GRADED BUNDLES

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May 30, 2021

Plan of the talk

- Multiplication by reals is enough
- Smooth actions of $(\mathbb{R}, \cdot)$ (homogeneity structures)
- Graded spaces (not graded vector spaces)
- Vector bundles and graded bundles (not graded vector bundles)
- Graded bundle=homogeneity structure
- Transition functions and the tower
- Splitting graded bundles
- Lifts of graded structures
- Some references
- Home work

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What is a vector space?

- A (real) vector space is a set E with a distinguished element 0^E, equipped with two operations:
  1. an addition
  $$+ : E \times E \to E\,, \quad (u,v) \mapsto u+v\,,$$
  2. and a multiplication by scalars
  $$h : \mathbb{R} \times E \to E\,, \quad h(t,v) = h_t(v) = t \cdot v = tv\,,$$

  satisfying a list of axioms.
- For instance, $(E,+)$ is a commutative group with 0^E being the neutral element, the homotheties h_t satisfy
  $$h_t \circ h_s = h_{ts}\,,$$
  and $h_0(v) = 0^E$ for all $v \in E$.

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One operation is enough

- To distinguish finite-dimensional real vector spaces among differentiable manifolds, a single operation of the above two is enough.
- If we know the addition, we get the multiplication by natural numbers in the obvious way:

$$nv = v + \cdots + v\,,$$

and we easily extend it to integers by $(-n)v = n(-v)$. The multiplication by rational numbers, $(m/n)v$ we obtain as the solution of the equation $nx = mv$.
Assuming differentiability (in fact, continuity) of h, we extend this multiplication to all reals uniquely.
- If we know the multiplication by reals h instead, we use a version of Euler's Homogeneous Function Theorem: any differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 1, i.e.

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Homogeneous Function Theorem

- Indeed, $t \cdot f(x) = f(t \cdot x)$ and differentiability gives
$$f(x) = \frac{\partial f}{\partial x^i}(t \cdot x) \cdot x^i \,.$$
Putting $t = 0$ we obtain further
$$f(x) = \frac{\partial f}{\partial x^i}(0) \cdot x^i$$
which means that f is linear. The other implication is obvious.
- Thus, from the multiplication by reals on E we get the dual space E^*, where the addition is well defined, and consequently the addition on $E = (E^*)^*$.
- All this can be reformulated for a vector bundle $\tau : E \to M$: the multiplication by reals h in E (homotheties) uniquely determines E with the projection $\tau = h_0$.

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Homogeneity structures

- We can consider now a general (smooth) action $h : \mathbb{R} \times F \to F$ of the multiplicative monoid $(\mathbb{R}, \cdot)$ on a manifold F, $h_t \circ h_s = h_{ts}$. Such an action we will call a homogeneity structure.
  A smooth function $f : F \to \mathbb{R}$ will be called homogeneous of degree w if
  $$f(h_t(x)) = t^w f(x) \quad \text{for} \quad t \geq 0\,.$$
- It is a nontrivial observation (we will come to it later) that homogeneity degrees can only be non-negative integers and that we can choose local coordinates which are homogeneous (and have non-negative integers as degrees).
- Note that it is crucial that h_t is defined for $t = 0$, since, for instance, the action $h : \mathbb{R}^\times \times \mathbb{R}_+ \to \mathbb{R}_+$, with $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, of the multiplicative group $\mathbb{R}^\times$ on $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$ given by $h_t(x) = |t|x$ admits 'homogeneous' functions of of arbitrary degree w, namely $f(x) = x^w$. Here $(tx)^w = t^w x^w$ for $t > 0$. However this is not homogeneity in the sense we consider, as the projection h_0 is not defined.

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Graded spaces

Assume now that for a homogeneity structure h on a manifold F there is a (necessary unique) point $0^F \in F$ such that $h_0(F) = \{0^F\}$. Such a structure we will call a graded space (they are not graded vector spaces) by the following reasons.

Theorem (Grabowski-Rotkiewicz)

Any graded space (F, h) is diffeomorphically equivalent (isomorphic) to a certain $(\mathbb{R}^{\mathbf{d}}, h^{\mathbf{d}})$, where $\mathbf{d} = (d_1, \dots, d_k)$, with positive integers d_i, and $\mathbb{R}^{\mathbf{d}} = \mathbb{R}^{d_1}[1] \times \cdots \times \mathbb{R}^{d_k}[k]$ is equipped with the action $h^{\mathbf{d}}$ of multiplicative reals given by

$$h^{\mathbf{d}}_t(y_1, \dots, y_k) = (t \cdot y_1, \dots, t^k \cdot y_k)\,, \quad y_i \in \mathbb{R}^{d_i}\,.$$

In other words, F can be equipped with a system of (global) coordinates (y_i^j), $i = 1 \dots, k$, $j = 1, \dots, d_i$, such that linear coordinates y_i^j in $\mathbb{R}^{d_i}[i]$ are homogeneous of degree i with respect to the homogeneity structure h, i.e.

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Of course, in these coordinates $0^F = (0, \dots, 0)$.

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How to recognize vector spaces?

- Note that the isomorphism in the above theorem is generally non-canonical. The number k, however, is uniquely determined and called the minimal degree of the graded space. By convention, a degree of h is any natural $k' \geq k$.
- How to recognize a vector space among graded spaces?
- Answer: Vector spaces are graded spaces of degree 1.
- Regularity condition: For any $y \in F$,

$$\frac{\mathrm{d}}{\mathrm{d}t_{\,|\,t=0}}(h_t(y)) = 0 \Leftrightarrow y = 0^F\,.$$

Theorem

The homogeneity structure in a graded space comes from a vector space structure if and only if it is regular. In this case, the vector space structure is uniquely determined by the homogeneity structure.

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Weight vector field

- It is natural to call a morphism between homogeneity structures (F_a, h^a), $a = 1, 2$, a smooth map $\Phi : F_1 \to F_2$ which intertwines the homogeneity structures: $\Phi \circ h^1_t = h^2_t \circ \Phi$.
- The $(\mathbb{R}, \cdot)$-action restricted to positive reals gives a one-parameter group of diffeomorphism of F, thus is generated by a vector field ∇_F. It is called the weight vector field as it completely determines the homogeneity structure. For a graded space with homogeneous global coordinates (y^j_w)
$$\nabla_F = \sum_w w\, y^j_w \partial_{y^j_w}\,.$$
- A function f is homogeneous of degree w if and only if $\nabla_F(f) = w \cdot f$, and a smooth map $\Phi : F_1 \to F_2$ is a morphism of homogeneity structures iff it relates the corresponding weight vector fields.
- Note that morphisms need not to be linear, so the category of graded spaces is different from that of vector spaces. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is an automorphism of the structure, but it is nonlinear.

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Weight vector field

- It is natural to call a morphism between homogeneity structures (F_a, h^a), $a = 1, 2$, a smooth map $\Phi : F_1 \to F_2$ which intertwines the homogeneity structures: $\Phi \circ h^1_t = h^2_t \circ \Phi$.
- The $(\mathbb{R}, \cdot)$-action restricted to positive reals gives a one-parameter group of diffeomorphism of F, thus is generated by a vector field ∇_F. It is called the weight vector field as it completely determines the homogeneity structure. For a graded space with homogeneous global coordinates (y^j_w)
$$\nabla_F = \sum_w w\, y^j_w \partial_{y^j_w}\,.$$
- A function f is homogeneous of degree w if and only if $\nabla_F(f) = w \cdot f$, and a smooth map $\Phi : F_1 \to F_2$ is a morphism of homogeneity structures iff it relates the corresponding weight vector fields.
- Note that morphisms need not to be linear, so the category of graded spaces is different from that of vector spaces. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is an automorphism of the structure, but it is nonlinear.

Vector bundles classically

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n \,,$$

$A(x) \in \mathrm{GL}(n, \mathbb{R})$.

- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0, and 'linear coordinates' y have degree 1. Linearity of changes of coordinates is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

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Graded bundles

- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ of rank $\mathbf{d}$, with a local trivialization by $U \times \mathbb{R}^{\mathbf{d}}$, and with the difference that the transition functions of local trivializations:
$$U \cap V \times \mathbb{R}^{\mathbf{d}} \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^{\mathbf{d}},$$
respect the weights of coordinates $(y^1, \dots, y^{|\mathbf{d}|})$ in the fibres. In other words, a graded bundle of rank $\mathbf{d}$ is a locally trivial fibration with fibers modelled on the graded space $\mathbb{R}^{\mathbf{d}}$.

Theorem

$A(x, y)$ must be polynomial in homogeneous fiber coordinates y's, i.e. any graded bundle is a polynomial bundle.

- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.
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Graded bundles - examples

- Note that, according to our convention, any differential manifold M can be viewed as a graded bundle of degree 0.
- A trivial example is of course
$$F = M \times \mathbb{R}^d = M \times (\mathbb{R}^{d_1}[1] \oplus \cdots \oplus \mathbb{R}^{d_k}[k]) .$$
- Another trivial example, is a split graded bundle, i.e. a graded vector bundle
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Homogeneity structure of a graded bundle

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- In local homogeneous coordinates,

$$h_t(x^A, y^a_w) = (x^A, t^w y^a_w) \,.$$

- We call a function $f : F \to \mathbb{R}$ homogeneous of degree (weight) w if

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- The whole information about the degree of homogeneity is contained in the weight vector field (called for vector bundles the Euler vector field)

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Mimicking the definition of a vector bundle morphism, we get the following.

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One can equivalently say that the fiber bundle morphism Φ is a smooth map which relates the weight vector fields ∇_{F^1} and ∇_{F^2}.

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The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

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Since morphisms of two homogeneity structures are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$, this describes also morphism of graded bundles.

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The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

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Associating the homogeneity structure with a graded bundle is an isomorphism of categories. In particular, for any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$ and a non-negative integer $k \in \mathbb{N}$ such that $h_0 : F \to M$ is canonically a graded bundle of degree k whose homogeneity structure coincides with h. In other words, there is an atlas on F consisting of local homogeneous functions.

Since morphisms of two homogeneity structures are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$, this describes also morphism of graded bundles.

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Consequences for vector bundles

Vector bundles can be recognized as graded bundles $\tau : F \to M$ of degree 1, i.e. satisfying the following regularity condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|\,t=0} h_t(p) = 0 \Leftrightarrow p \in M\,.$$

The principle multiplication by reals is enough has now the following consequences for vector bundles.

Corollary

A smooth map $\Phi : E_1 \to E_2$ between the total spaces of two vector bundles $\pi_i : E_i \to M_i$, $i = 1, 2$ is a morphism of vector bundles if and only if it intertwines the multiplications by reals:

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Graded bundles - further examples

- Example. Consider the second-order tangent bundle $\mathrm{T}^2 M$, i.e. the bundle of second jets of smooth maps $(\mathbb{R}, 0) \to M$.
  Writing paths in local coordinates (x^A) on M:
  $$x^A(t) = x^A(0) + \dot{x}^A(0)t + \ddot{x}^A(0)\frac{t^2}{2} + o(t^2)\,,$$
  we get local coordinates $(x^A, \dot{x}^B, \ddot{x}^C)$ on $\mathrm{T}^2 M$, which transform
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- This shows that associating with $(x^A, \dot{x}^B, \ddot{x}^C)$ the weights $0, 1, 2$, respectively, will give us a graded bundle structure of degree 2 on $\mathrm{T}^2 M$. Due to the quadratic terms above, this is not a vector bundle!

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Transition functions for graded bundles

- Let us go back to graded bundles. For a graded bundle F one can pick an atlas of F consisting of charts for which we have homogeneous local coordinates (x^A, y^a_w) with weight $\deg$, where $\deg(x^A) = 0$ and $\deg(y^a_w) = w$ with $1 \leq w \leq k$, where k is the degree of the graded bundle. Here, a should be considered as a 'generalised index' running over all the possible weights. The label w in this respect is somewhat redundant, but it will come in very useful.
- The local changes of coordinates are of the form

$$\begin{aligned} x'^A &= x'^A(x), \\ y'^a_w &= y^b_w T_b{}^a(x) + \sum_{\substack{1<n \\ w_1+\cdots+w_n=w}} \frac{1}{n!} y^{b_1}_{w_1} \cdots y^{b_n}_{w_n} T_{b_n \cdots b_1}{}^a(x), \end{aligned} \tag{1}$$

where $T_b{}^a$ are invertible and $T_{b_n \cdots b_1}{}^a$ are symmetric in indices b.
- In particular, the transition functions of coordinates of degree r involve only coordinates of degree $\leq r$, defining a reduced graded bundle F_r of degree r (we simply 'forget' coordinates of degrees $> r$).

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The tower of affine fibrations

- Transformations for the canonical projection $F_r \to F_{r-1}$ are linear modulo a shift by a polynomial in variables of degrees $< r$,

$$y'^a_r = y^b_r T_b{}^a(x) + \sum_{\substack{1<n \\ w_1+\cdots+w_n=r}} \frac{1}{n!} y^{b_1}_{w_1} \cdots y^{b_n}_{w_n} T_{b_n \cdots b_1}^{\ a}(x)\,,$$

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- In this way we get for any graded bundle F of degree k, like for jet bundles, a tower of affine fibrations

$$F = F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Example. In the case of the canonical graded bundle $F = \mathrm{T}^k M$, we get exactly the tower of projections of jet bundles

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Further constructions

- The reduced manifold F_r will also be denoted $F[\nabla \leq r]$ if we want to stress which weight vector field ∇ we have in mind (sometimes we will work with many).
- There is also a "dual" sequence of submanifolds and their inclusions

$$M := F_0 = F^{[k]} \hookrightarrow F^{[k-1]} \hookrightarrow \cdots \hookrightarrow F^{[0]} = F_k, \qquad (2)$$

where we define, locally but correctly,

$$F^{[i]} := \{ p \in F_k \,|\, y^a_w = 0 \ \text{ if } w \leq i \} .$$

- In words, "you project higher to lower, but set to 0 lower to higher".
- Note that the $C^\infty(M)$-module $\mathcal{A}^r(F)$ of homogeneous functions of degree r on F is finitely generated and projective, so it corresponds to sections of a vector bundle $A^r(F)$ over M. The graded algebra

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generated by homogeneous functions is called the polynomial algebra of F.

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- Note that the $C^\infty(M)$-module $\mathcal{A}^r(F)$ of homogeneous functions of degree r on F is finitely generated and projective, so it corresponds to sections of a vector bundle $A^r(F)$ over M. The graded algebra

$$\mathcal{A}(F) = \bigoplus_{i=0}^{\infty} \mathcal{A}^i(F)$$

generated by homogeneous functions is called the polynomial algebra of F.

Further constructions

- The reduced manifold F_r will also be denoted $F[\nabla \leq r]$ if we want to stress which weight vector field ∇ we have in mind (sometimes we will work with many).
- There is also a "dual" sequence of submanifolds and their inclusions

$$M := F_0 = F^{[k]} \hookrightarrow F^{[k-1]} \hookrightarrow \cdots \hookrightarrow F^{[0]} = F_k, \qquad (2)$$

where we define, locally but correctly,

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Splitting of graded bundles

Definition

A split graded bundle F of degree k over M is a graded bundle being a direct sum of vector bundles E_i over M, $i = 1, \ldots, k$:

$$F = E_1 \oplus \cdots \oplus E_k$$

such that the linear fiber coordinates in E_i are of degree i. In other words, split graded bundles are graded vector bundles.

Theorem

Any graded bundle F of degree k is isomorphic with the split graded bundle $\bar{F} = \bar{F}^1 \oplus \cdots \oplus \bar{F}^k$.

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Splitting of graded bundles – comments

- The situation is similar to the celebrated Batchelor Theorem in supergeometry stating that any supermanifold is (non-canonically) diffeomorphic with the 'superization' ΠE of a vector bundle E. Here, ΠE is a supermanifold with the same local affine coordinates (x, y) and transition functions as in E but the fiber linear coordinates y are regarded as odd functions:

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Of course, morphisms of such supermanifolds are different from that of vector bundles, so these categories are completely different.

- The Betchelor Theorem was actually proved first by Polish physicist Gawędzki, that provides therefore another example of the Arnold's law saying that *"Discoveries are rarely attributed to the correct person"*.
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Tangent lifts of graded structures

- Consider an arbitrary graded bundle F_k over M of minimal degree k with homogeneous coordinates (x^A, y^a_w), $1 \leq w \leq k$. The corresponding homogeneity structure is then

$$h_t(x^A, y^a_w) = (x^A,\, t^w y^a_w)$$

and the weight vector field: $\nabla_F := \sum_w w y^a_w \frac{\partial}{\partial y^a_w}$.

- Applying the tangent functor to all h_t, we get a homogeneity structure $(\mathrm{d_T} h)_t = \mathrm{T}(h_t)$ on $\mathrm{T}F$:

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$$\nabla_{\mathrm{T}F} = \mathrm{d_T}\nabla_F = \sum_w w y^a_w \frac{\partial}{\partial y^a_w} + \sum_w w \dot{y}^a_w \frac{\partial}{\partial \dot{y}^a_w}\,.$$

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Higher lifts and canonical isomorphisms

- Using higher tangent functors T^k, we can lift homogeneity structures on F to homogeneity structures on $\mathsf{T}^k F$ simply putting

$$(\mathrm{d}_{\mathsf{T}^k} h)_t = \mathsf{T}^k(h_t) : \mathsf{T}^k F \to \mathsf{T}^k F\,.$$

- We have fundamental isomorphisms between iterated higher tangent and cotangent bundles.

Theorem (Cantrijn-Crampin-Sarlet-Saunders-Tulczyjew)

For any manifold M and any $k \in \mathbb{N}$, there is a canonical isomorphism

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- The corresponding graded bundle structure $\mathsf{T}^k\mathsf{T}^*M \to \mathsf{T}^*M$ and the vector bundle structure $\mathsf{T}^*\mathsf{T}^k M \to \mathsf{T}^k M$ are compatible in a natural sense, so that $\mathsf{T}^*\mathsf{T}^k M \simeq \mathsf{T}^k\mathsf{T}^* M$ is a canonical example of a double graded bundle, which will be discussed in the next talk.

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Homework 1

- Problem 1. Prove that any real vector space structure on $\mathbb{R}^n$ with the same multiplication by reals coincides with the standard one.
- Problem 2. Prove directly that any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, which satisfies $f(t \cdot x) = t^k \cdot f(x)$ for some $k \in \mathbb{N}$ and all $t \in \mathbb{R}$, is a polynomial.
- Problem 3. Show that a submanifold E_0 of a vector bundle E over M is a vector subbundle (possibly covering a submanifold $M_0 \subset M$) if and only if it E_0 is invariant with respect to all homotheties, i.e. $h_t(E_0) \subset E_0$ for all $t \in \mathbb{R}$.
- Problem 4. Find a split graded bundle isomorphic to the graded bundle $\mathsf{T}^2 M$.
- Problem 5. Let $\tau : E \to M$ be a vector bundle. What is the base of the vector bundle structure on $\mathsf{T}^* E$ being the 1-phase lift of the vector bundle (graded bundle of degree 1) structure on E?

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THANK YOU FOR YOUR ATTENTION!