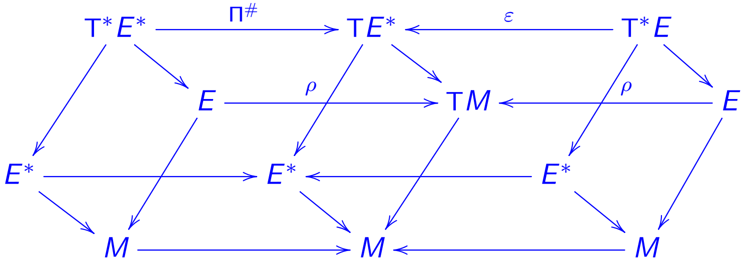
$$\mathcal{T}L(q,\dot{q}) = (q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q}), \dot{q}, \frac{\partial L}{\partial q}(q,\dot{q}))\,.$$

For $\gamma(t) = (q(t), \dot{q}(t))$ this implies the equations

$$\dot{q}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}(t)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}(q(t),\dot{q}(t)) = \frac{\partial L}{\partial q}(q(t),\dot{q}(t))\,.$$

- These equations are second-order equations for curves $q = q(t)$ in M.
- Regularity of the Lagrangian is completely irrelevant for this formalism. Singular Lagrangians just produce complicated and implicit dynamics, but the geometric model is the same.

Algebroid setting

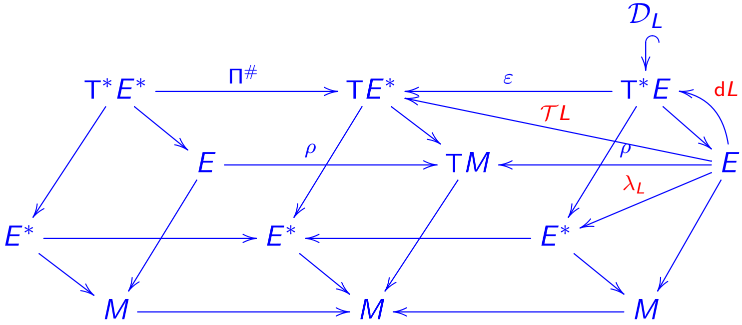


$H : E^* \longrightarrow \mathbb{R}$ $\mathcal{D} = \mathcal{T}L(E)$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\mathcal{D}_L \subset \mathsf{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Algebroid setting

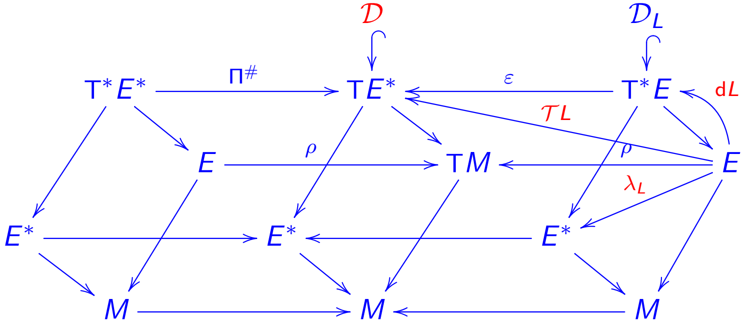


$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathrm{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathrm{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Algebroid setting

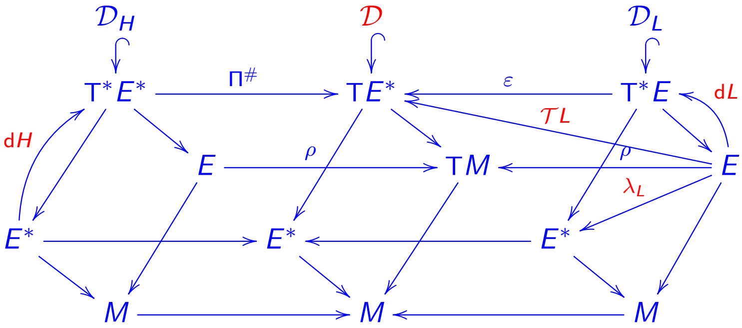


$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Euler-Lagrange equations for algebroids

If (q^a) are local coordinates in M,
(y^i) i (ξ_i) are linear coordinates in fibers of, respectively, E and E^*, and

$$\Pi = c^k_{ij}(q)\xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \rho^b_i(q)\partial_{\xi_i} \otimes \partial_{q^b} - \sigma^a_j(q)\partial_{q^a} \otimes \partial_{\xi_j}\,,$$

then the Euler-Lagrange equations read

$$(1) \qquad \frac{\mathrm{d}q^a}{\mathrm{d}t} = \rho^a_k(q)y^k\,,$$

$$(2) \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right)(q,y) = c^k_{ij}(q)y^i\frac{\partial L}{\partial y^k}(q,y) + \sigma^a_j(q)\frac{\partial L}{\partial q^a}(q,y)\,.$$

They are first-order differential equations (!) but for admissible curves in E, i.e. for curves satisfying (1). For $E = \mathsf{T}M$, they are exactly the tangent prolongations of curves in M, for which the equation is second-order.

Euler-Lagrange equations for algebroids

If (q^a) are local coordinates in M,
(y^i) i (ξ_i) are linear coordinates in fibers of, respectively, E and E^*, and

$$\Pi = c^k_{ij}(q)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho^b_i(q)\partial_{\xi_i}\otimes\partial_{q^b} - \sigma^a_j(q)\partial_{q^a}\otimes\partial_{\xi_j}\,,$$

then the Euler-Lagrange equations read

$$(1)\qquad \frac{\mathrm{d}q^a}{\mathrm{d}t} = \rho^a_k(q)y^k\,,$$

$$(2)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right)(q,y) = c^k_{ij}(q)y^i\frac{\partial L}{\partial y^k}(q,y) + \sigma^a_j(q)\frac{\partial L}{\partial q^a}(q,y)\,.$$

They are first-order differential equations (!) but for admissible curves in E, i.e. for curves satisfying (1). For $E = \mathsf{T}M$, they are exactly the tangent prolongations of curves in M, for which the equation is second-order.

Euler-Poincaré equations

A particular example of the equation (2) is not only the classical Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^a}(q,\dot q)=\frac{\partial L}{\partial q^a}(q,\dot q)\,.$$

but also the Lagrange-Poincaré equation for G-invariant Lagrangians on principal G-bundle

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^a}-\frac{\partial L}{\partial q^a}\right)(q,\dot q,v)-\left(B^k_{ba}(q)\dot q^b+D^k_{ia}(q)v^i\right)\frac{\partial L}{\partial v^k}(q,\dot q,v)=0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(q,\dot q,v)-\left(D^k_{aj}(q)\dot q^a+C^k_{ij}v^i\right)\frac{\partial L}{\partial v^k}(q,\dot q,v)=0\,,$$

and the Euler-Poincaré equations, for instance the rigid body equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(v)-C^k_{ij}v^i\frac{\partial L}{\partial v^k}(v)=0\,.$$

Euler-Poincaré equations

A particular example of the equation (2) is not only the classical Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a}(q,\dot{q}) = \frac{\partial L}{\partial q^a}(q,\dot{q})\,.$$

but also the Lagrange-Poincaré equation for G-invariant Lagrangians on principal G-bundle

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}\right)(q,\dot{q},v) - \left(B^k_{ba}(q)\dot{q}^b + D^k_{ia}(q)v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(q,\dot{q},v) - \left(D^k_{aj}(q)\dot{q}^a + C^k_{ij}v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

and the Euler-Poincaré equations, for instance the rigid body equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(v) - C^k_{ij}v^i\frac{\partial L}{\partial v^k}(v) = 0\,.$$

Euler-Poincaré equations

A particular example of the equation (2) is not only the classical Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a}(q,\dot{q}) = \frac{\partial L}{\partial q^a}(q,\dot{q})\,.$$

but also the Lagrange-Poincaré equation for G-invariant Lagrangians on principal G-bundle

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}\right)(q,\dot{q},v) - \left(B^k_{ba}(q)\dot{q}^b + D^k_{ia}(q)v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(q,\dot{q},v) - \left(D^k_{aj}(q)\dot{q}^a + C^k_{ij}v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

and the Euler-Poincaré equations, for instance the rigid body equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(v) - C^k_{ij}v^i\frac{\partial L}{\partial v^k}(v) = 0\,.$$

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathrm{T}M$ of n-vectors on M.

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathrm{T}M$ of n-vectors on M.

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathrm{T}M$ of n-vectors on M.

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathrm{T}M$ of n-vectors on M.

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n TM$ of n-vectors on M.

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathrm{T}M$ of n-vectors on M.

Digression on field theories

- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathrm{T}M$ of n-vectors on M.

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function

$$L : \wedge^2 \mathsf{T} M \to \mathbb{R} \, .$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2 \mathsf{T}^* M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \, .$$

- A submanifold S in the phase space $\wedge^2 \mathsf{T}^* M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha S$ at $\alpha \in \wedge^2 \mathsf{T}^* M$ is represented by a bivector from $\mathcal{D}_\alpha$.
If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function

$$L : \wedge^2 \mathsf{T}M \to \mathbb{R}\,.$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2\mathsf{T}^*M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^2\mathsf{T} \wedge^2 \mathsf{T}^*M\,.$$

- A submanifold S in the phase space $\wedge^2\mathsf{T}^*M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha S$ at $\alpha \in \wedge^2\mathsf{T}^*M$ is represented by a bivector from $\mathcal{D}_\alpha$.

If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function

$$L : \wedge^2 \mathsf{T}M \to \mathbb{R}\,.$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2 \mathsf{T}^* M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M\,.$$

- A submanifold S in the phase space $\wedge^2 \mathsf{T}^* M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha S$ at $\alpha \in \wedge^2 \mathsf{T}^* M$ is represented by a bivector from $\mathcal{D}_\alpha$.
If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function

$$L : \wedge^2 \mathrm{T}M \to \mathbb{R}\,.$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2 \mathrm{T}^* M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M\,.$$

- A submanifold S in the phase space $\wedge^2 \mathrm{T}^* M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathrm{T}_\alpha S$ at $\alpha \in \wedge^2 \mathrm{T}^* M$ is represented by a bivector from $\mathcal{D}_\alpha$.

If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function

$$L : \wedge^2 \mathsf{T} M \to \mathbb{R} .$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2 \mathsf{T}^* M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M .$$

- A submanifold S in the phase space $\wedge^2 \mathsf{T}^* M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha S$ at $\alpha \in \wedge^2 \mathsf{T}^* M$ is represented by a bivector from $\mathcal{D}_\alpha$.

If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

Dynamics of strings

- An evolution of strings is represented by surfaces in M. Passing to infinitesimal parts we will view a Lagrangian L as a function

$$L : \wedge^2 \mathsf{T}M \to \mathbb{R}\,.$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^2\mathsf{T}^*M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for 2-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M\,.$$

- A submanifold S in the phase space $\wedge^2\mathsf{T}^*M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha S$ at $\alpha \in \wedge^2\mathsf{T}^*M$ is represented by a bivector from $\mathcal{D}_\alpha$.
If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

The Hamiltonian side for multivector bundles

Recall that $\wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M$ is a double graded bundle (actually a GrL-bundle)

$$
\begin{array}{ccccc}
 & & \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M & & \\
 & \swarrow & & \searrow & \\
\wedge^2\mathrm{T}^*M & & & & \wedge^2\mathrm{T}M \\
 & \searrow & & \swarrow & \\
 & & M & &
\end{array}
.
$$

We have:

- the canonical Liouville 2-form on $\wedge^2\mathrm{T}^*M$:

$$\theta^2_M = \frac{1}{2} p_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,,\ \ p_{\mu\nu} = -p_{\nu\mu}\,;$$

- the canonical multisymplectic form

$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2}\,\mathrm{d}p_{\mu\nu} \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,;$$

- the vector bundle morphism

$$\beta^2_M\colon\ \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M \to \mathrm{T}^*\wedge^2\mathrm{T}^*M\,, \quad : u \mapsto i_u\omega^2_M\,.$$

The Hamiltonian side for multivector bundles

Recall that $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$ is a double graded bundle (actually a GrL-bundle)

$$\begin{array}{ccccc} & & \wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M & & \\ & \swarrow & & \searrow & \\ \wedge^2\mathsf{T}^*M & & & & \wedge^2\mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}\quad .$$

We have:

- the canonical Liouville 2-form on $\wedge^2\mathsf{T}^*M$:

$$\theta^2_M = \frac{1}{2}p_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,,\;\; p_{\mu\nu}=-p_{\nu\mu}\,;$$

- the canonical multisymplectic form

$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2}\,\mathrm{d}p_{\mu\nu}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,;$$

- the vector bundle morphism

$$\beta^2_M\colon\;\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M\to\mathsf{T}^*\wedge^2\mathsf{T}^*M\,,\qquad : u\mapsto i_u\omega^2_M\,.$$

The Hamiltonian side for multivector bundles

Recall that $\wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M$ is a double graded bundle (actually a GrL-bundle)

$$\begin{array}{ccccc} & & \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M & & \\ & \swarrow & & \searrow & \\ \wedge^2\mathrm{T}^*M & & & & \wedge^2\mathrm{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array} .$$

We have:

- the canonical Liouville 2-form on $\wedge^2\mathrm{T}^*M$:

$$\theta^2_M = \frac{1}{2}p_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,,\ \ p_{\mu\nu}=-p_{\nu\mu}\,;$$

- the canonical multisymplectic form

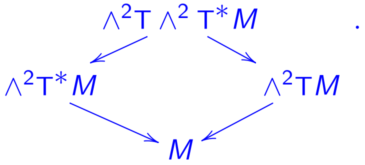
$$\omega^2_M=\mathrm{d}\theta^2_M=\frac{1}{2}\,\mathrm{d}p_{\mu\nu}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,;$$

- the vector bundle morphism

$$\beta^2_M\colon\ \wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M\to\mathrm{T}^*\wedge^2\mathrm{T}^*M\,,\quad : u\mapsto i_u\omega^2_M\,.$$

The Hamiltonian side for multivector bundles

Recall that $\wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is a double graded bundle (actually a GrL-bundle)



We have:

- the canonical Liouville 2-form on $\wedge^2 \mathsf{T}^* M$:

$$\theta^2_M = \frac{1}{2} p_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,,\ \ p_{\mu\nu} = -p_{\nu\mu}\,;$$

- the canonical multisymplectic form

$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2}\, \mathrm{d}p_{\mu\nu} \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,;$$

- the vector bundle morphism

$$\beta^2_M\colon\ \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T}^* M\,, \quad : u \mapsto i_u \omega^2_M\,.$$

The Hamiltonian side for multivector bundles

Recall that $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$ is a double graded bundle (actually a GrL-bundle)

$$\begin{array}{ccccc} & & \wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M & & \\ & \swarrow & & \searrow & \\ \wedge^2\mathsf{T}^*M & & & & \wedge^2\mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array} \quad .$$

We have:

- the canonical Liouville 2-form on $\wedge^2\mathsf{T}^*M$:

$$\theta^2_M = \frac{1}{2}p_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,,\ \ p_{\mu\nu} = -p_{\nu\mu}\,;$$

- the canonical multisymplectic form

$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2}\,\mathrm{d}p_{\mu\nu}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,;$$

- the vector bundle morphism

$$\beta^2_M\colon\ \wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M\to\mathsf{T}^*\wedge^2\mathsf{T}^*M\,,\quad :\ u\mapsto i_u\omega^2_M\,.$$

The Hamiltonian side for multivector bundles

Recall that $\wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M$ is a double graded bundle (actually a GrL-bundle)

$$\begin{array}{ccccc} & & \wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M & & \\ & \swarrow & & \searrow & \\ \wedge^2 \mathrm{T}^* M & & & & \wedge^2 \mathrm{T} M \\ & \searrow & & \swarrow & \\ & & M & & \end{array} .$$

We have:

- the canonical Liouville 2-form on $\wedge^2 \mathrm{T}^* M$:

$$\theta^2_M = \frac{1}{2} p_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \; p_{\mu\nu} = -p_{\nu\mu} \, ;$$

- the canonical multisymplectic form

$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2} \, \mathrm{d}p_{\mu\nu} \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \, ;$$

- the vector bundle morphism

$$\beta^2_M \colon \; \wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M \to \mathrm{T}^* \wedge^2 \mathrm{T}^* M \,, \quad : u \mapsto i_u \omega^2_M \, .$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma}) .$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M ,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M\colon \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M ,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa}) .$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M .$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma}) .$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathrm{T}^* \wedge^2 \mathrm{T}^* M \to \mathrm{T}^* \wedge^2 \mathrm{T}M ,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M \colon \wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M \to \mathrm{T}^* \wedge^2 \mathrm{T}M ,$$

(of double graded bundles over $\wedge^2 \mathrm{T}M$ and $\wedge^2 \mathrm{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa}) .$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathrm{T} \wedge^2 \mathrm{T}M \to \wedge^2 \mathrm{TT}M .$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma})\,.$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M\colon \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa})\,.$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M\,.$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma}) .$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M ,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M : \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M ,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa}) .$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M .$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma}).$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M : \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa}).$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M.$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma})\,.$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M\colon\ \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

(of double graded bundles over $\wedge^2\mathsf{T}M$ and $\wedge^2\mathsf{T}^*M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa})\,.$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M\,.$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma}).$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M \colon \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa}).$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M.$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma})\,.$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M \colon \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa})\,.$$

The map α^2_M can also be obtained as the dual of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M\,.$$

The Tulczyjew triple for strings

Combining the maps β^2_M and α^2_M, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:

The way of obtaining the implicit phase dynamics D, as a submanifold of $\wedge^2\mathrm{T} \wedge^2 \mathrm{T}^* M$, from a Lagrangian $L : \wedge^2\mathrm{T}M \to \mathbb{R}$ or from a Hamiltonian $H : \wedge^2\mathrm{T}^*M \to \mathbb{R}$ is now standard.

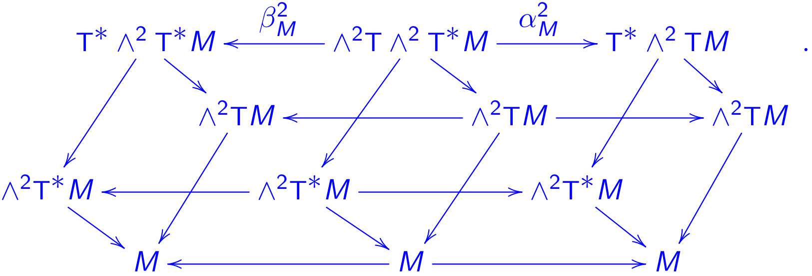
The Tulczyjew triple for strings

Combining the maps β^2_M and α^2_M, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:

The way of obtaining the implicit phase dynamics D, as a submanifold of $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, from a Lagrangian $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ or from a Hamiltonian $H:\wedge^2\mathsf{T}^*M\to\mathbb{R}$ is now standard.

The Tulczyjew triple for strings

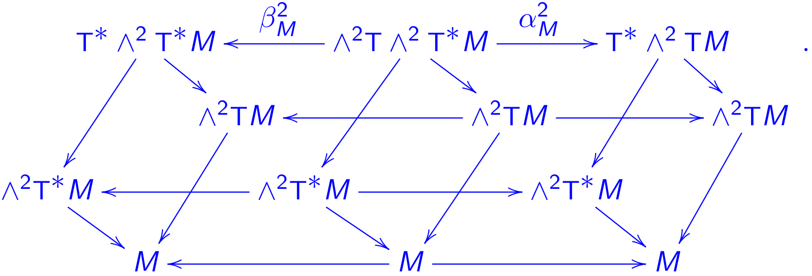
Combining the maps β^2_M and α^2_M, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics D, as a submanifold of $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, from a Lagrangian $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ or from a Hamiltonian $H:\wedge^2\mathsf{T}^*M\to\mathbb{R}$ is now standard.

The Tulczyjew triple for strings

Combining the maps β^2_M and α^2_M, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics D, as a submanifold of $\wedge^2\mathsf{T} \wedge^2 \mathsf{T}^* M$, from a Lagrangian $L : \wedge^2\mathsf{T}M \to \mathbb{R}$ or from a Hamiltonian $H : \wedge^2\mathsf{T}^*M \to \mathbb{R}$ is now standard.

The phase dynamics - Lagrangian side

$\wedge^2\mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2\mathsf{T}M \to \mathbb{R}$ - Lagrangian

$$\mathcal{D} = (\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\} .$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Lagrangian side

$\wedge^2\mathsf{T}M$ - (kinematic) configurations, $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ - Lagrangian

$$\mathcal{D}=(\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D}=\left\{(x^\mu,p_{\lambda\kappa},\dot{x}^{\nu\sigma},y^\eta_{\theta\rho},\dot{p}_{\gamma\delta\epsilon\zeta}):\quad y^\eta_{\eta\rho}=\frac{\partial L}{\partial x^\rho},\quad p_{\lambda\kappa}=\frac{\partial L}{\partial \dot{x}^{\lambda\kappa}}\right\}.$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Lagrangian side

$\wedge^2 \mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2 \mathsf{T}M \to \mathbb{R}$ - Lagrangian

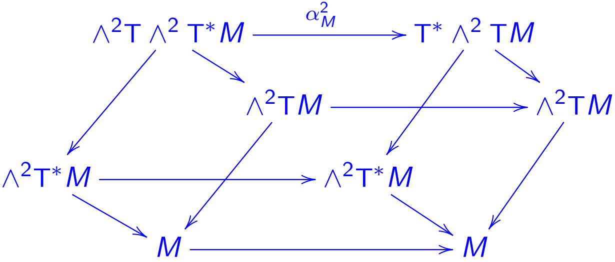
$$\mathcal{D} = (\alpha_M^2)^{-1}(\mathsf{d}L(\wedge^2 \mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\} .$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Lagrangian side

$\wedge^2 \mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2 \mathsf{T}M \to \mathbb{R}$ - Lagrangian



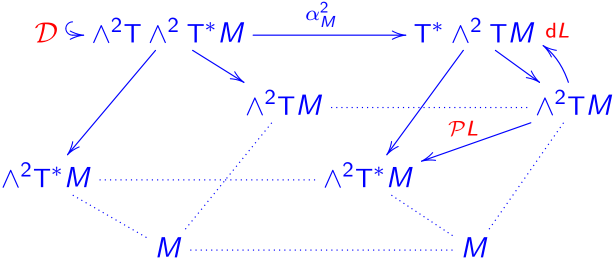
$$\mathcal{D} = (\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\}.$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Lagrangian side

$\wedge^2 \mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2 \mathsf{T}M \to \mathbb{R}$ - Lagrangian



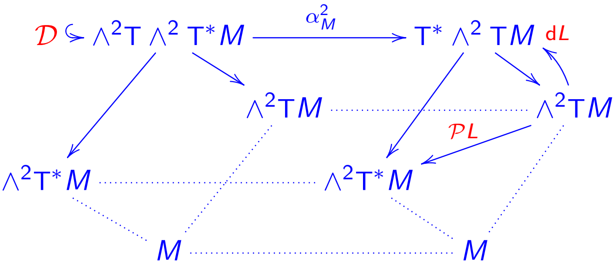
$$\mathcal{D} = (\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\}.$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Lagrangian side

$\wedge^2 \mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2 \mathsf{T}M \to \mathbb{R}$ - Lagrangian



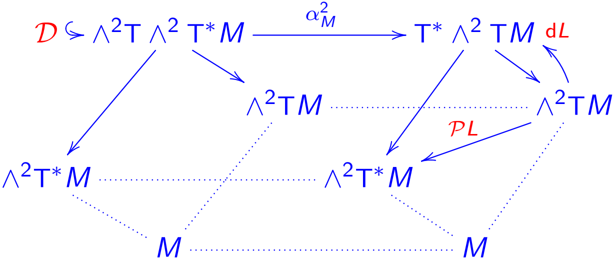
$$\mathcal{D} = (\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\}.$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Lagrangian side

$\wedge^2 \mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2 \mathsf{T}M \to \mathbb{R}$ - Lagrangian



$$\mathcal{D} = (\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\} .$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$

$$\mathcal{D} = (\beta^2_M)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\}.$$

Thus we get Hamilton equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$

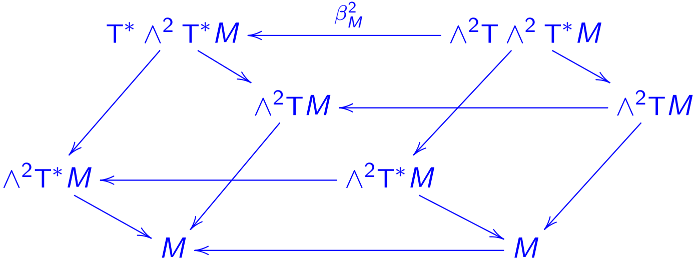
$$\mathcal{D} = (\beta^2_M)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\} .$$

Thus we get Hamilton equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$



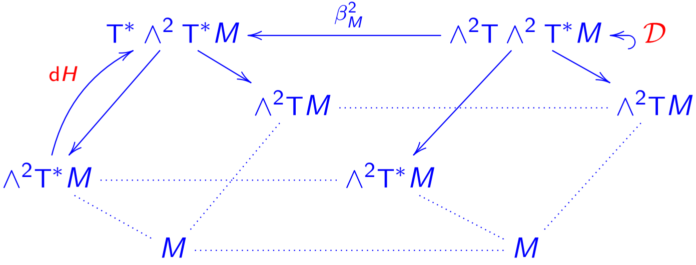
$$\mathcal{D} = (\beta^2_M)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\} .$$

Thus we get Hamilton equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$



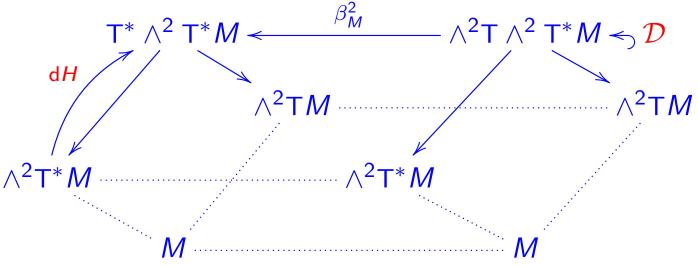
$$\mathcal{D} = (\beta^2_M)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\} .$$

Thus we get Hamilton equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$



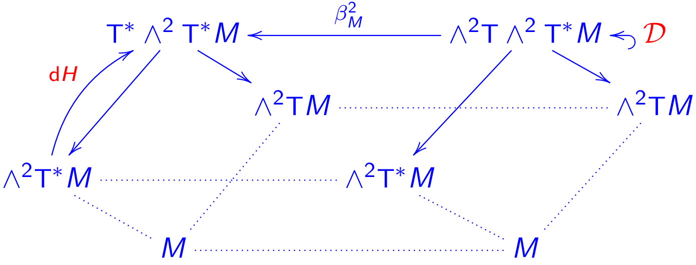
$$\mathcal{D} = (\beta_M^2)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\}.$$

Thus we get Hamilton equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$



$$\mathcal{D} = (\beta_M^2)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\}.$$

Thus we get Hamilton equations.

The Euler-Lagrange and Hamilton equations

For a surface in $\wedge^2 \mathsf{T}M$,

$$(t,s) \mapsto \left(x^\sigma(t,s)\,, \dot{x}^{\mu\nu}(s,t)\right),$$

the Euler-Lagrange equations read

$$\begin{aligned}
\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
\frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).
\end{aligned}$$

As for the Hamilton equations, we have

$$\begin{aligned}
\frac{\partial H}{\partial p_{\mu\nu}} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
-\frac{\partial H}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial p_{\mu\sigma}}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial p_{\mu\sigma}}{\partial t}.
\end{aligned}$$

The Euler-Lagrange and Hamilton equations

For a surface in $\wedge^2 \mathsf{T}M$,

$$(t,s) \mapsto \left(x^\sigma(t,s)\,, \dot{x}^{\mu\nu}(s,t)\right),$$

the Euler-Lagrange equations read

$$\begin{aligned}
\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
\frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).
\end{aligned}$$

As for the Hamilton equations, we have

$$\begin{aligned}
\frac{\partial H}{\partial p_{\mu\nu}} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
-\frac{\partial H}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial p_{\mu\sigma}}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial p_{\mu\sigma}}{\partial t}\,.
\end{aligned}$$

The Euler-Lagrange and Hamilton equations

For a surface in $\wedge^2 \mathsf{T}M$,

$$(t,s) \mapsto (x^\sigma(t,s)\,,\dot{x}^{\mu\nu}(s,t))\,,$$

the Euler-Lagrange equations read

$$\begin{aligned}
\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
\frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right)\,.
\end{aligned}$$

As for the Hamilton equations, we have

$$\begin{aligned}
\frac{\partial H}{\partial p_{\mu\nu}} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\,,\\
-\frac{\partial H}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial p_{\mu\sigma}}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial p_{\mu\sigma}}{\partial t}\,.
\end{aligned}$$

The Euler-Lagrange and Hamilton equations

For a surface in $\wedge^2 \mathsf{T}M$,

$$(t,s) \mapsto (x^\sigma(t,s), \dot{x}^{\mu\nu}(s,t)),$$

the Euler-Lagrange equations read

$$\begin{aligned} \dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}, \\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right). \end{aligned}$$

As for the Hamilton equations, we have

$$\begin{aligned} \frac{\partial H}{\partial p_{\mu\nu}} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}, \\ -\frac{\partial H}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial p_{\mu\sigma}}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial p_{\mu\sigma}}{\partial t}. \end{aligned}$$

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda},$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}},$$

which is defined on the open submanifold of positive bivectors.

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda},$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}},$$

which is defined on the open submanifold of positive bivectors.

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda},$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}},$$

which is defined on the open submanifold of positive bivectors.

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda},$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}},$$

which is defined on the open submanifold of positive bivectors.

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda}\,,$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\,.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}}\,,$$

which is defined on the open submanifold of positive bivectors.

An example

In the relativistic dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda}\,,$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\,.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}}\,,$$

which is defined on the open submanifold of positive bivectors.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathsf{T} M)$ and it is described by the Lagrange (phase) equations

$$
\begin{aligned}
y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^\nu}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma},\\
p_{\mu\nu} &= \frac{1}{\rho} h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa},
\end{aligned}
$$

where

$$
\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.
$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$
\begin{aligned}
H \quad &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}\,,\\
&: (p, r) \mapsto r(\sqrt{(p|p)} - 1)\,.
\end{aligned}
$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathsf{T} M)$ and it is described by the Lagrange (phase) equations

$$
\begin{aligned}
y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma}, \\
p_{\mu\nu} &= \frac{1}{\rho} h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa},
\end{aligned}
$$

where

$$
\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.
$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$
\begin{aligned}
H &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}\,, \\
&: (p, r) \mapsto r\left(\sqrt{(p|p)} - 1\right).
\end{aligned}
$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2\mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2\mathsf{T}M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho}h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$\begin{aligned} H &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}, \\ &: (p, r) \mapsto r(\sqrt{(p|p)} - 1)\,. \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathsf{T} M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho}h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$\begin{aligned} H &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}\,, \\ &: (p, r) \mapsto r(\sqrt{(p|p)} - 1)\,. \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathsf{T} M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho} \frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}} \dot{x}^{\mu\kappa} \dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho} h_{\mu\nu\lambda\kappa} \dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa} \dot{x}^{\mu\nu} \dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$\begin{aligned} H &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}, \\ &: (p, r) \mapsto r(\sqrt{(p|p)} - 1)\,. \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2\mathsf{T}M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho}h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^*\wedge^2\mathsf{T}^*M$, generated by the Morse family

$$\begin{aligned} H &: \wedge^2\mathsf{T}^*M \times \mathbb{R}_+ \to \mathbb{R}, \\ &: (p,r) \mapsto r\left(\sqrt{(p|p)} - 1\right). \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Nambu-Goto dynamics

The dynamics $\mathcal{D} \subset \wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathrm{T} M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^\alpha_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^\nu}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho} h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathrm{T}^* \wedge^2 \mathrm{T}^* M$, generated by the Morse family

$$\begin{aligned} H \quad &: \wedge^2 \mathrm{T}^* M \times \mathbb{R}_+ \to \mathbb{R}, \\ &: (p, r) \mapsto r\left(\sqrt{(p|p)} - 1\right). \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs of maps $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2) z_{yy} - 2 z_x z_y z_{xy} + (1 + z_y^2) z_{xx} = 0\,.$$

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs of maps $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs of maps $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs of maps $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0\,.$$

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs of maps $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:

.

The map

$$\beta^n_M \colon \wedge^n \mathrm{T} \wedge^n \mathrm{T}^* M \to \mathrm{T}^* \wedge^n \mathrm{T}^* M$$

comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathrm{T}^* M$, being the differential of the canonical Liouville n-form

$$\theta^n_M = p_{\mu_1 \mu_2 \dots \mu_n} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \,.$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathrm{T}^* \wedge^n \mathrm{T}^* M$ and $\mathrm{T}^* \wedge^n \mathrm{T} M$.

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:

The map

$$\beta^n_M \colon \wedge^n \mathrm{T} \wedge^n \mathrm{T}^* M \to \mathrm{T}^* \wedge^n \mathrm{T}^* M$$

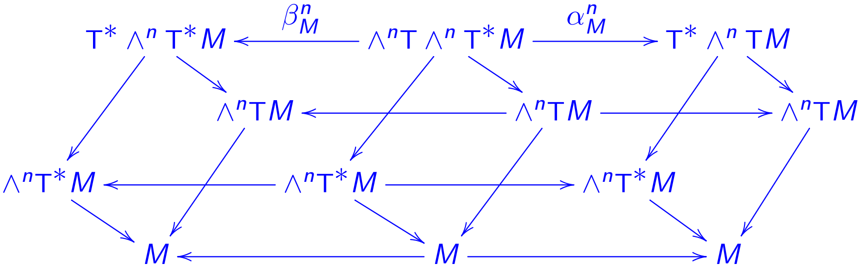
comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathrm{T}^* M$, being the differential of the canonical Liouville n-form

$$\theta^n_M = p_{\mu_1 \mu_2 \dots \mu_n} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \, .$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathrm{T}^* \wedge^n \mathrm{T}^* M$ and $\mathrm{T}^* \wedge^n \mathrm{T} M$.

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:



The map

$$\beta^n_M \colon \wedge^n \mathrm{T} \wedge^n \mathrm{T}^* M \to \mathrm{T}^* \wedge^n \mathrm{T}^* M$$

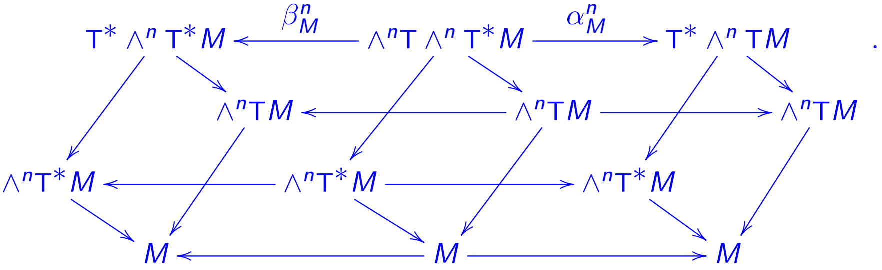
comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathrm{T}^* M$, being the differential of the canonical Liouville n-form

$$\theta^n_M = p_{\mu_1 \mu_2 \dots \mu_n} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \, .$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathrm{T}^* \wedge^n \mathrm{T}^* M$ and $\mathrm{T}^* \wedge^n \mathrm{T} M$.

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:



The map

$$\beta^n_M \colon \wedge^n \mathrm{T} \wedge^n \mathrm{T}^* M \to \mathrm{T}^* \wedge^n \mathrm{T}^* M$$

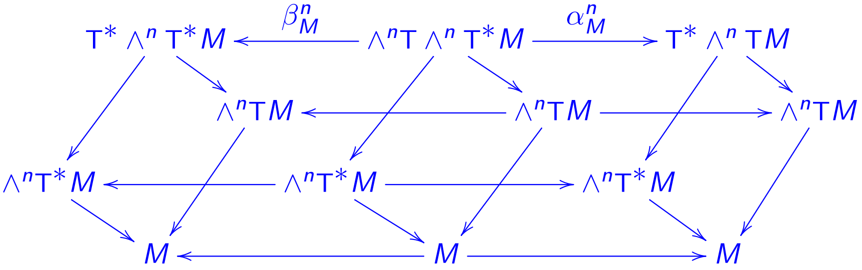
comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathrm{T}^* M$, being the differential of the canonical Liouville n-form

$$\theta^n_M = p_{\mu_1 \mu_2 \dots \mu_n} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \, .$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathrm{T}^* \wedge^n \mathrm{T}^* M$ and $\mathrm{T}^* \wedge^n \mathrm{T} M$.

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:



The map

$$\beta^n_M \colon \wedge^n \mathrm{T} \wedge^n \mathrm{T}^* M \to \mathrm{T}^* \wedge^n \mathrm{T}^* M$$

comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathrm{T}^* M$, being the differential of the canonical Liouville n-form

$$\theta^n_M = p_{\mu_1 \mu_2 \dots \mu_n} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n \,.$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathrm{T}^* \wedge^n \mathrm{T}^* M$ and $\mathrm{T}^* \wedge^n \mathrm{T} M$.

Some References

- **A.J. Bruce, K. Grabowska & J. Grabowski,** Geometrical mechanics on algebroids, *Int. J. Geom. Methods Mod. Phys.* **3** (2006), no. 3, 559–575.
- **J. Grabowski, M. de León, J. C. Marrero & D. Martín de Diego,** Nonholonomic constraints: a new viewpoint, *J. Math. Phys.* **50** (2009), no. 1, 013520, 17 pp.
- **K. Grabowska & J. Grabowski,** Variational calculus with constraints on general algebroids, *J. Phys. A* **41** (2008), no. 17, 175204, 25 pp.
- **K. Grabowska, J. Grabowski & P. Urbański,** Geometry of Lagrangian and Hamiltonian formalisms in the dynamics of strings, *J. Geom. Mech.* **6** (2014), 503–526.
- **J. Grabowski& M. Rotkiewicz,** Higher vector bundles and multi-graded symplectic manifolds, *J. Geom. Phys.* **59** (2009), 1285–1305.
- **J. Grabowski & M. Rotkiewicz,** Graded bundles and homogeneity structures, *J. Geom. Phys.* **62** (2012), 21–36.

Homework 3

- Problem 1. Find the phase dynamics $\mathcal{D} \subset \mathsf{TT}^*\mathbb{R}^3$ corresponding to the Lagrangian on $\mathsf{T}\mathbb{R}^3$:
$$L(x, \dot{x}) = \frac{1}{2}\sum_{i=1}^{3} x_i^2 + V(x)\,.$$
- Problem 2. Find the Legendre transformation $\lambda_{:}\mathsf{T}\mathbb{R}^3 \to \mathsf{T}^*\mathbb{R}^3$ for the above Lagrangian.
- Problem 3. Find the phase dynamics $\mathcal{D} \subset \mathsf{TT}^*\mathbb{R}$ corresponding to the Lagrangian on $\mathsf{T}\mathbb{R}$:
$$L(x, \dot{x}) = \dot{x} + V(x)\,.$$
- Problem 4. Find the Legendre map $\lambda_L : \mathsf{T}\mathbb{R} \to \mathsf{T}^*\mathbb{R}$ for the Lagrangian from Problem 3. Is λ_L a diffeomorphism?
- Problem 5. Let
$$\Pi = \xi\,\partial_\xi \otimes \partial\eta + 2\eta\,\partial_\xi \otimes \partial_\xi$$
be a linear tensor on $(\mathbb{R}^2)^* = \{(\xi, \eta)\}$. Find the bracket on $\mathbb{R}^2$ induced by Π, the phase dynamics $\mathcal{D} \subset \mathsf{T}(\mathbb{R}^2)^*$, corresponding to the Lagrangian $L = \frac{1}{2}(\xi^2 + \eta^2)$, and the Euler-Lagrange equations.

THANK YOU FOR YOUR ATTENTION!