DOUBLE STRUCTURES AND ALGEBROIDS

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May 30, 2021

Plan of the talk

- Double vector bundles
- n-fold graded bundles
- Canonical examples
- Canonical isomorphism
- Graded-linear bundles
- Lie algebroids
- General algebroids
- Non-holonomic reduction
- Groupoids
- Lie groupoids and their Lie algebroids
- Examples: Pair and Ehresmann groupoids
- Home work

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Double vector bundles

In geometry and applications one often encounters double vector bundles, i.e. manifolds equipped with two vector bundle structures which are compatible in a categorical sense. They were defined by Pradines and studied by Mackenzie, Konieczna (Grabowska), and Urbański as vector bundles in the category of vector bundles. More precisely:

Definition

A double vector bundle $(D; A, B; M)$ is a system of four vector bundle structures

$$\begin{array}{ccc} D & \xrightarrow{q^D_B} & B \\ {\scriptstyle q^D_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array}$$

in which D has two vector bundles structures, on bases A and B. The latter are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (namely the bundle projection, zero section, addition and scalar multiplication) is a morphism of vector bundles with respect to the other structures.

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The structure of double vector bundles

- In the above figure, we refer to A and B as the side bundles of D, and to M as the double base.
- In the two side bundles, the addition and scalar multiplication are denoted by the usual symbols $+$ and juxtaposition, respectively.
- We distinguish the two zero-sections, writing $0^A : M \to A$, $m \mapsto 0^A_m$, and $0^B : M \to B$, $m \mapsto 0^B_m$.
- In the vertical bundle structure on D with base A, the vector bundle operations are denoted by $+_A$ and $\cdot_A$, with $\tilde{0}^A : A \to D$, $a \mapsto \tilde{0}^A_a$, for the zero-section.
- Similarly, in the horizontal bundle structure on D with base B we write $+_B$ and $\cdot_B$, with $\tilde{0}^B : B \to D$, $b \mapsto \tilde{0}^B_b$, for the zero-section.
- The two structures on D, namely (D, q^D_B, B) and (D, q^D_A, A) will also be denoted, respectively, by $\tilde{D}_B$ and $\tilde{D}_A$, and called the horizontal bundle structure and the vertical bundle structure.

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Double vector bundles - compatibility conditions

The condition that each vector bundle operation in D is a morphism with respect to the other is equivalent to the following conditions, known as the interchange laws:

$$\begin{aligned}
(d_1 +_B d_2) +_A (d_3 +_B d_4) &= (d_1 +_A d_3) +_B (d_2 +_A d_4), \\
t \cdot_A (d_1 +_B d_2) &= t \cdot_A d_1 +_B t \cdot_A d_2, \\
t \cdot_B (d_1 +_A d_2) &= t \cdot_B d_1 +_A t \cdot_B d_2, \\
t \cdot_A (s \cdot_B d) &= s \cdot_B (t \cdot_A d), \\
\tilde{0}^A_{a_1+a_2} &= \tilde{0}^A_{a_1} +_B \tilde{0}^A_{a_2}, \\
\tilde{0}^A_{ta} &= t \cdot_B \tilde{0}^A_{a}, \\
\tilde{0}^B_{b_1+b_2} &= \tilde{0}^B_{b_1} +_A \tilde{0}^A_{b_2}, \\
\tilde{0}^B_{tb} &= t \cdot_A \tilde{0}^B_{b}.
\end{aligned}$$

The core

We denote by C the intersection of the two kernels:

$$C = \{c \in D \mid \exists\ m \in M \text{ such that } q^D_B(c) = 0^B_m, \quad q^D_A(c) = 0^A_m\},$$

which is called the core, and together with the map $q_C : c \mapsto m$, (C, q_C, M) is also a vector bundle over M.

Eventually we can write the diagram below to emphasis the core of the relevant double vector bundle.

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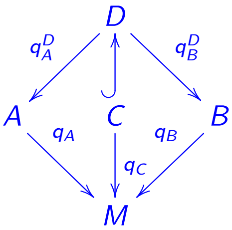

Double vector bundles - reference example

- Let $q_A : A \to M$, $q_B : B \to M$, $q_C : C \to M$ be vector bundles.
- Consider the manifold
$$D = A \times_M B \times_M C\,.$$
- D is a double vector bundle (with side bundles A and B, and the core C) with respect to the obvious projections
$$q^D_A : D \ni (a_m, b_m, c_m) \mapsto a_m \in A\,, \quad q^D_B : D \ni (a_m, b_m, c_m) \mapsto b_m \in B\,,$$
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and obvious vector space structures in fibers:
$$(a_m, b_m, c_m) +_A (a_m, b'_m, c'_m) = (a_m, b_m + b'_m, c_m + c'_m)\,, \text{ etc.}$$
- Actually, every double vector bundle is locally of this form.
- In particular, any Whitney direct sum $A \oplus_M B$, identified with $\simeq A \times_M B$, can be given a double vector bundle structure.

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Double Graded Bundles

- We can extend the concept of a double vector bundle of Pradines to double graded bundles.
- However, thanks to our simple description in terms of a homogeneity structure, the 'diagrammatic' definition of Pradines can be substantially simplified.
- As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following:

Definition (Grabowski-Rotkiewicz)

A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

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n-fold Graded Bundles

- The above condition can also be formulated as commutation of the corresponding weight vector fields, $[\nabla^1, \nabla^2] = 0$.
- For vector bundles this is equivalent to the concept of a double vector bundle in the sense of Pradines and Mackenzie.

Theorem (Grabowski-Rotkiewicz)

The concept of a double vector bundle, understood as a particular double graded bundle in the above sense, coincides with that of Pradines.

- All this can be extended to n-fold graded bundles in the obvious way:

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A n-fold graded bundle is a manifold equipped with n homogeneity structures $h^1, \ldots, h^n$ which are compatible in the sense that

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The concept of a double vector bundle, understood as a particular double graded bundle in the above sense, coincides with that of Pradines.

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A *n*-fold graded bundle is a manifold equipped with n homogeneity structures $h^1, \ldots, h^n$ which are compatible in the sense that

$$h^i_t \circ h^j_s = h^j_s \circ h^i_t \quad \text{for all } s, t \in \mathbb{R} \quad \text{and} \quad i, j = 1, \ldots, n\,.$$

n-fold Graded Bundles

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Proposition

The tangent and phase lifts of graded bundles are compatible with the vector bundle structures of the tangent (resp., cotangent) bundle.

$$\tau : E \longrightarrow M \qquad \tau_M : \mathsf{T}M \longrightarrow M$$

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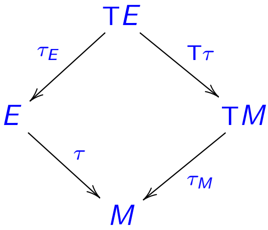

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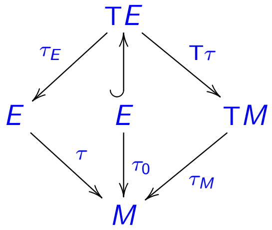

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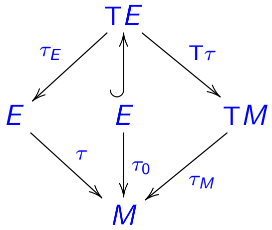

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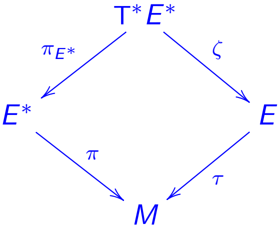

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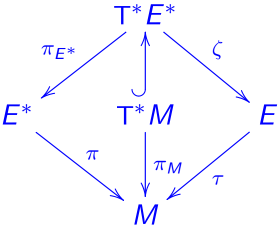

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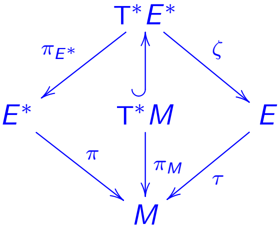

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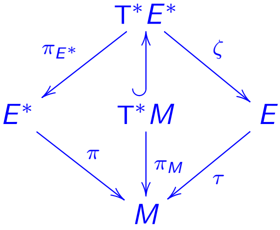

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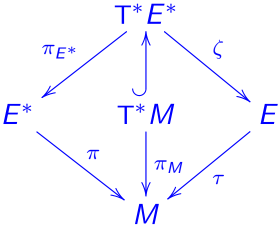

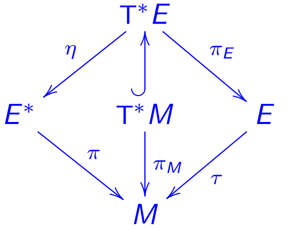

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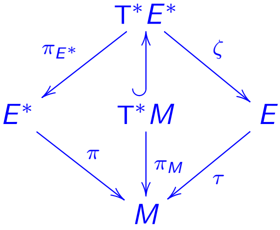

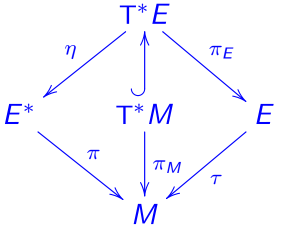

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Graded linear bundles

- A double graded bundle whose one structure is linear we will call a graded linear bundle (GrL-bundle). Canonical examples are $\mathsf{T}F$ and T^*F with the lifted and the vector bundle structures.
  Iterated lifts, $\mathsf{T}\mathsf{T}^*F \simeq \mathsf{T}^*\mathsf{T}F$ lead to triple structures of this kind.
- Example. The weight vector field of the lifted graded structure on $\mathsf{T}\mathsf{T}^2M$ with coordinates $(x^a, \dot{x}^b, \ddot{x}^c, \delta x^d, \delta\dot{x}^e, \delta\ddot{x}^f)$ is

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Linearity of different geometrical structures is usually related to some double vector bundle structures.

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- $\tau : E \to M$ is a rank-n vector bundle over an m-dimensional manifold M, and $\pi : E^* \to M$ its dual;
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We use affine coordinates (x^a, ξ_i) on E^* and the dual coordinates (x^a, y^i) on E.

Definition

A Lie algebroid structure on E is given by a linear Poisson tensor Π on E^*, $[\Pi, \Pi]_{Schouten} = 0$. In local coordinates,

$$\Pi = \frac{1}{2} c_{ij}^k(x)\xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x)\partial_{\xi_i} \wedge \partial_{x^b}\,,$$

where $c_{ij}^k(x) = -c_{ji}^k(x)$.

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A Lie algebroid structure on E is given by a linear Poisson tensor Π on E^*, $[\Pi, \Pi]_{Schouten} = 0$. In local coordinates,

$$\Pi = \frac{1}{2} c_{ij}^k(x) \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x) \partial_{\xi_i} \wedge \partial_{x^b} ,$$

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- $\tau : E \to M$ is a rank-n vector bundle over an m-dimensional manifold M, and $\pi : E^* \to M$ its dual;
- $\mathcal{A}^i(E) = \mathrm{Sec}(\wedge^i E)$, for $i = 0, 1, 2, \ldots$, the module of sections of the bundle $\wedge^i E$.
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The bivector field Π defines a Poisson bracket $\{\cdot,\cdot\}_\Pi$ on the algebra $C^\infty(E^*)$ of smooth functions on E^* by $\{\phi,\psi\}_\Pi = \langle \Pi, \mathrm{d}\phi\wedge\mathrm{d}\psi\rangle$.

Theorem

A Lie algebroid structure (E,Π) can be equivalently defined as

- *a Lie bracket $[\cdot,\cdot]_\Pi$ on the space $\mathrm{Sec}(E)$, together with a vector bundle morphisms $\rho\colon E\to TM$ (the anchor), such that*

$$[X,fY]_\Pi = \rho(X)(f)Y + f[X,Y]_\Pi, \tag{1}$$

  for all $f\in C^\infty(M)$, $X,Y\in\mathrm{Sec}(E)$,

- *or as a homological derivation d^Π of degree 1 in the Grassmann algebra $\mathcal{A}(E^*)$ (de Rham derivative). The latter is a map $\mathrm{d}^\Pi:\mathcal{A}(E^*)\to\mathcal{A}(E^*)$ such that $\mathrm{d}^\Pi:\mathcal{A}^i(E^*)\to\mathcal{A}^{i+1}(E^*)$, $(\mathrm{d}^\Pi)^2=0$, and that, for $\alpha\in\mathcal{A}^a(E^*)$, $\beta\in\mathcal{A}^b(E^*)$ we have*

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These objects are related to Π according to the formulae

$$\begin{aligned} \iota([X,Y]_\Pi) &= \{\iota(X),\iota(Y)\}_\Pi,\\ \pi^*(\rho(X)(f)) &= \{\iota(X),\pi^*f\}_\Pi\,,\\ (\mathrm{d}^\Pi\mu)^v &= [\Pi,\mu^v]_S\,. \end{aligned}$$

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- A Lie algebroid over a single point, with the zero anchor, is a Lie algebra.
- The tangent bundle, $\mathsf{T}M$, of a manifold M, with bracket the Lie bracket of vector fields and with anchor the identity of $\mathsf{T}M$, is a Lie algebroid over M. Any integrable sub-bundle of $\mathsf{T}M$, in particular the tangent bundle along the leaves of a foliation, is also a Lie algebroid.
- If (M, Λ) is a Poisson manifold, then the cotangent bundle T^*M is a Lie algebroid over M. The anchor is the map $\Lambda^{\#} : \mathsf{T}^*M \to \mathsf{T}M$ The Lie bracket $[,]_\Lambda$ of differential 1-forms satisfies $[\mathrm{d}f, \mathrm{d}g]_\Lambda = \mathrm{d}\{f, g\}_\Lambda$.
- If P is a principal bundle with structure group G, base M and projection p, the G-invariant vector fields on P are the sections of a vector bundle with base M, denoted $E = \mathsf{T}P/G$, and called the Atiyah algebroid of the principal bundle P. This vector bundle is a Lie algebroid, with bracket induced by the Lie bracket of G-invariant vector fields on P, and with surjective anchor induced by $\mathsf{T}p : \mathsf{T}P \to \mathsf{T}M$.

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General algebroids

- We know that the linear bivector field Π on E^* induces a morphism of double vector bundles $\Pi^\# : \mathsf{T}^*E^* \to \mathsf{T}E^*$, covering the identity on E^*. Composing it with the canonical isomorphism $\mathcal{R} : \mathsf{T}^*E \to \mathsf{T}^*E^*$, we get a morphism of double vector bundles $\varepsilon_\Pi : \mathsf{T}^*E \to \mathsf{T}E^*$ covering the identity on E^*.
- A general algebroid is a double vector bundle morphism covering the identity on E^*:

In local coordinates,

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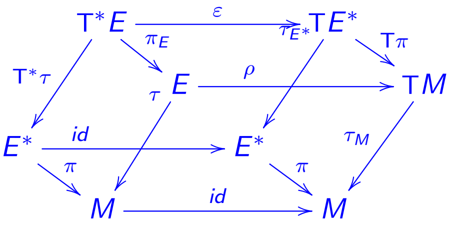

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Algebroids

Any such morphism is associated with a linear tensor field on E^*,

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We speak about a skew algebroid (resp., Lie algebroid) if the tensor Π_ε is skew-symmetric (resp., Poisson tensor).

Theorem

An algebroid structure (E,ε) can be equivalently defined as a bilinear bracket $[\cdot,\cdot]_\varepsilon$ on sections of $\tau\colon E\to M$, together with vector bundle morphisms $a_l^\varepsilon, a_r^\varepsilon\colon E\to \mathsf{T}M$ (left and right anchors), such that

$$[fX,gY]_\varepsilon = f(a_l^\varepsilon\circ X)(g)Y - g(a_r^\varepsilon\circ Y)(f)X + fg[X,Y]_\varepsilon$$

for $f,g\in C^\infty(M)$, $X,Y\in \mathrm{Sec}(E)$.
For skew-algebroids the bracket is skew-symmetric, thus $a_l^\varepsilon = a_r^\varepsilon = \rho^\varepsilon$, and for Lie algebroids it satisfies the Jacobi identity,

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Non-holonomic reduction

Let ε be a Lie algebroid structure on a vector bundle E over M associated with the tensor Π_ε. We assume additionally that the vector bundle is Riemannian, i.e. E is equipped with a tensor which defines scalar products on fibers of E. For a linear subbundle D in E, supported on the whole M, consider a decomposition

$$E = D \oplus_M D^\perp \tag{3}$$

and the associated projection $p : E \to D$. With such a decomposition we can associate a skew-algebroid structure on D. The projection P induces a map on sections: $p : \mathrm{Sec}(E) \to \mathrm{Sec}(D)$ and thus a bracket

$$[X, Y]_{\varepsilon_p} = p[X, Y]_\varepsilon \tag{4}$$

on sections of D – the nonholonomic restriction of $[\cdot, \cdot]$ along p. This is a skew algebroid bracket with the original anchor.

A particular case of this construction can be applied to a vector subbundle D of $\mathsf{T}M$, for M equipped with a Riemannian structure, e.g. nonholonomic systems with mechanical Lagrangians.

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Groupoids

Definition

A groupoid over a set Γ_0 is a set Γ equipped with source and target mappings $\alpha, \beta : \Gamma \to \Gamma_0$, a multiplication map m from $\Gamma_2 \stackrel{\text{def}}{=} \{(g,h) \in \Gamma \times \Gamma \mid \beta(g) = \alpha(h)\}$ to Γ, an injective map $\epsilon : \Gamma_0 \to \Gamma$, and an involution $\iota : \Gamma \to \Gamma$, satisfying the following properties (where we write gh for $m(g,h)$ and g^{-1} for $\iota(g)$):

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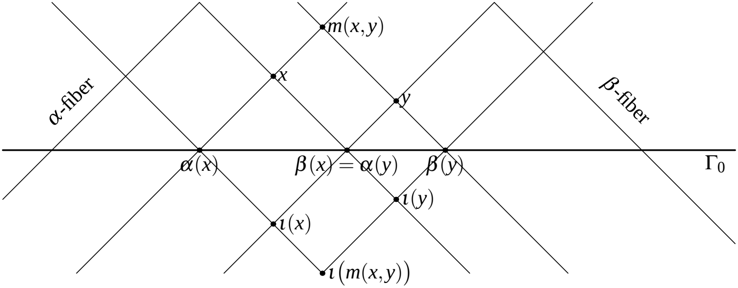

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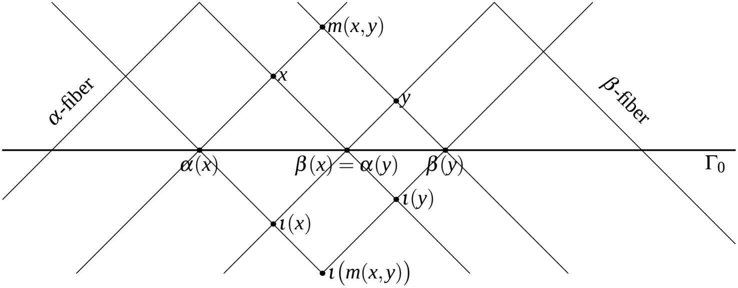

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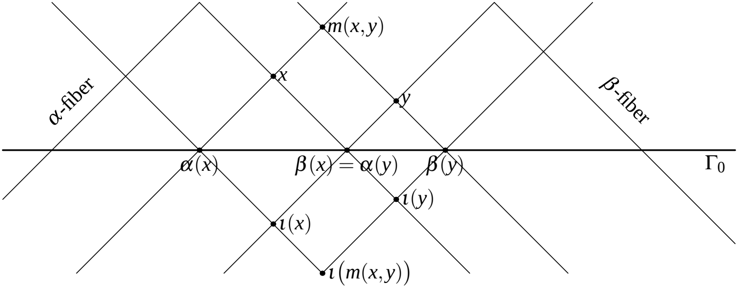

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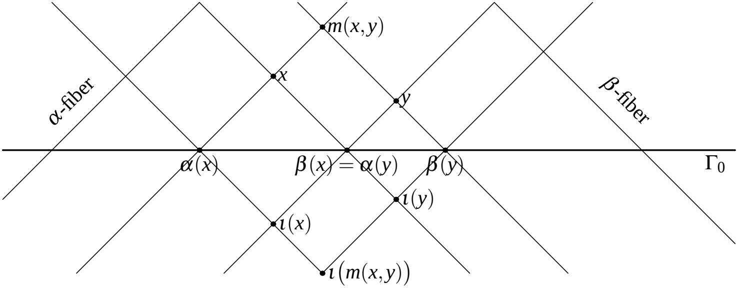

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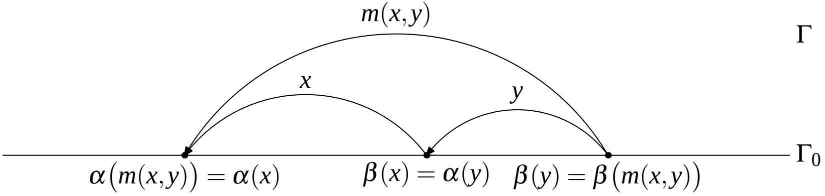
Groupoid as a small category

- The full information about the groupoid is contained in the multiplication relation:

$$\Gamma_3 = \{(x,y,z) \in \Gamma \times \Gamma \times \Gamma \mid (x,y) \in \Gamma_2 \text{ and } z = xy\} .$$

- Alternatively, a groupoid $\Gamma \rightrightarrows \Gamma_0$ is defined as a small category, i.e. a category whose objects form a set Γ_0, in which every morphism is an isomorphism. Elements of Γ represent morphisms in this category.



- Any group G is a groupoid over its neutral element, $G \rightrightarrows \{e\}$. Here, any morphism is an automorphism.

Groupoid as a small category

- The full information about the groupoid is contained in the multiplication relation:

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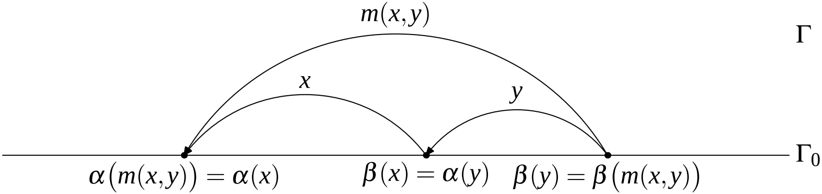

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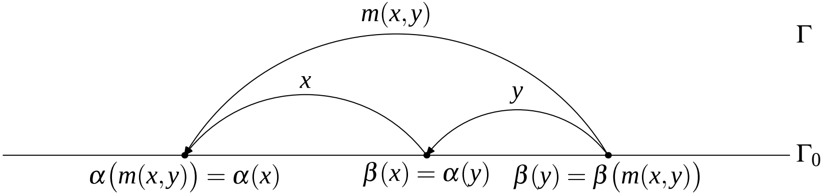

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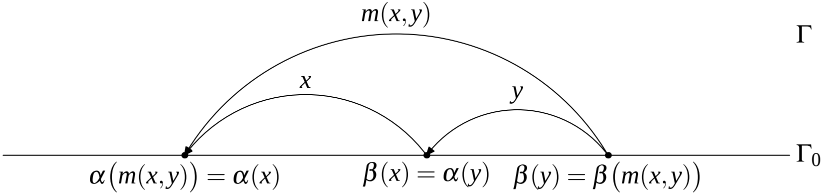

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Lie groupoids

- In differential geometry we consider differentiable (Lie) groupoids (introduced by Ehresmann), i.e. groupoids $G \rightrightarrows M$, where G, G_2, G_3, M are smooth manifolds, α, β are smooth submersions, ϵ is an immersion and ι is a diffeomorphism.
- The anchor property implies that each element g of G determines the left and right translation maps

$$l_g : \mathcal{F}^\alpha(\beta(g)) \to \mathcal{F}^\alpha(\alpha(g)), \quad r_g : \mathcal{F}^\beta(\alpha(g)) \to \mathcal{F}^\beta(\beta(g)),$$

- Let us consider the vector bundle $\tau : A(G) \to M$, whose fiber at a point $x \in M$ is $A_x G = V_{\epsilon(x)}\alpha = Ker(T_{\epsilon(x)}\alpha)$.
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Lie algebroid of a Lie groupoid

- We can now introduce a Lie algebroid structure $([\cdot,\cdot],\rho)$ on $A(G)$, which is defined by

$$\overleftarrow{[X,Y]} = [\overleftarrow{X},\overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)), \tag{5}$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$.

- We recall that a Lie algebroid A over a manifold M is a real vector bundle $\tau : A \to M$ together with a skew-symmetric bracket $[\cdot,\cdot]$ on the space $\Gamma(\tau)$ of sections of $\tau : A \to M$ and a bundle map, called the anchor map, such that

$$[X, fY] = f[X,Y] + \rho(X)(f)Y,$$

for $X, Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$. Here we denoted by ρ also the map induced by ρ on sections.

Theorem

For any groupoid $G \rightrightarrows M$, the formulae (5) define on $\tau : A(G) \to M$ the structure of a Lie algebroid.

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Pair groupoid

Example

- Let M be a set and $\Gamma = M \times M$. define the source and target maps as
$$\alpha(u,v) = u\,, \quad \beta(u,v) = v\,.$$
- Then, $M \times M$ is a groupoid over M with the units mapping $\epsilon(u) = (u,u)$, and the partial composition by $(u,v)(v,z) = (u,z)$. In other words,
$$\Gamma_3 = \{(u,v,v,z,u,z) \in \Gamma \times \Gamma \times \Gamma \mid u,v,z \in M\}\,.$$
- Note that M can be viewed as embedded into $M \times M$ as the diagonal.
- If M is a manifold, we deal with a Lie groupoid. We can identify α-fibers as
$$\mathcal{F}^{\alpha}(u,u) = \{(u,v) \mid v \in M\} \simeq M\,,$$
so $A(\Gamma)$ with $\mathsf{T}M$. The left invariant vector field $\overleftarrow{X}$ tangent to α-fibers in Γ and corresponding to $X \in \mathcal{X}(M)$ is, under this identification, $\overleftarrow{X}(u,v) \simeq X(v)$. In consequence, the Lie algebroid of Γ is $\mathsf{T}M$ with the bracket of vector fields.

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Ehresmann gauge groupoid

Example

- For $p : P \to M$ being a principal bundle with the structure group G, consider the set $\Gamma = (P \times P)/G$ of G-orbits, where G acts on $P \times P$ diagonally, $(v, u)g = (vg, ug)$.
- For the coset $\langle v|u\rangle$ of (v, u), define the source and target maps
$$\alpha\langle v|u\rangle = p(u) \quad \beta\langle v|u\rangle = p(v)\,,$$
and the (partial) multiplication $\langle w|v\rangle\langle v|u\rangle = \langle w|u\rangle$.
- It is well defined, as
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In this way we obtained a Lie groupoid $\Gamma = (P \times P)/G \rightrightarrows M = M$, the Ehresmann gauge groupoid of P.
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Homework 2

- Problem 1. Prove that the tangent and cotangent bundles of a double graded bundle are canonically triple graded bundles.
- Problem 2. On the space of curves $\gamma : \mathbb{R} \to M$ in a manifold M, consider the $(\mathbb{R}, \cdot)$-action $\hat{h}_t(\gamma)(s) = \gamma(ts)$.
  Prove that this action induces the canonical homogeneity structure on the space T^2M of second jets of curves in M.
- Problem 3. Show that the second tangent lift of a homogeneity structure h on F, defined by $(\mathsf{T}^2 h)_t = \mathsf{T}^2(h_t)$, is a homogeneity structure on $\mathsf{T}^2 F$. Here $\mathsf{T}^2\phi : \mathsf{T}^2 M \to \mathsf{T}^2 N$ denotes the obvious second-jet prolongation of $\phi : M \to N$ to the second tangent bundles.
- Problem 4. Prove that the lifted homogeneity structure $\mathsf{T}^2 h$ from the previous problem is compatible with the canonical homogeneity structure on the second tangent bundle $\mathsf{T}^2 F$.
- Problem 5. Show that the anchor map induces, for any Lie algebroid E, a homomorphism of the Lie algebroid bracket into the Lie bracket of vector fields:

$$\rho([X, Y]_\varepsilon) = [\rho(X), \rho(Y)]_{vf} \,.$$

THANK YOU FOR YOUR ATTENTION!