1 What is the Integrability of Nonlinear Dynamical Systems: analytical and Lie-algebraic aspects

1.1 Introduction: A View on the State of Art

In recent times it has been stated that many dynamical systems of classical mathematical physics and mechanics are endowed with symplectic structures, given in the majority of cases by the related Poisson brackets. Very often such Poisson structures on corresponding manifolds are canonical, which gives rise to the possibility of producing their hidden group theoretical essence for many completely integrable dynamical systems. It is a well understood fact that great part of comprehensive integrability theories of nonlinear dynamical systems on manifolds is based on Lie-algebraic ideas, by means of which, in particular, the classification of such very popular nowadays compatibly bi-Hamiltonian and isospectrally Lax type integrable systems has been carried out. Some of Lectures are devoted to their description, but to our regret so far the work has not been completed. Hereby our main goal in each analyzed case consists in separating the basic algebraic essence responsible for the complete integrability, and which is, at the same time, in some sense universal, i.e., characteristic for all of them.

Integrability analysis in the framework of an effective enough gradient-holonomic algorithm, devised during the past century, is fulfilled through three stages: 1) finding a symplectic structure (Poisson bracket) transforming an original dynamical system into a Hamiltonian form; 2) finding first integrals (action variables or conservation laws); 3) defining an additional set of variables and some functional operator quantities with completely controlled evolutions (for instance, as Lax type representation). Making use of the small parameter method developed in the book and an asymptotic approach to finding explicit forms of symplectic structures and conservation laws, we have succeeded in directly proving the complete Lax type integrability of many nonlinear dynamical systems on functional manifolds important for applications.

1.2 Introduction: Hamiltonian Action and Momentum Mapping

Consider a group Lie G (it may be a Lie group, a Banach group etc), some smooth *symplectic* manifold ($M;\omega^{(2)}\in\Lambda^2(M)$ is closed $d\omega^{(2)}=0$ and nondegenerate) and some smooth *group-action* $G\times M\rightarrow M$, that is for any fixed $u\in M$ there is defined the mapping $M\ni u\rightarrow g\circ u\in M$, such that $(gh)\circ u=g\circ(h\circ u)$. The exists a dual to $G\times M\rightarrow M$ mapping at a fixed $u\in M:G\ \ni g\rightarrow g\circ u\in M$, which is naturally lifted to their tangent spaces at the unit element $e\in G$ and $u\in M$, respectively:

$$\begin{array}{ccc} T_e(G) & \stackrel{u_*}{\rightarrow} & T_u(M) \\ \downarrow p_G & & \downarrow p_M \\ G & \stackrel{u}{\rightarrow} & M \end{array}, \tag{1}$$

where the linear tangent mapping $u_*:T_e(G)\rightarrow T_u(M)$ maps the elements $a\in\mathcal{G}$ of the Lie algebra $\mathcal{G}\simeq T_e(G)$ into the vector fields $X_a\in\Gamma(T(M))$ on M at point $u\in M:$

$$u_*(a)=X_a. \tag{2}$$

There is accepetd the following

Condition 1 ***Assumption 1:*** *The group action $G\times M\rightarrow M$ is symplectic, that is the Lie derivative $L_{X_a}\omega^{(2)}=0$ for all elements $a\in\mathcal{G}$. This, in particular, means that for any $a\in\mathcal{G}$ there exist smooth mappings $H_a:M\rightarrow\mathbb{R}$, such that $i_{X_a}\omega^{(2)}=-dH_a$.*

The set of *linear with respect to the elements* $a\in\mathcal{G}$ functions $H_a:M\rightarrow\mathbb{R}$ makes it possible to determine the so called *Souriau momentum mapping* $\mathrm{l}:M\rightarrow\mathcal{G}^*$,

$$\mathrm{l}(u)(a):=H_a(u) \tag{3}$$

for any $a \in \mathcal{G}$ at arbitrary $u \in M$, where $\mathcal{G}^*$ denotes the corresponding dual space to the Lie algebra $\mathcal{G}$, that is the set of continuous linear functionals on $\mathcal{G}$, acting on $\mathcal{G}$ via the natural convolution $\mathcal{G}^* \times \mathcal{G} \ \rightarrow \mathbb{R}$.

There also is accepted the following **Assumption 2**:

Condition 2 *The group action $G \times M \rightarrow M$ is* ***Hamiltonian****, that is* ***the equivariance*** *property*

$$\mathrm{Ad}^*_{g^{-1}} \mathrm{l}(u) = \mathrm{l}(g \circ u) \tag{4}$$

holds for all $g \in G$ and any $u \in M$, where $\mathrm{Ad}^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ denotes the corresponding co-adjoint action of the group G on the dual space $\mathcal{G}^*$. Its direct consequences are the following: a) the functions $H_a : M \rightarrow \mathbb{R}, a \in \mathcal{G}$, are invariant with respect to the group action $G \times M \rightarrow M$: $H_a(g \circ u) = H_a(u)$ for any $u \in M$; b) the related ***Poisson bracket*** *$\{H_a, H_b\} := -\omega^{(2)}(X_a, X_b)$ on M satisfies the Lie algebra* ***homomorphism*** *property:*

$$H_{[a,b]} = \{H_a, H_b\} \tag{5}$$

for any $a, b \in \mathcal{G}$, where we denoted by $[\cdot, \cdot]$ the natural commutator on the Lie algebra $\mathcal{G}$.

Consider now the momentum mapping $\mathrm{l} : M \rightarrow \mathcal{G}^*$ and assume that some smooth vector field $K : M \rightarrow T(M)$ generates some *nonlinear Hamiltonian system* on the manifold M, that is for some smooth function $H : M \rightarrow \mathbb{R}$

$$du/dt = \{H, u\} \tag{6}$$

for any $u \in M$ and some evolution parameter $t \in \mathbb{R}$. The following lemma holds.

Lemma 3 *Let a smooth function $H : M \rightarrow \mathbb{R}$ be a) invariant with respect to the group action $G \times M \rightarrow M$ and representable as $H = h \circ \mathrm{l}$ for some smooth mapping $h : \mathcal{G}^* \rightarrow \mathbb{R}$; b) the Poisson brackets $\{H, H_a\} = 0$ for all $a \in \mathcal{G}$. Then the momentum mapping $\mathrm{l} : M \rightarrow \mathcal{G}^*$ satisfies the following linear evolution equation on $\mathcal{G}^*$:*

$$\frac{d}{dt}\mathrm{l}(u) = -\mathrm{ad}^*_{\nabla h(l)} \mathrm{l}(u), \tag{7}$$

where $\mathrm{ad}^* : \mathcal{G} \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ denotes the coadjoint action of the Lie algebra $\mathcal{G}$ on its adjoint space $\mathcal{G}^*$ and the element $\nabla h(\mathrm{l}) \in \mathcal{G}$ is, by definition, determined from the condition $\frac{d}{d\varepsilon} h(\mathrm{l} + \varepsilon m)|_{\varepsilon=0} := m(\nabla h(\mathrm{l}))$ for all $m \in \mathcal{G}^*$.

Assume now *additionlly* that the Lie algebra $\mathcal{G}$ of the group G is *metrized, that is there exists such a nondegenerate symmetric and ad-invariant metric $(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}$ on the space $\mathcal{G}$* that

$$(a, [b, c]) = ([a, b], c) = 0 \tag{8}$$

for all a, b and $c \in \mathcal{G}$. Then, evidently one can identify, owing to the classical Riesz theorem, the adjoint space $\mathcal{G}^*$ with $\mathcal{G}$, that is $\mathcal{G}^* \simeq \mathcal{G}$, and respectively rewrite the linear evolution equation (7) as that on $\mathcal{G}$ in the following classical Lax type form:

$$\frac{d}{dt}\mathrm{l}(u) = [\nabla h(\mathrm{l}), \mathrm{l}(u)] \tag{9}$$

for all $u \in M$. The form (9) realizes the corresponding **linearization** of the *nonlinear Hamiltonian system* (6) on the manifold M. Its much more concrete realization will be presented in what follows belows.

1.3 AKS-theorem and R-structure

Let's consider an arbitrary Lie algebra $(\mathcal{G}, [\cdot, \cdot])$ over the field $\mathbb{K}(\mathbb{R}\mathrm{or}\mathbb{C})$ and the related adjoint action $\mathrm{ad}_a : \mathcal{G} \rightarrow \mathcal{G}$, $\mathrm{ad}_a b = [a, b]$ for a fixed element $a \in \mathcal{G}$ and all $b \in \mathcal{G}$. Denote also by $\mathcal{G}^*$ the natural adjoint space to the linear space $\mathcal{G}$ and construct on $\mathcal{G}^*$ for a fixed element $a \in \mathcal{G}$ the corresponding co-adjoint action $\mathrm{ad}^*_a : \mathcal{G}^* \rightarrow \mathcal{G}^*$, where, by definition,

$$\mathrm{ad}^*_a \mathrm{l}(b) := \mathrm{l}(\mathrm{ad}_a b]) = \mathrm{l}([a, b]) \tag{10}$$

of all elements $l \in \mathcal{G}^*$ and $b \in \mathcal{G}$.

Consider now such a modul homomorfism $R : \mathcal{G} \rightarrow \mathcal{G}$, the so called R-structure on $\mathcal{G}$, that the bracket

$$[a, b]_R := [Ra, b] + [a, Rb] \tag{11}$$

is for any $a, b \in \mathcal{G}$ is also a Lie bracket, that is

$$[[a, b]_R c] + cycle = 0 \tag{12}$$

for all a, b and $c \in \mathcal{G}$.

The space $\mathcal{G}_R^*$ related with the Lie bracket is very interesting being Poissonian, that is it can be endowed with the canonical Lie-Poisson structure

$$\{f, g\}_R := l([\nabla f(l), \nabla g(l)]_R) \tag{13}$$

at any element $l \in \mathcal{G}_R^*$ for arbitrary smooth functions $f, g : \mathcal{G}^* \rightarrow \mathbb{K}$, where the gradients $\nabla f(l), \nabla g(l) \in \mathcal{G}$ are defined via the weak expression $\frac{d}{d\varepsilon} f(l + p\varepsilon)|_{\varepsilon=0} = p(\nabla h(l))$ for all $p \in \mathcal{G}^*$. The bracket (13) is, by construction, skew-symmetric and satisfying the classical Jacobi condition:

$$\{h, \{f, g\}_R\}_R + cycle = 0 \tag{14}$$

for any f, g and $h : \mathcal{G}^* \rightarrow \mathbb{K}$.

Based on the bracket (13) one can naturally construct for any smooth functional $H : \mathcal{G}^* \rightarrow \mathbb{K}$ the corresponding *Hamiltonian flow* on $\mathcal{G}^*$:

$$dl/dt := \{H, l\}_R = -\mathrm{ad}^*_{R\nabla H(l)} l - R^* \mathrm{ad}^*_{\nabla H(l)} l, \tag{15}$$

where $l \in \mathcal{G}^*$ and $t \in \mathbb{K}$ is the related evolution parameter.

Definition 4 *The Hamiltonian flow (15) is called "integrable", if there is such a Poissonian bracket $\{\cdot, \cdot\}_R$ on $\mathcal{G}^*$ jointly with a countable set $\mathcal{H}$ of smooth and functionally independent mappings $H_j : \mathcal{G}^* \rightarrow \mathbb{K}$, $j = \overline{1, |\mathcal{H}|}$, which satisfy the commutation conditions*

$$\{H, H_j\}_R = 0 = \{H_j, H_k\}_R \tag{16}$$

for all $j, k = \overline{1, |\mathcal{H}|}$, equivalent to the following commuting to each other flows

$$dl/dt_j = -\mathrm{ad}^*_{R\nabla H_j(l)} l - R^* \mathrm{ad}^*_{\nabla H_j(l)} l \tag{17}$$

of $l \in \mathcal{G}^*$ for all evolution parameters $t_j \in \mathbb{K}, j = \overline{1, |\mathcal{H}|}$.

The main **problem** is: *how to construct this set $\mathcal{H}$ of invariants, satisfying the conditions (16)?*

A special answer to this problem was done in 1978 by American mathematicians Adler, Kostant and Symes, who stated the following theorem.

Theorem 5 *(KMS) Let a Lie algebra $\mathcal{G}$ allow splitting $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, where $\mathcal{G}_+$ and $\mathcal{G}_-$ are its subalgebras, and $\mathcal{K}$ denotes its set of Casimir functionals:*

$$\mathcal{K} := \{H_j : \mathcal{G}^* \rightarrow \mathbb{K} : \mathrm{ad}^*_{\nabla H_j(l)} l = 0, j = \overline{1, |\mathcal{K}|}\}. \tag{18}$$

Denote also by $\mathrm{Pr}_\pm : \mathcal{G} = \mathcal{G}_\pm \subset \mathcal{G}$ the corresponding projections on the subalgebras $\mathcal{G}_\pm$, respectively. Then the modul mapping $R = (\mathrm{Pr}_+ - \mathrm{Pr}_-)/2$ generates an R-structure on the Lie algebra $\mathcal{G}$, subject to which the set $\mathcal{K} \subset \mathcal{H}$, and flows

$$dl/dt_j := \{H_j, l\}_R = -\mathrm{ad}^*_{R\nabla H_j(l)} l \tag{19}$$

are commuting to each other for all $j = \overline{1, |\mathcal{K}|}$.

Corollary 6 *Let a Lie algebra $\mathcal{G}$ is metrized, that is there exists such a trace-type symmetric, nondegenerate and bilinear $\mathbb{R}$-valued form $(\cdot|\cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ on $\mathcal{G}$:*

$$(a|b) := \mathrm{Tr}(ab) \tag{20}$$

for any $a, b \in \mathcal{G}$, which is ad-invariant, that is

$$(a, |[b, c]) = ([a, b]|c) \tag{21}$$

for all a, b and $c \in \mathcal{G}$. Then the basis set $\mathcal{K}$ of Casimir invariants coincides with the set of functionals

$$H_j := (a^j|a) \tag{22}$$

for all $j \in \mathbb{Z}_+$.

Based on Corollary 6 one can easily construct an infinite hierarchy of integrable Hamiltonian flows (19) in the following commutator Lax type form:

$$d\mathrm{l}/dt_j = [\mathrm{Pr}_+ \nabla H_j(\mathrm{l}), \mathrm{l}] \tag{23}$$

for all $j \in \mathbb{Z}_+$.

Consider now such a Lie algebra $\mathcal{G}$, which can not be a *a priori metrized, that is the* trace-type symmetric, nondegenerate and bilinear $\mathbb{K}$-valued form $(\cdot|\cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$ on $\mathcal{G}$ does not exist. This is exactly a case when the Lie algebra $\mathcal{G}$ is constructed as an algebra A-valued matrices with an associative noncommutative algebra A taken to be either a finitely generated free associative and noncommutrative space or a finitely generated group G algebra. Exactly the latter case was considered by Kontsevich [18, 19, 20] and which is a topic of the next Section.

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