

On the conformal Lie superalgebras $K(1; N = 1; 2; 3)$, their super-pseudodifferential generalizations and related semi-supersymmetric integrable systems

Anatolij K. Prykarpatski (Cracow University of Technology, Poland)
 Oksana Ye. Hentosh (IIAPMM of NAS, Lviv, Ukraine)
 Victor A. Bovdi (UAE University, Al Ain, UAE)

Abstract

As one of most important legacy of modern mathematical physics is deemed nowadays a new fruitful conception of supersymmetry, whose main idea [9, 11] is to treat elementary bosonic and fermionic particles equally, what mathematically amounts to incorporating [2, 7] anticommuting variables of Grassmann type together with the usual commuting variables. In such way a number of well known mathematical physical equations have been generalized into the supersymmetric analogues, amongst which we find [5, 8, 12] supersymmetric versions of sine-Gordon, Korteweg-de Vries, Burgers, Kadomtsev-Petviashvili hierarchy, Boussinesq, nonlinear Schrödinger equation and many others. The investigation of the problem of particle-like behavior in supersymmetric field theories naturally leads to a theory of super-integrable systems and studying their properties, which can be helpful in part in analysis of modern super-string mathematical physics problems. It is nowadays well known [3] that there exist integrable fermionic extensions [12] of the completely integrable field theory systems on functional supermanifolds, related with conformal superalgebra symmetries, and which are not supersymmetric. There was also observed [6] that in most cases the supersymmetric integrable extensions were associated [8, 12] to evolution superflows, generated by means of the super-Lax type representations, yet as it was clearly demonstrated in the work [3], there exist also supersymmetric integrable extensions, related with semi-supersymmetric systems, generated by the centrally extended $N = 2$ - superconformal loop Lie superalgebra symmetry and allowing reduction to the supersymmetric flows. Here is worth also to mention that, in fact, almost all of so called "new" nonlinear $N = 1$ - superintegrable dynamical systems, published [3] during past decades, are related to coadjoint flows of the affine conformal $N = 1$ - symmetry Lie superalgebra $\mathcal{K}(1|1)$, described in detail first in [4], and in general, are related [3] to the affine Sturm-Liouville type superconformal spectral problem $(-D_\theta^3 + (\sum_{j=-m}^{m+p-1} (u_j\theta + v_j)\lambda^j) + \theta\lambda^{m+p})f(x, \theta) = 0, m, p \in \mathbb{N}, \lambda \in \mathbb{C}$, on the supercircle $\mathbb{S}^{1|1} \simeq \{(x, \theta) \in \mathbb{S}^1 \times \Lambda_1^{(1)}\}$ for a smooth function $f \in C^\infty(\mathbb{S}^{1|1}; \Lambda_0^{(1)})$, where $\Lambda_0^{(1)} \oplus \Lambda_1^{(1)} := \Lambda^{(1)}$ - the corresponding \mathbb{Z}_2 - graded superalgebra. Meanwhile, as the description of $N = 1$ - supersymmetric Lax type flows was elaborated [8, 12] are known widely enough, the $N \geq 2$ - supersymmetric integrable flows, analyzed in [6, 8], still should be paid more attention.

Owing to the interesting observation in the work [3], based on the affine Sturm-Liouville type superconformal spectral problem $(D_{\theta_1}D_{\theta_2} + \sum_{j=-m}^{m+p-1} u_j(x, \theta)\lambda^j + \lambda^{m+p})f(x, \theta) = 0, m, p \in \mathbb{N}, \lambda \in \mathbb{C}$, on the supercircle $\mathbb{S}^{1|2} \simeq \{(x, \theta) \in \mathbb{S}^1 \times \Lambda_1^{(2)}\}$ for $f \in C^\infty(\mathbb{S}^{1|2}; \Lambda_0^{(2)})$ the special reductions of the related nonlinear integrable superconformal evolution flows prove to be supersymmetric dynamical systems on functional supermanifolds. Another interesting Backlund type construction of nonlinear $N = 2$ - superconformal semi-supersymmetric dynamical systems was suggested in [3], generalizing in part those obtained before in [6].

Within our Report we successively analyze modern Lie algebraic approaches, lying in the background of effective constructions of integrable in general semi-supersymmetric Hamiltonian systems on functional $N \geq 2$ - supermanifolds, possessing rich yet hidden super-symmetries and endowed with suitably related super-Poisson structures. As an application, we describe countable hierarchies of new Lax type integrable nonlinear $N = 3$ - semi-supersymmetric dynamical systems. In particular, we analyze the suitably central extended super-conformal affine Lie superalgebra $\hat{\mathcal{K}}(1|3)$ and its finite-dimensional coadjoint orbits, generated by the related Casimir functionals on the supercoalgebra $\hat{\mathcal{K}}(1|3)^*$, and construct an infinite hierarchy of completely integrable super-Hamiltonian systems on smooth functional supermanifolds, which also prove to be supersymmetric. Moreover, we generalized these results subject to the suitably factorized super-pseudodifferential Lax type representations, taking into account the devised before algebro-analytic constructions both in mentioned above works [3, 6] and in devoted to Lie algebraic properties of factorized Lax type representations [10] and the respectively factorized Hamiltonian systems. As a new interesting

result, we succeed in algorithmic construction of integrable super-Hamiltonian factorized systems, generated by Casimir invariants of centrally extended pseudo-differential operator superalgebras.

References

- [1] Agrebaoui B., Hattab R., 1-cocycles on the group of contactomorphisms on the supercircle $\mathbb{S}^{1,3}$ generalizing the Schwarzian derivative. Czechoslovak Mathematical Journal, 66(4) (2016), 1143–1163
- [2] Berezin F.A., The method of second quantization, Pure & Applied Physics 24, Academic Press, New York-London, 1966. logs of the Lax-integrable nonlinear dynamical systems. Ukrainian Mathematical Journal, 58(7) 2006, 1001-1015.
- [3] Hentosh O.E., The Lax integrable Laberge–Mathieu hierarchy of supersymmetric nonlinear dynamical systems and its finite-dimensional Neumann type reduction. Ukrainian Math. J. 61(7) (2009), 1075–1092; <https://doi.org/10.1007/s11253-009-0260-7>.
- [4] Kulish P.P., Analog of the Korteweg–de Vries equation for the superconformal algebra. J. Soviet Math. 41(2), (1988), 970–975
- [5] Kupershmidt B.A., Integrable systems, " Proc. Nat. Acad. Sci. USA, 81 (1984), 6562-6563.
- [6] Laberge C.-A., Mathieu P., $N = 2$ superconformal algebra and integrable $O(2)$ fermionic extensions of the Korteweg – de Vries equation, Phys. Lett. B. 215(4) 1988, 718 – 722.
- [7] Leites D., Introduction to supermanifolds, Russian Math. Surveys 35(1) (1980), 1-64
- [8] Popowicz Z., $N = 2$ Super-complexification of the Korteweg-de Vries, Sawada-Kotera and Kaup-Kupershmidt Equations. Journal of Nonlinear Mathematical Physics, 26(2) (2019), 294-312; <https://doi.org/10.1080/14029251.2019.1591732>
- [9] Volkov D.V. and Akulov V.P., Is the neutrino a Goldstone particle?, Phys. Lett. 46B (1973), 109-110.
- [10] Vovk M., Pukach, Hentosh O., Prykarpatsky Ya., The structure of rationally factorized Lax type flows and their analytical integrability. WSEAS Transactions on Mathematics, 16 (2017), 323-330
- [11] Wess J. and Zumino B., Super-gauge transformations in four dimensions, Nucl. Phys. B70 (1974) 39-50.
- [12] Yamanaka I., Sasaki R., Super Virasoro Algebra and Solvable Supersymmetric Quantum Field Theories. Progress of Theoretical Physics, 79(5) 1988, 1167-1184