Grassmann geometry in spaces of functions

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First, we will briefly survey the main aspects of the Grasmann manifold of a Hilbert space. Let $\mathcal{H}$ be a complex Hilbert space, to each closed subspace $\mathcal{S} \subset \mathcal{H}$ corresponds a unique orthogonal projection $P_{\mathcal{S}}$ onto $\mathcal{S}$. Thus, $Gr(\mathcal{H})$, the set of closed subspaces of of $\mathcal{H}$, can be parametrized

$$Gr(\mathcal{H}) \simeq \{P \in \mathcal{B}(\mathcal{H}) : P = P^2 = P^*\},$$

by means of $P \sim R(P)$ (the range of P). Here $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators in $\mathcal{H}$. The benefit of this approach is that $Gr(\mathcal{H})$ is naturally a closed subset of a Banach space: $Gr(\mathcal{H})$ turns out to be a complemented real analytic submanifold of $\mathcal{B}(\mathcal{H})$, and therefore the tangent spaces of $Gr(\mathcal{H})$ inherit a natural ambient norm. In geometric terms, $Gr(\mathcal{H})$ has a natural Finsler metric (non Riemannian, non smooth). The natural action of the unitary group of $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ on subspaces, namely

$$U \bullet \mathcal{S} = U(\mathcal{S}) \quad (U \in \mathcal{U}(\mathcal{H}), \mathcal{S} \subset \mathcal{H})$$

translates into

$$U \bullet P_{\mathcal{S}} = U P_{\mathcal{S}} U^*.$$

A natural linear connection was introduced in $Gr(\mathcal{H})$ ([5], [4]). The geodesics of this connection can be explicitely computed. They are of the form

$$\delta(t) = e^{itX} P e^{-itX},$$

where $X = X^*$ is *co-diagonal* with respect to P: $PXP = P^{\perp} X P^{\perp} = 0$. Porta and Recht [5] noted that if $\|P - Q\| < 1$ ($P, Q \in Gr(\mathcal{H})$) then there exists a unique geodesic joining P and Q, and this geodesic has minimal length (for the Finsler metric described above, i.e., for the usual norm in $\mathcal{B}(\mathcal{H})$). Later (see [1]) this was refined:

1. There exists a geodesic between $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ ($\mathcal{S}, \mathcal{T} \subset \mathcal{H}$) if and only if

   $$\dim(\mathcal{S} \cap \mathcal{T}^{\perp}) = \dim(\mathcal{S}^{\perp} \cap \mathcal{T}).$$

   This (eventually non unique) geodesic can be chosen minimal. The length of this geodesic (if parametrized in the interval $[0, 1]$) equals $\|[X, P]\|$ (here $[X, P] = XP - PX$).

2. The geodesic is unique if and only if

   $$\mathcal{S} \cap \mathcal{T}^{\perp} = \mathcal{S}^{\perp} \cap \mathcal{T} = \{0\}.$$

In the second part we examine concrete examples of Hilbert spaces and closed subspaces. This part of the talk is joint work with E. Chiumiento, G. Larotonda and A. Varela [2], [3]. For instance, we consider:

- $\mathcal{H} = L^2(\mathbb{T})$, $\mathcal{S}, \mathcal{T}$ of the form $\varphi H^2(\mathbb{T})$, where $H^2(\mathbb{T})$ is the Hardy space of the unit circle and φ is a continuous unimodular function. It is shown that there exists a unique geodesic between $\varphi H^2(\mathbb{T})$ and $\psi H^2(\mathbb{T})$ if and only if $w(\varphi) = w(\psi)$ (w=winding number), if and only if $N(T_{\bar{\varphi}\psi}) = \{0\}$. Here $T_{\bar{\varphi}\psi}$ is the Toeplitz operator with symbol $\bar{\varphi}\psi$. In such case a unique selfadjoint operator $X_{\varphi,\psi}$ is defined (the exponent in the geodesic joining both subspaces). The minimality propery of this geodesics permits one to establish operator norm inequalities which we shall specify in the talk.

- $\mathcal{H}$ a reproducing kernel Hilbert space of analytic functions: among these $H^2(\mathbb{T})$, or the Bergman space $B_2(\mathbb{D})$ of the unit disk; the subspaces are of the form

$$\mathcal{Z}_{\mathbf{a}} = \{f \in \mathcal{H} : f|_{\mathbf{a}} = 0\},$$

  where $\mathbf{a} = \{a_1, a_2, \dots\}$ is a finite or infinite (countable) set. For finite sets, the answer to the question of existence of geodesics between subspaces of this type is affirmative for the Hardy space, if and only if both sets have the same cardinality; for the Bergman space, for $n \geq 3$, there are negative examples. We have some results for infinite sets, and again, operator norm inequalities are deduced from the minimality condition.

References

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[3] E. Andruchow; E. Chiumiento; A. Varela. *Grassmann geometry of zero sets in reproducing kernel Hilbert spaces.* J. Math. Anal. Appl. 500 (2021), no. 1, Paper No. 125107, 31 p

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