Quantum symmetric spaces from reflection equation and module categories

Makoto Yamashita (山下 真)
joint works with Kenny De Commer, Sergey Neshveyev, Lars Tuset

University of Oslo

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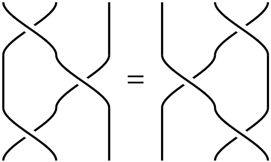
Yang–Baxter equation

Braided tensor categories in mathematical physics:

- quantum integrable systems (scattering in quantized setting)
- quantum field theory (intrinsic symmetry of quantum fields)
- quantum groups (quantized spaces with group law)



Yang–Baxter equation

braided tensor category

Hopf algebra
universal R-matrix

Knizhnik–Zamolodchikov equation

Often: deformation of *simple Lie groups* $\rightsquigarrow$ matrix solution to Yang–Baxter equation

Next to quantize: symmetric spaces

Definition (Symmetric space)

- Riemannian manifold M
- can model $(x_1, \dots, x_n) \mapsto (-x_1, \dots, -x_n)$ by *isometry*

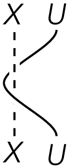
1. $U = \mathrm{Iso}^+(M)$: orientation preserving isometries
2. $M \cong U/\mathrm{Stab}(p)$ for any $p \in M$

Example (2-sphere $S^2 \subset \mathbb{R}^3$)

- symmetry around $(1, 0, 0)$: $(x, y, z) \mapsto (x, -y, -z)$
- $S^2 \cong \mathrm{SU}(2)/\mathrm{SO}(2)$

Spoiler: how it will go

Quantization will be *ribbon twist braided module category*:
add *reflection operator* to Yang–Baxter equation



"(quasi)particle bouncing off the boundary wall"

U: quasiparticle following braid statistics

X: quasiparticle on "boundary"

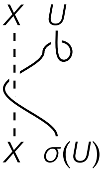
Mathematically:

- U: an object of a braided monoidal category $\mathscr{C}$
- X: an object of a *module category* $\mathscr{D}$, $X \otimes U$ makes sense in $\mathscr{D}$
- reflection operator: natural isomorphism $X \otimes U \to X \otimes U$

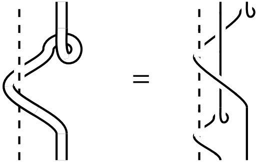
or better…

Spoiler: how it will go

Quantization will be *ribbon twist braided module category*: add *ribbon twist-braid operator* to Yang–Baxter equation



such that



with braided automorphism σ

For irreducible compact symmetric spaces:

- purely algebraic quantization: Letzter–Kolb coideals of $\mathcal{U}_q(\mathfrak{u})$
- transcendental quantization: cyclotomic Knizhnik–Zamolodchikov equations
- classification through co-Hochschild cohomology

Poisson manifolds

Definition (Poisson manifold)

Poisson bracket $\{f_1, f_2\} \in C^\infty(M)$ for $f_i \in C^\infty(M)$:

- $(C^\infty(M), \{\cdot, \cdot\})$: Lie algebra
- Leibniz rule $\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + \{f_1, f_3\} f_2$

Example (Sklyanin bracket on SU(2))

$C^\infty(\mathrm{SU}(2)) = \langle x_{i,j} \mid 1 \le i,j \le 2 \text{ coordinate functions}\rangle$

$$\{x_{ij}, x_{kl}\} = \delta_{2i}\delta_{1k}x_{1j}x_{2l} - \delta_{1i}\delta_{2k}x_{2j}x_{1l} - \delta_{j1}\delta_{l2}x_{i2}x_{k1} + \delta_{j2}\delta_{l1}x_{i1}x_{k2}$$

Problem (deformation quantization)

Can we find associative products $\star_\hbar$ *on (large subspace of)* $C^\infty(M)$ *such that* $\{f_1, f_2\} = \lim_{\hbar \to 0} \frac{1}{\hbar}(f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1)$*?*

Poisson–Lie groups and homogeneous spaces

With Poisson structure on $U, M, N, \ldots$

- $f\colon M \to N$ is a *Poisson map* if $f^\#\colon C^\infty(N) \to C^\infty(M)$ respects the brackets
- U is a *Poisson–Lie group* if the product map $U \times U \to U$ is Poisson
- action $U \curvearrowright M$ is *Poisson* if $U \times M \to M$ is Poisson

Lie algebraic characterizations (Drinfeld)

Poisson–Lie group structure on $U \leftrightarrow$ Lie bialgebra $\delta\colon \mathfrak{u} \to \bigwedge^2 \mathfrak{u}$
$\leftrightarrow$ *classical Yang–Baxter equation* for r s.t. $\delta(x) = [r, \Delta(x)]$
Poisson action $U \curvearrowright U/K \leftrightarrow$ coisotropic subalg $\mathfrak{k} \subset \mathfrak{u}$: $\delta(\mathfrak{k}) \subset \mathfrak{k} \otimes \mathfrak{u} + \mathfrak{u} \otimes \mathfrak{k}$

Foth–Lu: possible for $U = \mathrm{Iso}^+(M) \curvearrowright M$ compact symmetric space

Poisson–Lie groups and homogeneous spaces

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Example (Poisson homogeneous structures on S^2)

(SU(2), Sklyanin bracket) acts on $(S^2, \{x_i, x_{i+1}\} = (x_1 + t)x_{i+2})$ (index mod3) $\quad (x_i)_{i=1}^3$: coordinate on $\mathbb{R}^3 \supset S^2$, $t \in \mathbb{R}$

Quantum groups

Drinfeld–Jimbo deformation of universal enveloping algebra

$U(\mathfrak{u})$ can be deformed as a Hopf algebra $\mathscr{U}_\hbar(\mathfrak{u})$; 'just' deform coproduct: $\Delta_\hbar(x) - \mathrm{flip}\,\Delta_\hbar(x) = 2\hbar\delta(x)$ + higher order in $\hbar$ for $x \in \mathfrak{u}$

Example ($\mathscr{U}_\hbar(\mathfrak{su}(2))$)

- generators: $E_\hbar, F_\hbar, K_\hbar = e^{\pi\sqrt{-1}\hbar H}$
- relations: $K_\hbar E_\hbar K_\hbar^{-1} = e^{2\pi\sqrt{-1}\hbar} E_\hbar, K_\hbar F_\hbar K_\hbar^{-1} = e^{-2\pi\sqrt{-1}\hbar} F_\hbar,$
  $E_\hbar F_\hbar - F_\hbar E_\hbar = (K_\hbar - K_\hbar^{-1})/(e^{\pi\sqrt{-1}\hbar} - e^{-\pi\sqrt{-1}\hbar})$
- coproduct: $\Delta_\hbar(K_\hbar) = K_\hbar \otimes K_\hbar$
  $\Delta_\hbar(E_\hbar) = E_\hbar \otimes K_\hbar + 1 \otimes E_\hbar, \Delta_\hbar(F_\hbar) = F_\hbar \otimes 1 + K_\hbar^{-1} \otimes F_\hbar$

Unitary structure (when $\hbar \in \sqrt{-1}\mathbb{R}$): $K_\hbar^* = K_\hbar, E_\hbar^* = F_\hbar K_\hbar, F_\hbar^* = K_\hbar^{-1} E_\hbar$

Deformation quantization from matrix coefficients

dual Hopf algebra of matrix coefficients

$\mathscr{O}_\hbar(U) = \langle \mathscr{U}_\hbar(\mathfrak{u}) \to \mathbb{C}, T \mapsto \phi(T\xi) \mid$
$\xi \in V$: (admissible) finite dimensional $\mathscr{U}_\hbar(\mathfrak{u})$-module, $\phi \in V^* \rangle$

$\mathscr{O}_\hbar(U)$ is a Hopf algebra by:

- linear combination from direct sum modules
- product, coproduct by transpose of $\Delta_\hbar$, product in $\mathscr{U}_\hbar(\mathfrak{u})$
- unitary structure from unitary structure and antipode of $\mathscr{U}_\hbar(\mathfrak{u})$

$\langle K_{i,\hbar} \mid i$: vertex of Dynkin diagram$\rangle \subset \mathscr{U}_\hbar(\mathfrak{u}) \rightsquigarrow$ highest weight theory

- 'same classification' of irreducible finite dimensional modules
- $\mathscr{O}_\hbar(U) \cong \bigoplus_{\pi\colon \operatorname{Irr} \mathscr{U}_\hbar(\mathfrak{u})} V_\pi^* \otimes V_\pi$: 'same' coalgebra as $\mathscr{O}(U)$

$\rightsquigarrow$ Coalgebra identification $\mathscr{O}_\hbar(U) = \mathscr{O}(U)$ solves deformation quantization problem

Equivariant quantization

Given Poisson action $U \curvearrowright M$:

Problem (Quantization as $U_\hbar$-algebras)

$\exists$ *deformation quantization* $\mathcal{O}_\hbar(M)$ *as* $\mathcal{O}_\hbar(U)$*-comodule algebra?*

Example (Podleś spheres, $c \geq 0$)

$\mathcal{O}(S^2_{\hbar,c}) = \langle A = A^*, B, B^* \mid BA = q^2 AB, B^*B = A - A^2 + c, BB^* = q^2 - q^4 + c \rangle$

If $c = c(\hbar)$ depends on $\hbar$, $\mathcal{O}(S^2_{\hbar,c(\hbar)})$ is a deformation quantization for

$$\{x_i, x_{i+1}\} = \left(x_1 + \frac{1}{\sqrt{4c(0)+1}} \right) x_{i+2} \; (i \bmod 3)$$

Problem (Quantization of module categories)

$\exists$ *deformation of module category*

{equivariant vector bundle over M} $\curvearrowleft$ Rep U*?*

Categorical framework

Definition (module category)

A (right) *module category* over a tensor category $\mathscr{C}$ is given by:

- linear category $\mathscr{D}$
- bifunctor $\mathscr{D} \times \mathscr{C} \to \mathscr{D}$, $(X, V) \mapsto X \otimes V$
- natural isomorphisms $X \otimes 1 \to X$, $\Psi\colon (X \otimes V) \otimes W \to X \otimes (V \otimes W)$

satisfying pentagon equation, …

Ostrik, De Commer–Y., Neshveyev: (in fin. dim. or unitary setting)
module category $\mathscr{D}$ over Rep $\mathscr{U}_\hbar(\mathfrak{u})$ & $X \in \mathscr{D}$
$\leftrightarrow$ $\mathscr{O}_\hbar(U)$-comodule algebra A s.t. $\mathscr{D}$ ={equivariant A-modules}

- deformation of Rep $\mathfrak{k} \curvearrowleft$ Rep $\mathfrak{u}$ gives a quantization of U/K
- action of Ψ and Drinfeld twist on matrix coefficients
  = coefficients of $\star_\hbar$-product

Braided category from quantum groups

Universal R-matrix $\mathscr{R} = 1 + 2\hbar r + \cdots \in \mathscr{U}_\hbar(\mathfrak{u}) \otimes \mathscr{U}_\hbar(\mathfrak{u})$

$\mathscr{R}\Delta_\hbar(x)\mathscr{R}^{-1} = \text{flip}\, \Delta_\hbar(x)$
$(\Delta_\hbar \otimes \text{id})(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i y_j$ for $\mathscr{R} = \sum_i x_i \otimes y_i$,
$(\text{id} \otimes \Delta_\hbar)(\mathscr{R}) = \mathscr{R}_{12}\mathscr{R}_{13} = \sum_{i,j} x_i x_j \otimes y_i \otimes y_j$

$\beta(\xi \otimes \eta) = \text{flip}\, (\pi \otimes \pi')(\mathscr{R})(\xi \otimes \eta)$ for $\xi \in V_\pi, \eta \in V_{\pi'}$:

- is an intertwiner of (finite dimensional) $\mathscr{U}_\hbar(\mathfrak{u})$-modules
- satisfies the Yang-Baxter equation

The category of finite-dimensional $\mathscr{U}_\hbar(\mathfrak{u})$-modules is:

1. a tensor category, with tensor product module by $\Delta_\hbar$
2. with *braiding* by the action of β

Two viewpoints on deformation

Rep $\mathscr{U}_\hbar(\mathfrak{u})$ is a deformation of Rep U by:

- keeping the structure of $\mathbb{C}$-linear category
- but change how $\pi_1 \otimes \pi_2$ decomposes $\leftrightarrow$ deformation of coproduct

Drinfeld: we could instead:

- keep how $\pi_1 \otimes \pi_2$ decomposes
  $\equiv$ use the same bifunctor $\otimes\colon$ Rep $U \times$ Rep $U \to$ Rep U
- but change the associator $\Phi\colon (\pi_1 \otimes \pi_2) \otimes \pi_3 \cong \pi_1 \otimes (\pi_2 \otimes \pi_3)$

$\rightsquigarrow$ should consider *quasi-Hopf algebra* $(\mathscr{U}(\mathfrak{u}), \Delta, \Phi)$

1. associator Φ: *pentagon equation*
2. braiding from R-matrix $\mathscr{R}$: *hexagon equation* with Φ
   $\rightsquigarrow$ *quasi-triangular* quasi-Hopf algebra
3. deformation of $\mathscr{U}(\mathfrak{u})$ is controlled by *co-Hochschild cohomology*

How to use co-Hochschild cohomology

If $\Phi = \sum_{k=0}^{\infty} \hbar^k \Phi^{(k)} \in (\mathscr{U}(\mathfrak{u})^{\otimes 3})^{\mathfrak{u}}[\![\hbar]\!]$ is an associator in Rep U:

$$\begin{array}{ccccc} & & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 & & \\ & \swarrow^{\Phi_{1,2,3}} & & \searrow^{\Phi_{12,3,4}} & \\ (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 & & \circlearrowleft & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \\ \downarrow{\scriptstyle \Phi_{1,23,4}} & & & & \downarrow{\scriptstyle \Phi_{1,2,34}} \\ V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) & & \xrightarrow[\Phi_{2,3,4}]{} & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) \end{array}$$

and if Φ' is another such, with $\Phi^{(k)} = \Phi'^{(k)}$ for $0 \leq k < n$

$\Rightarrow$ *linearized* condition

$\psi_{1,2,3} + \psi_{1,23,4} + \psi_{2,3,4} = \psi_{12,3,4} + \psi_{1,2,34}$ for $\psi \in \Phi'^{(n)} - \Phi^{(n)}$

$\leftrightarrow$ 3-cocycle in the complex:

$((\mathscr{U}(\mathfrak{u})^{\otimes k})^{\mathfrak{u}} \to (\mathscr{U}(\mathfrak{u})^{\otimes k+1})^{\mathfrak{u}}$, alt. sum of Δ on different legs) $\xrightarrow[\text{chm}]{} (\bigwedge_{\mathbb{C}}^{*} \mathfrak{u})^{\mathfrak{u}}$

$\dim(\bigwedge^3 \mathfrak{u})^{\mathfrak{u}} = 1$ for simple $\mathfrak{u}$ (essential parameter space)

Braided category from configuration spaces

Knizhnik–Zamolodchikov equations

$(\pi_1, V_1), \dots, (\pi_n, V_n)$: representations of U
t_{ij}: action of invariant 2-tensor $t = -\sum_k x_k \otimes x_k \in \mathfrak{u} \otimes \mathfrak{u}$ on $V_i \otimes V_j$
$\rightsquigarrow$ KZ_n-equation

$$\frac{\partial v}{\partial z_i}(z) = \hbar \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} v(z) \quad (1 \le i \le n)$$

on $Y_n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$, $v \colon Y_n \to V_1 \otimes \cdots \otimes V_n$

- $n = 3$: associator $\Phi_\hbar$ by monodromy from '$z_1 = z_2$' to '$z_2 = z_3$'
- $n = 2$: braiding by 180° monodromy $e^{\pi\sqrt{-1}\hbar t_{12}}$ around '$z_1 = z_2$'

Drinfeld: $\mathrm{Rep}\, \mathcal{U}_\hbar(\mathfrak{u}) \cong (\mathrm{Rep}\, U, \Phi_\hbar)$ as braided tensor categories
$\beta^2 \sim e^{2\pi\sqrt{-1}\hbar t_{12}} \sim \mathscr{R}_{21}\mathscr{R}$ contains all information

Involution

Understanding $K = \mathrm{Stab}(p)$ as fixed point subalgebra

Involution $\sigma \in \mathrm{Aut}(\mathfrak{u})$: $\sigma \neq \mathrm{id}, \sigma^2 = \mathrm{id}$
$\rightsquigarrow \mathfrak{k} = \mathrm{Lie}(K)$ is $\mathfrak{u}^\sigma = \mathrm{Lie}(U)^\sigma$; K is U^σ

Irreducible compact symmetric space is either:

type I above construction with compact *simple* Lie group U
type II $M \simeq U \simeq (U \times U)/U$ (Poisson torsor; we don't care)

Type I case:

- $\mathfrak{k} = \mathfrak{u} \cap \mathfrak{g}$ for some *real form* $\mathfrak{g} \subset \mathfrak{u} \otimes \mathbb{C}$
- M has *Hermitian* structure $\Leftrightarrow M \subset U.\mu$ for some $\mu \in \mathfrak{u}^*$

Example (2-sphere)

- $\mathfrak{so}(2;\mathbb{R}) = \mathfrak{su}(2) \cap \mathfrak{sl}(2;\mathbb{R}) \subset \mathfrak{sl}(2;\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$
- $\mathrm{SU}(2) \to \mathrm{SO}(3) \curvearrowright S^2 \subset \mathbb{R}^3 \simeq \mathfrak{su}(2)^*$

Coideal quantization of symmetric spaces

Real form $\mathfrak{g} \subset \mathfrak{u} \otimes \mathbb{C} \rightsquigarrow$ classification by Satake diagrams $\rightsquigarrow$ try to mimic the 'Satake involution' $\theta \in \mathrm{Aut}(\mathfrak{u})$ by $\theta_\hbar \in \mathrm{Aut}(\mathscr{U}_\hbar(\mathfrak{u}))$

- replace Weyl group action $W \curvearrowright \mathfrak{u} \otimes \mathbb{C}$ by *Lusztig braid automorphisms*
- use $\langle K_{i,\hbar} \mid i \rangle \subset \mathscr{U}_\hbar(\mathfrak{u})$ as substitute of maximal torus

$\rightsquigarrow$ Letzter, Kolb: coideal subalgebra $\mathscr{U}_\hbar^\theta(\mathfrak{k}) \subset \mathscr{U}_\hbar(\mathfrak{u})$

- Specializes to $\mathscr{U}(\mathfrak{k}) \subset \mathscr{U}(\mathfrak{u})$
- Dual coideal $\mathscr{O}_\hbar(U/K) = \{f \in \mathscr{O}_\hbar(U) \mid T \triangleleft f = \varepsilon(T)f \text{ for } T \in \mathscr{U}_\hbar^\theta(\mathfrak{k})\}$ is a deformation quantization of U/K
- Extra parameter for Hermitian case: $\mathscr{U}_\hbar^{\theta,t}(\mathfrak{k})$

Example (Dual coideal of Podleś spheres)

$*$-coideals $\mathscr{U}_\hbar^{\theta,t}(\mathfrak{so}(2)) = \langle F_\hbar - e^{-2\pi\sqrt{-1}\hbar} E_\hbar K_\hbar^{-1} + \sqrt{-1} t K_\hbar^{-1} \rangle \subset \mathscr{U}_\hbar(\mathfrak{su}(2))$ for $t \in \mathbb{R}$

Reflection equation

Gurevich, Donin, Mudrov …: *reflection equation*

$$\beta_{V,V} K_1 \beta_{V,V} K_1 = K_1 \beta_{V,V} K_1 \beta_{V,V} \quad (\beta\colon \text{braiding})$$

for $K \in \operatorname{End}_{\mathbb{C}}(V) \rightsquigarrow$ quantum homogeneous spaces

Example (K-matrix for $\mathscr{U}_\hbar(\mathfrak{su}(2))$, $V = \mathbb{C}^2$ (defining representation))

$$K = \begin{pmatrix} \sqrt{-1}t(q^{-1} - q) & -q^{-1/2} \\ q^{-1/2} & 0 \end{pmatrix}, \beta = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1} - q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, q = e^{\pi\sqrt{-1}\hbar}$$

Balagović, Kolb: *universal K-matrix* $\mathscr{K} \in \mathscr{U}_\hbar^{\theta(,t)}(\mathfrak{k}) \otimes \mathscr{U}_\hbar(\mathfrak{u})$

- we should think of $\mathscr{K}$ as natural transformation $\pi_1 \otimes \pi_2^{\sigma_\hbar} \to \pi_1 \otimes \pi_2$ for $\pi_1 \in \operatorname{Rep} \mathscr{U}_\hbar^\theta(\mathfrak{k})$, $\pi_2 \in \operatorname{Rep} \mathscr{U}_\hbar(\mathfrak{u})$, $\sigma_\hbar \in \operatorname{Aut}(\mathscr{U}_\hbar(\mathfrak{u}))$ quantization of involution σ
- $(\varepsilon \otimes \mathrm{id})(\mathscr{K}) = K$ solves the (modified) reflection equation

Braiding on module categories

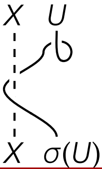
Definition (Ribbon twist braid on module categories)

Given

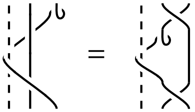
- $(\mathscr{C}, \beta)$: braided tensor category
- $\sigma \colon \mathscr{C} \to \mathscr{C}$: braided autoequivalence
- $\mathscr{D}$: $\mathscr{C}$-module category

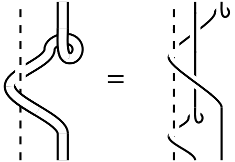
ribbon σ-braid on $\mathscr{D}$ is: $\eta \colon X \otimes \sigma(V) \to X \otimes V$ satisfying

- reflection equation against braiding β
- multiplicativity in V



such that





Type B configuration space

Original KZ equation: on $Y_n = \mathbb{C}^n \setminus$ (type A hyperplanes "$z_i = z_j$")
Cherednik, Leibman, Golubeva–Leksin, Eniquez–Etingof
Adding ribbon twist $\eta \colon X \otimes \sigma(V) \to X \otimes V$: adding "$z_1 = 0$"

Cyclotomic KZ$_n$-equation

- $(\pi_i, V_i)_{i=1}^n$: representations of U, (π_0, X): representation of K
- $t_{0i}^{\mathfrak{k}}$: invariant 2-tensor of $\mathfrak{k}$ on $X \otimes V_i$
- $C_i^{\mathfrak{k}}$: Casimir of $\mathfrak{k}$ on V_i

$$\frac{\partial v}{\partial z_i}(z) = \hbar \left(\frac{2t_{0i}^{\mathfrak{k}} + C_i^{\mathfrak{k}}}{z_i} + \sum_{j \neq i} \frac{t_{ij}^{\mathfrak{k}} \pm t_{ij}^{\mathfrak{u} \ominus \mathfrak{k}}}{z_i \mp z_j} \right) v(z) \quad (1 \leq i \leq n)$$

on $Y_n' = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, z_i \neq 0\}$, $v \colon Y_n' \to X \otimes V_1 \otimes \cdots \otimes V_n$

Type B configuration space

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$\rightsquigarrow$ module category $(\operatorname{Rep} K, \Psi_\hbar) \curvearrowleft (\operatorname{Rep} U, \Phi_\hbar)$, with ribbon σ-twist braid

- $n = 2$: associator $\Psi_\hbar$ by monodromy from '$z_1 = 0$' to '$z_1 = z_2$'
- $n = 1$: $\eta = e^{\pi\sqrt{-1}\hbar(2t_{01}^{\mathfrak{k}} + C_1^{\mathfrak{k}})}$ by 180° monodromy around '$z_1 = 0$'

Picard's approximation

Associator: monodromy of cyclotomic KZ_2 equation from '$w = z_1/z_2 = 0$' to '$w = 1$': $\Psi_\hbar = \lim_{a\to 0} a^{-\hbar A_1} H_a(1-a) a^{\hbar A_0}$;

$$H_a(w) = 1 + \sum_{n=1}^{\infty} \hbar^n \sum_{i_* \in \{-1,0,1\}^n} \int_a^w \omega_{i_1} \cdots \omega_{i_n} A_{i_1} \cdots A_{i_n}$$

with

- $A_{-1} = t^{\mathfrak{k}}_{12} - t^{\mathfrak{u}\ominus\mathfrak{k}}_{12}$, $A_1 = t^{\mathfrak{u}}_{12}$, $A_0 = 2t^{\mathfrak{k}}_{01} + C^{\mathfrak{k}}_1$ (residues)
- $\omega_k = \frac{dx}{x-k}$
- $\int_a^w \omega_{i_1} \cdots \omega_{i_n}$: iterated integral

$$\Rightarrow \Psi_\hbar = 1 + \hbar(\log 2) A_{-1} + \hbar^2 \left(T + \frac{\pi^2}{12}[A_0, A_{-1}] + \frac{\pi^2}{6}[A_1, A_0] \right) + O(\hbar^3)$$

for symmetric element $T \in \mathbb{C}1_{\mathcal{U}(\mathfrak{k})} \otimes \mathcal{U}(\mathfrak{u})^{\otimes 2}$

Conjectural correspondence

Conjecture (DC.–N.–T.–Y.)

For $\hbar \in \sqrt{-1}\mathbb{R}$*:* $\exists$ unitary *module category equivalence*

$$\operatorname{Rep} \mathcal{U}_\hbar^{\theta,t}(\mathfrak{k}) \curvearrowleft \operatorname{Rep} \mathcal{U}_\hbar(\mathfrak{u}) \equiv (\mathbb{C}_\chi . \operatorname{Rep} U, \Psi_\hbar) \curvearrowleft (\operatorname{Rep} U, \Phi_\hbar)$$

up to Drinfeld's equivalence $\operatorname{Rep} \mathcal{U}_\hbar(\mathfrak{u}) \equiv (\operatorname{Rep} U, \Phi)$

Hermitian case: parameter correspondence t *(for Poisson structures)*

$\leftrightarrow$ *character* $\chi\colon \mathfrak{z}(\mathfrak{k}) \to \sqrt{-1}\mathbb{R}$ *(for KZ equation)*

Why care about unitary setting?

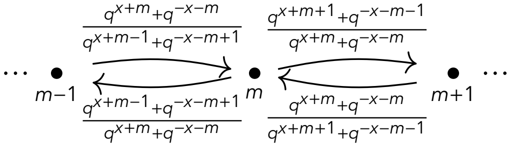
- Good duality between $\mathcal{O}_\hbar(U)$-coactions and module categories
- Hermitian case: $\mathcal{U}_\hbar^{\theta,t}(\mathfrak{k})$ for different t are *isomorphic* comodule algebras, but correspond to different Poisson structures
  $\rightsquigarrow$ they have different $*$-structures (and different unitary reps)
  $\rightsquigarrow$ will give "correct" basepoint of module category

Sanity check: 2-spheres

Theorem (DC.-Y.; cf. Etingof-Ostrik)

Unitary module categories over Rep $\mathscr{U}_\hbar(\mathfrak{su}(2))$ *are classified by certain weighted graphs*

- Rep $\mathscr{U}_\hbar(\mathfrak{su}(2))$: universal rigid unitary category generated by $U_{1/2} = (\pi_{\mathrm{nat}}, \mathbb{C}^2)$ (Temperley–Lieb category)
- weighted graph describing how $U_{1/2}$ and duality morphism $R\colon U_0 \to U_{1/2} \otimes U_{1/2}$
- Podleś spheres correspond to the graphs



$\rightsquigarrow$ brute force classification

Getting relative co-Hochschild cohomology

For general (U, K): better to work over formal Laurent series $\mathbb{C}[h^{-1}, h]\!]$

- work over quasi-Hopf algebra $(\mathscr{U}(\mathfrak{u}), \Phi)$, *quasi-coactions* $(\mathscr{U}(\mathfrak{k}), \Psi')$
- degree-wise comparison of associators Ψ, Ψ' for same Φ

$\rightsquigarrow$ difference is a 2-cocycle in the co-Hochschild complex

$\mathscr{U}(\mathfrak{k})^{\mathfrak{k}} \to (\mathscr{U}(\mathfrak{k}) \otimes \mathscr{U}(\mathfrak{u}))^{\mathfrak{k}} \to (\mathscr{U}(\mathfrak{k}) \otimes \mathscr{U}(\mathfrak{u})^{\otimes 2})^{\mathfrak{k}} \to \cdots$

$\sim_{\mathrm{qis}} (\mathrm{Sym}^c_{\mathbb{C}}(\mathfrak{k}) \otimes_{\mathrm{tw}} \Omega(\mathrm{Sym}^c_{\mathbb{C}}(\mathfrak{u})))^{\mathfrak{k}} \sim_{\mathrm{qis}} (\bigwedge^*_{\mathbb{C}}(\mathfrak{u} \ominus \mathfrak{k}))^{\mathfrak{k}}$ (Koszul duality)

Lemma

For any symmetric pair $(\mathfrak{u}, \mathfrak{k})$, $(\bigwedge^2_{\mathbb{C}}(\mathfrak{u} \ominus \mathfrak{k}))^{\mathfrak{k}} \cong H^1(\mathfrak{k}; \mathbb{C}) \cong \mathfrak{z}(\mathfrak{k}) \otimes \mathbb{C}$

- U/K irred. $\Rightarrow (\mathfrak{u} \ominus \mathfrak{k})^{\mathfrak{k}} = 0 \Rightarrow (\bigwedge^2(\mathfrak{u} \ominus \mathfrak{k}))^{\mathfrak{k}} \simeq H^2(\mathfrak{u}, \mathfrak{k})$ up to linear dual
- look at Serre spectral sequence for $K \to U \to U/K$

Classification at formal setting

Theorem (DC.–N.–Y., in prep.; classification for non-Hermitian $(\mathfrak{u}, \mathfrak{k})$)

all quasi-coaction of $(\mathcal{U}(\mathfrak{u})[\![\hbar]\!], \Phi)$ on $\mathcal{U}(\mathfrak{k})[\![\hbar]\!]$ (with possibly deformed product) are equivalent

non-Hermitian $\Leftrightarrow$ $\mathfrak{k}$ semisimple implies (via Whitehead's lemma):

- $H^2(\mathcal{U}(\mathfrak{k}); \mathcal{U}(\mathfrak{k})) \cong H^2(\mathfrak{k}; {}_{\mathrm{ad}}\mathcal{U}(\mathfrak{k})) = 0 \Rightarrow \mathcal{U}(\mathfrak{k})[\![\hbar]\!]$ does not deform
- $H^1(\mathcal{U}(\mathfrak{k}); M) \cong H^1(\mathfrak{k}; {}_{\mathrm{ad}}M) = 0 \Rightarrow$ coaction does not deform

Theorem (DC.–N.–Y., in prep.; classification for Hermitian $(\mathfrak{u}, \mathfrak{k})$)

Quasi-coaction $(\mathcal{U}(\mathfrak{k})[\![\hbar]\!], \Psi_{\mathrm{KZ}} \lhd \chi)$ for $\chi \in \mathfrak{z}(\mathfrak{k})^*[\![\hbar]\!]$ exhaust the quasi-coactions (Δ, Ψ) by $(\mathcal{U}(\mathfrak{u})[\![\hbar]\!], \Phi_{\mathrm{KZ}})$ with '$\Psi^{(1)} = 0$' up to equivalence

So we should in fact take $\chi = \chi^{(-1)}\hbar^{-1} + \chi^{(0)} + \chi^{(1)}\hbar + \cdots \in \hbar^{-1}\mathfrak{z}(\mathfrak{k})^*[\![\hbar]\!]$?

Classification at formal setting

Theorem (DC.–N.–Y., in prep.; classification for non-Hermitian $(\mathfrak{u}, \mathfrak{k})$)

all quasi-coaction of $(\mathscr{U}(\mathfrak{u})[\![\hbar]\!], \Phi)$ *on* $\mathscr{U}(\mathfrak{k})[\![\hbar]\!]$ *(with possibly deformed product) are equivalent*

Theorem (DC.–N.–Y., in prep.; classification for Hermitian $(\mathfrak{u}, \mathfrak{k})$)

Quasi-coaction $(\mathscr{U}(\mathfrak{k})[\![\hbar]\!], \Psi_{\mathrm{KZ}} \triangleleft \chi)$ *for* $\chi \in \mathfrak{z}(\mathfrak{k})^*[\![\hbar]\!]$ *exhaust the quasi-coactions* (Δ, Ψ) *by* $(\mathscr{U}(\mathfrak{u})[\![\hbar]\!], \Phi_{\mathrm{KZ}})$ *with '*$\Psi^{(1)} = 0$*' up to equivalence*

Need to check $\Psi_{\mathrm{KZ}} \rightsquigarrow \Psi_{\mathrm{KZ}} \triangleleft \chi$ can 'move' cohomology class in $H^2(\mathscr{U}(\mathfrak{k})^{\mathfrak{k}} \to (\mathscr{U}(\mathfrak{k}) \otimes \mathscr{U}(\mathfrak{u}))^{\mathfrak{k}} \to (\mathscr{U}(\mathfrak{k}) \otimes \mathscr{U}(\mathfrak{u})^{\otimes 2})^{\mathfrak{k}} \to \cdots)$

- $(\mathscr{U}(\mathfrak{k}) \otimes \mathscr{U}(\mathfrak{u})^{\otimes *})^{\mathfrak{k}} \sim_{\mathrm{qis}} (\mathrm{Sym}^c_{\mathbb{C}}(\mathfrak{u} \ominus \mathfrak{k}))^{\mathfrak{k}} \cong D^*_{\mathrm{pol}}(U/K)^U \sim_{\mathrm{qis}} T^*_{\mathrm{pol}}(U/K)^U$
- pair $T^2_{\mathrm{pol}}(U/K)^U$ with the canonical 2-form on $U/K \simeq U.\mu$ for $\mu \in \mathfrak{u}^*$
- ribbon σ-braid $\eta = e^{\pi\sqrt{-1}\hbar(2t^{\mathfrak{k}}_{01} + 2(\chi \otimes \mathrm{id})(t)_1 + C^{\mathfrak{k}}_1)}$ is a *complete* invariant