School on Geometry and Physics

28 June — 2 July 2021

1. **Janusz GRABOWSKI** — *Instytut Matematyczny Polskiej Akademii Nauk, Poland*

   Graded bundles in geometry and physics

   Talk 1: **Graded bundles**

   We start with showing that the multiplication by reals completely determines a smooth real vector bundle. Then, we consider a general smooth actions on the monoid of multiplicative reals on smooth manifolds. In this way homogeneity structures are defined. The vector bundles are homogeneity structures which are regular in a certain sense. It can be shown that homogeneity structures are manifolds whose local coordinates have associated degrees taking values in non-negative integers — graded bundles are born. A canonical example are the higher tangent bundles. We show also how to lift canonically homogeneity structures (graded bundle structures) to tangent and cotangent fibrations.

   Talk 2: **Double structures and algebroids**

   We define double graded bundles (in general n-tuple graded bundles) in terms of homogeneous structures. Classical examples are double vector bundles obtained from lifts, especially to TE and T^*E for a vector bundle E. We show the canonical isomorphism of double vector bundles T^*E^* and T^*E. We define general algebroids (in particular, Lie algebroids) in terms of double vector bundle morphisms. We introduce also the concept of a Lie groupoid.

   Talk 3: **Tulczyjew triples and geometric mechanics on algebroids**

   Starting with the classical Tulczyjew triple involving TT^*M, T^*TM and T^*T^*M, we define the triple associated with a general algebroid involving TE^*, T^*E and T^*E^*. Using now Lagrangian and Hamiltonian functions we explain how to construct dynamics and Euler-Lagrange equations out of them. We end up with dynamics of strings as mechanics on double graded bundles and the Plateau problem.

2. **Andrzej KRASIŃSKI** — *Centrum Astronomiczne im. Mikołaja Kopernika Polskiej Akademii Nauk, Poland*

   Relativistic cosmology from F to Sz (F = Friedmann, Sz = Szekeres)

   1: **A quick (and superficial) introduction to general relativity**

   It will give the participants the idea of the field equations of general relativity (the Einstein equations). They are the basis from which all cosmological models were derived.

   2: **The Friedmann–Lemaître (FL) models and their basic implications**

   This lecture will introduce the evolution equations of the FL models, their relation to observational cosmology, the definition and properties of redshift, the description of horizons, the "history of the Universe" based on these models, and redshift drift as apossible test of the hypothesis of accelerated expansion of the Universe

3: **The Lemaître–Tolman (LT) models and their relation to some of thecosmological observations**

These models are the simplest existing generalisation of the FL models. In spite of being spherically symmetric, they allow for several interesting insights, within the exact theory, into the consequences of existence of matter condensations and rarefactions (voids) observed in the Universe. This lecture will show how the description of formation of such structures follows from the evolution equations of the LT models. It will also show how inhomogeneities in matter distribution can mimic the accelerated expansion of the Universe by perturbing the propagation of light rays. Finally, it will be shown that nonradial rays propagating through a mass inhomogeneity undergo a direction drift, which may become a measure of inhomogeneity in large-scale matter distribution when the observations become sufficiently precise. The light rays reaching the present observer that were emitted radially soon after the initial singularity (the Big Bang) get blueshifted rather than redshifted, i.e. their observed frequency is higher than at the emission point.

4: **The Szekeres (Sz) models and their observational implications**

These models are fully nonsymmetric generalisations of the LT models. They arise by making the spheres, invariantly defined in the LT geometry, nonconcentric. This allows for describing a still larger collection of phenomena observed in the actual Universe. One of them is mimicking the observed gamma-ray bursts by blueshifting the photons of the relic radiation (unlike in LT models, the blueshift in the Sz models occurs only along two opposite preferred directions).

3. **Andrew MCKEE** — *Uniwersytet w Białymstoku, Poland*

Crossed product operator algebras and approximation properties

In these talks I plan to give an overview of some aspects of the study of crossed product operator algebras, formed from the action of a group on an operator algebra. I will begin by introducing group actions and the corresponding operator algebras built from these actions, in particular the group operator algebras which we obtain when the action is trivial.

In the second part of the talk I will discuss the properties of operator algebras which we are interested in, namely approximation properties such as nuclearity. I will also explain how these properties have been studied for operator algebras arising from groups. In the final part I will present some results on approximation properties of crossed product operator algebras; I aim to show how the techniques we use in this work generalise existing techniques used to study operator algebras arising from groups.

4. **Anatolij PRYKARPATSKI** — *Politechnika Krakowska, Poland*

What is the Integrability of Nonlinear Dynamical Systems: analytical and Lie-algebraic aspects

In recent times it has been stated that many dynamical systems of classical mathematical physics and mechanics are endowed with symplectic structures, given in the majority of cases by the related Poisson brackets. Very often such Poisson structures on corresponding manifolds are canonical, which gives rise to the possibility of producing their hidden group theoretical essence for many completely integrable dynamical systems. It is a well understood fact that great part of comprehensive integrability theories of nonlinear dynamical systems on manifolds is

based on Lie-algebraic ideas, by means of which, in particular, the classification of such very popular nowadays compatibly bi-Hamiltonian and isospectrally Lax type integrable systems has been carried out. Some of Lectures are devoted to their description, but to our regret so far the work has not been completed. Hereby our main goal in each analyzed case consists in separating the basic algebraic essence responsible for the complete integrability, and which is, at the same time, in some sense universal, i.e., characteristic for all of them.

Integrability analysis in the framework of an effective enough gradient-holonomic algorithm, devised during the past century, is fulfilled through three stages: 1) finding a symplectic structure (Poisson bracket) transforming an original dynamical system into a Hamiltonian form; 2) finding first integrals (action variables or conservation laws); 3) defining an additional set of variables and some functional operator quantities with completely controlled evolutions (for instance, as Lax type representation). Making use of the small parameter method developed in the book and an asymptotic approach to finding explicit forms of symplectic structures and conservation laws, we have succeeded in directly proving the complete Lax type integrability of many nonlinear dynamical systems on functional manifolds important for applications.

[**Abstract continued on separate sheet**]

5. **Piotr SOŁTAN** — *Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, Poland*

Unbounded operators on Hilbert spaces

In the series of lectures I will introduce the topic of unbounded operators. The lectures will begin with a brief summary of the theory of bounded operators. Next we will discuss the elementary theory of unbounded operators and introduce a versatile tool called the "z-transform" of the "bounded transform" of an operator. Using this tool we will extend such notions as the functional calculus and polar decomposition to unbounded operators. We will conclude with the topic of self-adjoint extensions of symmetric operators.

Plan:

1: Basic operator theory (bounded operators on Hilbert spaces, the spectrum, the adjoint operator, functional calculus, positive operators, polar decomposition)

2: Unbounded operators (domains, graphs, the spectrum, the adjoint operator)

3: The z-transform of an operator, functional calculus, polar decomposition

4: Self-adjoint extension of symmetric operators

6. **Piotr STACHURA** — *Szkoła Główna Gospodarstwa Wiejskiego, Poland*

A short stroll around Groupoid Land

I'm going to discuss the following topics:

- Zakrzewski's morphisms of groupoids;
- differential groupoids, their Lie algebroids and Poisson structure on the dual bundles;
- symplectic groupoids and cotangent lift of differential groupoids;
- groupoid algebra as quantization of Poisson structure.

7. **Henryk ŻOŁĄDEK** — *Uniwersytet Warszawski, Poland*

Qualitative and analytic analysis of some differential equations

Talk 1: **Completely and partially integrable mechanical systems and their perturbations**

We consider the Lagrange and Hess–Appelrot (H–A) cases of the Euler–Poisson (E–P) system $\dot{M} = M \times \Omega - \Gamma \times K$, $\dot{\Gamma} = \Gamma \times \Omega$, which describes the dynamics of a rigid body about a fixed point. The Lagrange case is completely integrable (with a family of invariant 2–tori), while the H–A case is partially integrable in the sense that the system has an invariant torus (so-called Hess surface).

We construct explicitly the invariant tori and describe dynamics on them.

Next, we consider perturbations of these cases within the E–P class. In the Lagrange case, the condition for periodic orbits leads to Melnikov type functions. In the H–A case one investigates the normal hyperbolicity property, which implies perturbation of the Hess surface; next, limit cycles on it are studied.

Qualitative results are obtained for the situation with a critical circle (when one of the radii of the invariant torus tends to 0) and the situation when the Hess surface degenerates to a separatrix connection of a saddle. In the latter case the perturbation leads to a chaotic dynamics.

Talk 2: **Normal forms of vector fields and their analytic properties**

A singular point $x = y = 0$ of a planar vector field $V(x, y)$ is elementary if at least one of the eigenvalues of its linearization is nonzero. The normal forms for such germs, with respect to the action of the group of formal changes of the coordinates and of changes of the time, were obtained long ago. But the non-elementary cases have turned out extremely hard to solve. Only recently an effective tool was created to deal with this problem.

It relies upon a splitting the perturbation W in $V = V_0 + W$ into a part transversal to V_0 and a part tangent to V_0. This leads to a splitting of the homological operator ad_{V_0}, which acts on vector fields, into two homological operators acting on functions.

In this way complete formal normal forms for the cases with nilpotent linear part (the Bogdanov–Takens singularity) and with zero linear part were obtained.

Next, one studies the analyticity of the obtained normal forms. Here we obtain bounds for the 1–dimensional homological operators (in the analytic case) and we use a criterion of Ilyashenko (in the non-analytic case).

Talk 3: **Invariants of group actions, dimension/degree duality and normal forms of vector fields** We develop a constructive approach to the problem of polynomial first integrals for linear vector fields. As an application we obtain a new proof of the theorem of Wietzenböck about finiteness of the number of generators of the ring of constants of a linear derivation in the polynomial ring.

In the case of nilpotent linear vector field X with one Jordan cell we deal with an irreducible representation $\mathrm{Sym}^n \mathbb{V}$, $\mathbb{V} \simeq \mathbb{C}^2$, of the Lie algebra $\mathrm{sl}(2)$, where X is a highest weight vector. The homogeneous polynomial first integrals of degree d of X correspond to highest weight vectors in the representation $\mathrm{Sym}^d(\mathrm{Sym}^n \mathbb{V})$. We obtain a generating function for the multiplicities of the splitting of the latter representation into irreducible ones.

Moreover, we show that the ring of invariants of the latter representation is a polynomial ring, i.e., without relations between generators.

The dim/deg duality is an isomorphism $\mathrm{Sym}^d(\mathrm{Sym}^n\mathbb{V}) \simeq \mathrm{Sym}^n\left(\mathrm{Sym}^d\mathbb{V}\right)$. We construct the duality map explicitly.

Moreover, we propose an alternative approach to the analyticity property of the normal form reduction of a germ of vector field with nilpotent linear part in a case considered by Stolovich and Verstringe.

Contributed lectures

8. **Zohreh RAVANPAK** — *Instytut Matematyczny Polskiej Akademii Nauk, Poland*

Discrete mechanics on octonions

I will briefly talk about the discrete Lagrangian mechanics on Lie groups which has been developed in many papers. Then I will show how the discrete mechanics can be constructed on non-associative objects, smooth loops. This shows that the associativity assumption is not crucial for mechanics and opens new perspectives. At the end, I will explain the process of the formulation of the discrete Lagrangian on unitary octonions.

9. **Elwira WAWRENIUK** — *Uniwersytet w Białymstoku, Poland*

Integrable Hamiltonian systems on the symplectic realizations of $\mathbf{e}(3)^*$

The phase space of a gyrostat with a fixed point and a heavy top is the Lie-Poisson space $\mathbf{e}(3)^* \cong \mathbb{R}^3 \times \mathbb{R}^3$ dual to the Lie algebra $\mathbf{e}(3)$ of Euclidean group $E(3)$. One has three naturally distinguished Poisson submanifolds of $\mathbf{e}(3)^*$: (i) the dense open submanifold $\mathbb{R}^3 \times \dot{\mathbb{R}}^3 \subset \mathbf{e}(3)^*$ which consists of all 4-dimensional symplectic leaves ($\vec{\Gamma}^2 > 0$); (ii) the 5-dimensional Poisson submanifold of $\mathbb{R}^3 \times \dot{\mathbb{R}}^3$ defined by $\vec{J} \cdot \vec{\Gamma} = \mu||\vec{\Gamma}||$; (iii) the 5-dimensional Poisson submanifold of $\mathbb{R}^3 \times \dot{\mathbb{R}}^3$ defined by $\vec{\Gamma}^2 = \nu^2$, where $\dot{\mathbb{R}}^3 := \mathbb{R}^3 \backslash \{0\}$, $(\vec{J}, \vec{\Gamma}) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbf{e}(3)^*$ and $\nu < 0$, μ are some fixed real parameters. Basing on the $U(2,2)$-invariant symplectic structure of Penrose twistor space we find full and complete $E(3)$-equivariant symplectic realizations of these Poisson submanifolds which are 8-dimensional for (i) and 6-dimensional for (ii) and (iii). As a consequence of the above Hamiltonian systems on $\mathbf{e}(3)^*$ lift to the ones on the above symplectic realizations. In such a way after lifting integrable cases of gyrostat with a fixed point, as well as of heavy top, we obtain a large family of integrable Hamiltonian systems on the phase spaces defined by these symplectic realizations.

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Monday, June 28

LECTURES 09:00–13:10

09:00–09:50 *Relativistic cosmology from F to Sz (F = Friedmann, Sz = Szekeres)*
Andrzej KRASIŃSKI, Centrum Astronomiczne im. Mikołaja Kopernika Polskiej Akademii Nauk, Poland

10:00–10:50 *Qualitative and analytic analysis of some differential equations*
Henryk ŻOŁĄDEK, Uniwersytet Warszawski, Poland

10:50–11:20 Coffee break

11:20–12:10 *Graded bundles in geometry and physics*
Janusz GRABOWSKI, Instytut Matematyczny Polskiej Akademii Nauk, Poland

12:20–13:10 *A short stroll around Groupoid Land*
Piotr STACHURA, Szkoła Główna Gospodarstwa Wiejskiego, Poland

Tuesday, June 29

LECTURES 09:00–13:10

09:00–09:50 *Qualitative and analytic analysis of some differential equations*
Henryk ŻOŁĄDEK, Uniwersytet Warszawski, Poland

10:00–10:50 *Crossed product operator algebras and approximation properties*
Andrew MCKEE, Uniwersytet w Białymstoku, Poland

10:50–11:20 Coffee break

11:20–12:10 *Unbounded operators on Hilbert spaces*
Piotr SOŁTAN, Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, Poland

12:20–13:10 *Relativistic cosmology from F to Sz (F = Friedmann, Sz = Szekeres)*
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Wednesday, June 30

LECTURES 10:00–13:10

10:00–10:50 *Graded bundles in geometry and physics*
Janusz GRABOWSKI, Instytut Matematyczny Polskiej Akademii Nauk, Poland

10:50–11:20 Coffee break

11:20–12:10 *A short stroll around Groupoid Land*
Piotr STACHURA, Szkoła Główna Gospodarstwa Wiejskiego, Poland

12:20–13:10 *What is the Integrability of Nonlinear Dynamical Systems: analytical and Lie-algebraic aspects*
Anatolij PRYKARPATSKI, Politechnika Krakowska, Poland

AFTERNOON LECTURE 15:00–17:40

15:00–15:50 *A short stroll around Groupoid Land*
Piotr STACHURA, Szkoła Główna Gospodarstwa Wiejskiego, Poland

15:50–16:05 Coffee break

16:05–16:55 *Unbounded operators on Hilbert spaces*
Piotr SOŁTAN, Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, Poland

17:00–17:40 *Discrete mechanics on octonions*
Zohreh RAVANPAK, Instytut Matematyczny Polskiej Akademii Nauk, Poland

Thursday, July 1

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09:00–09:50 *Relativistic cosmology from F to Sz (F = Friedmann, Sz = Szekeres)*
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AFTERNOON LECTURE 15:00–17:40

15:00–15:50 *Unbounded operators on Hilbert spaces*
Piotr SOŁTAN, Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, Poland

15:50–16:05 Coffee break

16:05–16:55 *Crossed product operator algebras and approximation properties*
Andrew MCKEE, Uniwersytet w Białymstoku, Poland

17:00–17:40 *Integrable Hamiltonian systems on the symplectic realizations of* $e(3)^*$
Elwira WAWRENIUK, Uniwersytet w Białymstoku, Poland

Friday, July 2

LECTURES 09:00–13:10

09:00–09:50 *What is the Integrability of Nonlinear Dynamical Systems: analytical and Lie-algebraic aspects*
Anatolij PRYKARPATSKI, Politechnika Krakowska, Poland

10:00–10:50 *Unbounded operators on Hilbert spaces*
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10:50–11:20 Coffee break

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12:20–13:10 *Crossed product operator algebras and approximation properties*
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