Ogólny tytuł: Qualitative and analytic analysis of some differential equations

Title 1: Completely and partially integrable mechanical systems and their perturbations

Abstract 1: We consider the Lagrange and Hess–Appelrot (H–A) cases of the Euler–Poisson (E–P) system $\dot{M} = M \times \Omega - \Gamma \times K$, $\dot{\Gamma} = \Gamma \times \Omega$, which describes the dynamics of a rigid body about a fixed point. The Lagrange case is completely integrable (with a family of invariant 2–tori), while the H–A case is partially integrable in the sense that the system has an invariant torus (so-called Hess surface).

We construct explicitly the invariant tori and describe dynamics on them.

Next, we consider perturbations of these cases within the E–P class. In the Lagrange case, the condition for periodic orbits leads to Melnikov type functions. In the H–A case one investigates the normal hyperbolicity property, which implies perturbation of the Hess surface; next, limit cycles on it are studied.

Qualitative results are obtained for the situation with a critical circle (when one of the radii of the invariant torus tends to 0) and the situation when the Hess surface degenerates to a separatrix connection of a saddle. In the latter case the perturbation leads to a chaotic dynamics.

Title 2: Normal forms of vector fields and their analytic properties

Abstract 2: A singular point $x = y = 0$ of a planar vector field $V(x, y)$ is elementary if at least one of the eigenvalues of its linearization is nonzero. The normal forms for such germs, with respect to the action of the group of formal changes of the coordinates and of changes of the time, were obtained long ago. But the non-elementary cases have turned out extremely hard to solve. Only recently an effective tool was created to deal with this problem.

It relies upon a splitting the perturbation $W$ in $V = V_0 + W$ into a part transversal to $V_0$ and a part tangent to $V_0$. This leads to a splitting of the homological operator $\mathrm{ad}_{V_0}$, which acts on vector fields, into two homological operators acting on functions.

In this way complete formal normal forms for the cases with nilpotent linear part (the Bogdanov–Takens singularity) and with zero linear part were obtained.

Next, one studies the analyticity of the obtained normal forms. Here we obtain bounds for the 1–dimensional homological operators (in the analytic case) and we use a criterion of Ilyashenko (in the non-analytic case).

Title 3: Invariants of group actions, dimension/degree duality and normal forms of vector fields

Abstract 3. We develop a constructive approach to the problem of polynomial first integrals for linear vector fields. As an application we obtain a new proof of the theorem of Wietzenböck about finiteness of the number of generators of the ring of constants of a linear derivation in the polynomial ring.

In the case of nilpotent linear vector field $X$ with one Jordan cell we deal with an irreducible representation $\mathrm{Sym}^n \mathbb{V}$, $\mathbb{V} \simeq \mathbb{C}^2$, of the Lie algebra $\mathrm{sl}(2)$, where $X$ is a highest weight vector. The homogeneous polynomial first integrals of degree $d$ of $X$ correspond to highest weight vectors in the representation $\mathrm{Sym}^d \left( \mathrm{Sym}^n \mathbb{V} \right)$.

We obtain a generating function for the multiplicities of the splitting of the latter representation into irreducible ones.

Moreover, we show that the ring of invariants of the latter representation is a polynomial ring, i.e., without relations between generators.

The dim/deg duality is an isomorphism $\mathrm{Sym}^d\left(\mathrm{Sym}^n\mathbb{V}\right) \simeq \mathrm{Sym}^n\left(\mathrm{Sym}^d\mathbb{V}\right)$. We construct the duality map explicitly.

Moreover, we propose an alternative approach to the analyticity property of the normal form reduction of a germ of vector field with nilpotent linear part in a case considered by Stolovich and Verstringe.