Algebraic geometric properties of spectral surfaces of quantum integrable systems and their isospectral deformations

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Quantum completely integrable systems

The *quantum completely integrable system* (QCIS) (with n degrees of freedom) over a n-dimensional algebraic variety X is a pair (Λ, θ), where

- Λ is an irreducible n-dimensional affine algebraic variety;
- $\mathcal{O}_\Lambda$ is the ring of regular functions on Λ
- $D(X)$ is the ring of differential operators on a variety X (without loss of generality X may be taken to be a formal polydisc Spec$(k[[x_1, \dots, x_n]])$
- $$\theta : \mathcal{O}_\Lambda \to D(X)$$

  is an embedding.

Recall that for a commutative K-algebra R the filtered ring $D(R)$ is generated by $Der_K(R)$ and R inside the ring $\mathrm{End}_K(R)$.

$$D_0(R) \subset D_1(R) \subset D_2(R) \subset \dots, \quad D_i(R)D_j(R) \subset D_{i+j}(R),$$

where $D_i(R)$ are defined inductively. In particular, the usual function ord is defined on $D(R)$.

From now on we introduce the following notation:
$\hat{R} := K[[x_1, \ldots, x_n]]$, where $\mathrm{char}\, K = 0$ — the ring of regular functions on a formal polydisc.
$D_n := \hat{R}[\partial_1, \ldots, \partial_n]$ — the ring of differential operators on a formal polydisc.
Then QCIS are just subrings of commuting operators in D_n. We'll study such subrings and their isospectral deformations.

Case $n = 1$: commuting ordinary differential operators

Definition

An ordinary differential operator $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in D_1$ of positive order n is called *(formally) elliptic* if $a_n \in K^*$. A ring $B \subset D_1$ containing an elliptic element is called *elliptic*.

- (reduction to the elliptic case)
  If $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in D_1$, where $a_n(0) \neq 0$, then there is a change of variables $\varphi \in \mathrm{Aut}(D_1)$ such that

$$Q := \varphi(P) = \partial^n + b_{n-2}\partial^{n-2} + \cdots + b_0 \qquad (1)$$

  for some $b_0, \ldots, b_{n-2} \in K[[x]]$.
- Let B be a commutative subalgebra of D_1 containing an elliptic element P. Then *all* elements of B are elliptic.

Krichever-Mumford classification in $n = 1$ case.

Theorem

There is a one-to-one correspondence

$$[B \subset D_1 \quad \textit{of rank } r] \longleftrightarrow [(C, p, \mathcal{F}, z, \phi) \quad \textit{of rank } r]/\simeq$$

$$[B \subset D_1 \quad \textit{of rank } 1]/\sim \longleftrightarrow [(C, p, \mathcal{F}) \quad \textit{of rank } 1]/\simeq$$

where

- $[B]$ *means a class of equivalent commutative elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in D_1^*$.
- $\sim$ *means "up to linear changes of variables"*
- $(C, p, \mathcal{F}, z, \phi)$ *means the algebraic-geometric spectral data of rank* r

Here the *rank* of B is

$$\mathrm{rk}(B) := GCD\{\mathrm{ord}(P),\ P \in B\}.$$

Spectral data

Definition

- C is an integral projective curve over K;
- $p \in C$ is a closed regular K-point;
- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on C with

$$h^0(C, \mathcal{F}) = h^1(C, \mathcal{F}) = 0;$$

- z is a local coordinate at p;
- $\phi : \hat{\mathcal{F}}_p \simeq (K[[z]])^{\oplus r}$ is a trivialisation (i.e. an $\hat{\mathcal{O}}_p \simeq K[[z]]$-module isomorphism).

Recall: Isospectral deformations of rank one commutative rings of ODOs determine the KP flows on the compactified Jacobian of the spectral curve C. If C is smooth, $K = \mathbb{C}$ and $\mathrm{rk}(\mathcal{F}) = 1$, then there are explicit formulae due to Krichever. If C is singular and rational, then there are explicit formulae due to Wilson.

The $n > 1$ cases are much more complicated. To explain the corresponding results we need to introduce new notation.

The ring $\hat{D}_n^{sym}$ and its order function

Consider the K–vector space:

$$\mathcal{M} := \hat{R}[[\partial_1, \dots, \partial_n]] = \left\{ \sum_{\underline{k} \geq 0} a_{\underline{k}} \underline{\partial}^{\underline{k}} \;\middle|\; a_{\underline{k}} \in \hat{R} \text{ for all } \underline{k} \in \mathbb{N}_0^n \right\}.$$

Definition

For any $0 \neq P := \sum_{\underline{k} \geq 0} a_{\underline{k}} \underline{\partial}^{\underline{k}} \in \mathcal{M}$ we define its *order* to be

$$\mathbf{ord}(P) := \sup\{ |\underline{k}| - v(a_{\underline{k}}) \ \} \in \mathbb{Z} \cup \{\infty\},$$

where $v(a_{\underline{k}}) := \max\{n| \quad a_{\underline{k}} \in \mathfrak{m}^n, \mathfrak{m} = (x_1, \dots, x_n)\}$, and $|\underline{k}| = k_1 + \ldots + k_n$.

Definition

$$\hat{D}_n^{sym} := \left\{ Q \in \mathcal{M} \,\middle|\, \mathbf{ord}(Q) < \infty \right\}.$$

Properties of $\hat{D}_n^{sym}$:

- $\hat{D}_n^{sym}$ is a ring (with natural operations $\cdot$, $+$ coming from D_n); $\hat{D}_n^{sym} \supset D_n$.
- $\hat{R}$ has a natural structure of a left $\hat{D}_n$-module, which extends its natural structure of a left D_n-module.
- Operators from $\hat{D}_n^{sym}$ can realize arbitrary endomorphisms of the K-algebra $\hat{R}$ which are continuous in the $\mathfrak{m}$-adic topology: e.g. for $n = 1$ the operator

$$\exp(u * \partial) := \sum_{k=0}^{\infty} \frac{u^k}{k!} \partial^k, \quad u \in xK[[x]]$$

acts as

$$\exp(u * \partial) \circ f(x) = f\big(u + x\big).$$

- There are Dirac delta functions: for $\delta_i := \exp((-x_i) * \partial_i)$

  $\delta_i \circ f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = f\big(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n\big);$

- Operators of integration:

$$\int_i := \big(1 - \exp((-x_i) * \partial_i)\big) \cdot \partial_i^{-1} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} (-\partial)^k,$$

$$\int_i \circ x_i^m = \frac{x_i^{m+1}}{m+1}$$

- Difference operators ($n = 1$):

$$\sum_{i=0}^{M} f_i(n) T^i \hookrightarrow \hat{D}_n \quad \text{via} \quad T \mapsto x, n \mapsto -x\partial,$$

etc.

The ring $\hat{D}_n$ and the order function ord_n

Definition

We define $\hat{D}_1 = \hat{D}_1^{sym}$ and define $\hat{D}_n = \hat{D}_{n-1}^{sym}[\partial_n]$.
Obviously, $\hat{D}_n \subset \hat{D}_n^{sym}$.

Definition

We define the *order function* ord_n on $\hat{D}_n$ as

$$\mathrm{ord}_n(P) = l \quad \text{if} \quad \hat{D}_n \ni P = \sum_{s=0}^{l} p_s \partial_n^s.$$

The coefficient p_l is called *the highest term* and will be denoted by $HT_n(P)$ (as the term naturally associated with the function ord_n).

The notion of Γ-order

The Γ-order ord_Γ is defined on *some elements* of the algebra $\hat{D}_n^{sym}$:
Let's denote by $\hat{D}_n^{i_1,\dots,i_q}$ the subalgebra in $\hat{D}_n^{sym}$ consisting of operators *not depending on* $\partial_{i_1},\dots,\partial_{i_q}$. The Γ-order is defined recursively.

Definition

We say that $\mathrm{ord}_\Gamma(P)=k_1$, where $0\neq P\in\hat{D}_n^{2,3,\dots n}$, if $P=\sum_{s=0}^{k_1}p_s\partial_1^s$, where $0\neq p_{k_1}\in\hat{R}$.
We say that $\mathrm{ord}_\Gamma(P)=(k_1,\dots,k_i)$, where $P\in\hat{D}_n^{i+1,i+2,\dots,n}$, if $P=\sum_{s=0}^{k_i}p_s\partial_i^s$, where $p_s\in\hat{D}_n^{i,i+1,\dots,n}$, and $\mathrm{ord}_\Gamma(p_{k_i})=(k_1,\dots,k_{i-1})$.
We say that

$$\mathrm{ord}_\Gamma(P)=(k_1,\dots,k_n),$$

where $P\in\hat{D}_n^{sym}$, if $P=\sum_{s=0}^{k_n}p_s\partial_n^s$, where $p_s\in\hat{D}_n^n$, and $\mathrm{ord}_\Gamma(p_{k_n})=(k_1,\dots,k_{n-1})$.
In this situation we say that the operator P is *monic* if the highest coefficient $p_{k_1,\dots,k_n}$ (defined recursively in analogous way) is 1.

Quasi-elliptic algebras

Now we define the algebras that admit an effective description in terms of its algebro-geometric spectral data (which will be defined below).

Definition

The subalgebra $B \subset \hat{D}_n \subset \hat{D}_n^{sym}$ of commuting operators is called 1-*quasi elliptic* if there are n operators $P_1, \ldots, P_n$ such that

- For $1 \leq i < n$

$$\mathrm{ord}_{\Gamma}(P_i) = (0, \ldots 0, 1, 0 \ldots 0, l_i),$$

  where 1 stands at the i-th place and $l_i \in \mathbb{Z}_+$;
- $\mathrm{ord}_{\Gamma}(P_n) = (0, \ldots, 0, l_n)$, where $l_n > 0$;
- For $1 \leq i \leq n$ $\mathbf{ord}(P_i) = |\mathrm{ord}_{\Gamma}(P_i)|$.
- P_i are monic.

The notion of rank

Definition

The analytic rank of $B \subset \hat{D}_n$ is

$$An.rank(B) := \operatorname{rk} F = \dim(Q(B) \otimes_B F) =$$

$$\dim\{\psi| \quad P \circ \psi = \chi(P)\psi \quad \forall P \in B, \chi - \text{generic point}\}.$$

The algebraic rank is

$$Alg.rank(B) = GCD\{\mathbf{ord}(P)| \quad P \in B\}.$$

Fact: $An.rank(B) \geq Alg.rank(B)$. We'll say that $B \subset \hat{D}_n$ is of rank r if $An.rank(B) = Alg.rank(B) = r$.

Classification theorem

For $n > 1$ practically all known examples of commutative rings of *PDOs* can be made 1-quasi-elliptic after a change of coordinates and conjugation by a unity.

Theorem (Z.)

There is a one-to-one correspondence

$$[B \subset \hat{D}_2 \quad \textit{of rank } r] \longleftrightarrow [(X, C, p, \mathcal{F}, \pi, \phi) \quad \textit{of rank } r]/ \simeq$$

$$[B \subset \hat{D}_2 \quad \textit{of rank } 1]/ \sim \longleftrightarrow [(X, C, \mathcal{F}) \quad \textit{of rank } 1]/ \simeq$$

where

- $[B]$ *means a class of equivalent commutative* 1*-quasi-elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in \hat{D}_2^*$.
- $\sim$ *means:* $B_1 \sim B_2$ *if there is a linear change of variables* φ *and a unity* $U \in \hat{D}_2^{sym}$, $\mathbf{ord}(U) = 0$ *such that* $B_1 = U^{-1}\varphi(B_2)U$.
- $(X, C, p, \mathcal{F}, \pi, \phi)$ *are algebro-geometric spectral data of rank* r*:*

Spectral data

Definition

- X is an integral projective algebraic surface over K;
- C is an integral ample Cartier divisor on X. Moreover, $C^2 = r$.
- $p \in C$ is a closed K-point, which is regular on C and on X;
- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on X, which is Cohen-Macaulay along C, (*i.e. for each point $q \in C$ the $\mathcal{O}_{X,q}$-module $\mathcal{F}_q$ is a Cohen-Macaulay module*), and for $n \geq 0$

$$h^0(X, \mathcal{F}(nC)) = \frac{(nr+1)(nr+2)}{2}$$

- $\pi : \hat{\mathcal{O}}_{X,p} \simeq K[[u,t]]$ and $\phi : \hat{\mathcal{F}}_p \simeq \hat{\mathcal{O}}_{X,p}^{\oplus r}$ are some trivialisations of the local ring and module correspondingly.

Remark: We can additionally assume that X is *Cohen-Macaulay* because of the following result:

Proposition

If $B \subset \hat{D}_2$ is a commutative subring, then there exist a Cohen-Macaulay commutative subring $\tilde{B} \supset B$.
Moreover, if $B \subset D_2$, then $\tilde{B} \subset D_2$.

Cohen-Macaulay surfaces may have singularities: the singular locus is a union of codimension one curves.

Analogy with $n = 1$ case: Isospectral deformations of rank one commutative rings of ODOs determine the KP flows on the *Jacobian* of the spectral curve. Isospectral deformations of rank one commutative rings of PDOs determine some flows on the *moduli space M_χ of torsion free sheaves with fixed Hilbert polynomial $\chi(n) = \frac{(n+1)(n+2)}{2}$*.

A dense open subset of this moduli space parametrises *Cohen-Macaulay sheaves*. Cohen-Macaulay sheaves on Cohen-Macaulay surfaces can be effectively described with the help of *matrix-problem approach* due to Burban and Drozd. Then the higher-dimensional version of the *Sato theory* ("algebraic inverse scattering method") is used to obtain explicit examples or explicit deformations of known examples of commuting PDOs. Below I recall some examples obtained with the help of these techniques.

Examples of exact solutions

Example. Consider a commutative subring B generated by 3 operators:

$$P = \partial_2^2 - 2\frac{1}{(1-x_2)^2}\delta_1,$$

$$Q = \partial_1\partial_2 + \frac{1}{1-x_2}\delta_1\partial_1,$$

$$P' = \partial_2^3 - 3\frac{1}{(1-x_2)^2}\delta_1\partial_2 - 3\frac{1}{(1-x_2)^3}\delta_1.$$

Its spectral surface is a rational singular, with normalisation $\mathbb{P}^2$. If we derive equations of isospectral deformations of these operators (equations of a generalized SW system with the initial condition = Schur operator of B), we obtain the following equations:

$$\frac{\partial s_1}{\partial t_1} = \frac{1}{4}(s_1)_{x_2x_2x_2} - \frac{3}{2}(s_1)^2_{x_2}, \quad \frac{\partial s_1}{\partial t_2} = -(s_1)_{x_2}(s_1)_{x_1} - \frac{1}{2}(s_1)_{x_2x_2}\partial_1,$$
$$\frac{\partial s_1}{\partial t_3} = -(s_1)^2_{x_1} - (s_1)_{x_1x_2}\partial_1 - (s_1)_{x_2}\partial_1^2, \tag{2}$$

where $s_1(x_1, x_2, t_1, t_2, t_3) = s_1(t)$ is the first coefficient of the operator $S(t) = 1 + s_1(t)\partial_2^{-1} + \ldots$, and $S(0) = S$ is the Schur operator of B. Notably

$$s_1(0) = \frac{1}{1 - x_2}\delta_1$$

is a solution of the equations above.

Example: Quantum Calogero-Moser systems

Consider the Calogero–Moser operator

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - 2\left(\frac{1}{(x_1-\xi_1)^2} + \frac{1}{(x_2-\xi_2)^2}\right),$$

where $(\xi_1, \xi_2) \in \mathbb{C}^2$ is such that $\xi_1\xi_2 \neq 0$. In this case we have due to Chalykh, Veselov, Styrkas:

- There is a commutative subring $B_H \subset D_2$, $B_H \simeq A = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, the isomorphism is given with the help of the Berest BA-function:

$$\Psi_{Be} = z_1z_2 + \frac{z_1}{\xi_2 - x_2} + \frac{z_2}{\xi_1 - x_1} + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

  s.t. for any $q \in A$ there exists a unique $L_q \in B_H$

$$L_q\Psi_{Be} = q\Psi_{Be}.$$

Example (Burban-Z.)

Deformed BA-function:

$$\Psi(x_1, x_2, z_1, z_2) = \Psi_{Be} + \beta\overline{\Psi},$$

where Ψ_{Be} is the Berest function and

$$\overline{\Psi} = \frac{1 + \beta\left(\frac{z_1}{\xi_2} + \frac{z_2}{\xi_1}\right)}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} +$$

$$\frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_1 z_1)z_1 + (\xi_1 - x_1)\exp(x_1 z_1)z_1^2\right) +$$

$$\frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_2 z_2)z_2 + (\xi_2 - x_2)\exp(x_2 z_2)z_2^2\right).$$

Example

The simplest deformations of differential operators from the B_H:
for any $q \in z_1^2 z_2^2 A$ denote $q'(z_1, z_2) := q/(z_1^2 z_2^2)$.

$$\hat{D}_2^{sym} \ni L_q = Sq'(\partial_1, \partial_2)(\partial_1 - \frac{1}{1-x_1})(\partial_2 - \frac{1}{1-x_2}), \quad \text{where}$$

$$S = S_0 + \beta T,$$

$$S_0 = \partial_1\partial_2 + \frac{1}{\xi_2 - x_2}\partial_1 + \frac{1}{\xi_1 - x_1}\partial_2 + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

$$T = \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)}\left(\frac{1}{\xi_2}\left(\delta_2\partial_1 + (\xi_1 - x_1)\delta_2\partial_1^2\right) + \frac{1}{\xi_1}\left(\delta_1\partial_2 + (\xi_2 - x_2)\delta_1\partial_2^2\right)\right) + \frac{1}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)}\delta_1\delta_2\left(1 + \beta\left(\frac{\partial_1}{\xi_2} + \frac{\partial_2}{\xi_1}\right)\right)$$

In the matrix problem approach it is important to know what are the Cohen-Macaulay sheaves with special properties on the *normalisation* of the spectral surface. So, it is important to know what are the possible *normal* surfaces X such that a pre-spectral datum $(X, C, \mathcal{F})$ from classification theorem exists. I'll call such surfaces *normal forms*.

Question

What are the normal forms? Can they be smooth? Can they be classified?

Normal forms of commuting PDOs

Q: Which geometric data describe commutative subrings $B \subset D_2$ of *PDOs*?

Theorem (Kurke, Z.)

If $B \subset D_2$ is 1-quasi-elliptic of rank 1, with constant highest symbols, then

- *The sheaf $\mathcal{F}$ is Cohen-Macaulay of rank 1;*
- *The divisor C is a rational curve;*
- *If $n : \mathbb{P}^1 \to C$ is the normalisation map, then $\mathcal{F}|_C = (n_*(\mathcal{O}_{\mathbb{P}^1}))$.*

Conjecture

The conditions from theorem are sufficient.

Proposition

If X is a smooth normal form of a commutative ring of PDOs, then $X \simeq \mathbb{P}^2$ (and then $C \simeq \mathbb{P}^1$, $\mathcal{F} \simeq \mathcal{O}_X$).

Smooth normal forms

Q: Are there smooth normal forms of commutative subrings from $\hat{D}_2$?

Question

Find a smooth surface X such that there is a curve C and a divisor D with the following properties:

1. *C is ample (i.e. the sheaf $\mathcal{O}_X(C)$ is ample), $C^2 = 1$ and $h^0(X, \mathcal{O}_X(C)) = 1$;*
2. *$(D, C)_X = g(C) - 1$;*
3. *$h^i(X, \mathcal{O}_X(D)) = 0$, $i = 0, 1, 2$ and $h^0(X, \mathcal{O}_X(D + C)) = 1$.*

Remark: The condition $h^0(X, \mathcal{O}_X(C)) = 1$ means that we are looking for normal forms of "non-trivial" commutative subrings.

Definition

The subring $B \subset \hat{D}_2$ is "trivial", if it contains the operator ∂_1 or the operator ∂_2, i.e. B consists of operators not depending on x_1 or x_2.

Smooth normal forms

The examples of such algebras naturally arise from examples of commuting *ordinary differential operators* just by adding one extra derivation.

Proposition

The subring $B \subset \hat{D}_2$ *is "trivial" iff* $h^0(X, \mathcal{O}_X(C)) \geq 2$.

Proposition

Let $(X, C, \mathcal{F})$ *be a pre-spectral data of rank one with a smooth surface* X *and* $g(C) \leq 1$. *Then* $h^0(X, \mathcal{O}_X(C)) \geq 2$.

Conjecture

If X is a smooth normal form, then it is either rational (and corresponds to a "trivial" subring) or of general type.

Smooth normal forms

Theorem (Kulikov-Z.)

There is an eight-dimensional family of pairwise non-isomorphic Godeaux surfaces X such that on each X from this family there are at least 840 different divisors D_j and four curves C_i satisfying the conditions from Question.

Each of these Godeaux surfaces is a factor of a quintic in $\mathbb{P}^3$ by the group $\mathbb{Z}^5$.

Conjecture

All normal forms have the property $q = H^1(X, \mathcal{O}_X) = 0$. There are no other smooth normal forms of general type corresponding to "non-trivial" subrings.

Amazingly, the commutative rings of operators corresponding to the smooth normal forms do not have isospectral deformations!

On the other hand, there are many non-smooth normal forms:

Theorem (Z.)

For any smooth curve C there is a normal cone X (with the only singularity at the cone top) which is a normal form.