Banach Poisson-Lie groups, Bruhat-Poisson structure of the restricted Grassmannian and KdV hierarchy

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Outline

(M, ω) symplectic manifold with symplectic action of a Lie group G

Moment map

1. with values in the **dual of the Lie algebra of** G
   Application to the eigenvalues of the sum of two Hermitian matrices
2. with values in the **dual of a Poisson-Lie group**
   Application to the singular values of the product of two matrices

From Quantum groups to Poisson-Lie groups

Kirillov, 99, Merits and Demerits of the orbit method:

The first approximation to a quantum group is a so-called **Poisson-Lie group** which is an ordinary Lie group G with an additional structure: the Poisson brackets in the space of functions on G. It arises when we consider the (non-commutative) multiplication in the function algebra on a quantum group as a deformation of the ordinary multiplication in the function algebra on G.

Namely, this deformation has the form

$$f \circ g = fg + h\{f, g\} + o(h),$$

where h is the deformation parameter (considered either as a real number or as a formal variable) and $\{\ ,\ \}$ is the Poisson bracket on G compatible with group multiplication.

In terms of the corresponding bivector c compatibility means

$$c(xy) = L_x^* c(y) + R_y^* c(x),$$

where L_x, R_y are the operators of left and right shifts on G. It follows that c vanishes at the unit element $e \in G$ and is completely determined by its first order jet at e. The latter gives a Lie algebra structure $[\ ,\ ]_*$ on the space $T_e^*(G) = \mathfrak{g}^*$.

Finite-dimensional Poisson Lie groups

M a finite-dimensional manifold. **Poisson bracket** = bilinear $\{\cdot,\cdot\} : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ with

- skew-symmetry $\{f,g\} = -\{g,f\}$
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$

Definition of Poisson-Lie groups

Definition : A Poisson-Lie group B is a Lie group equipped with a Poisson structure compatible with the group multiplication.

Example

Any Lie group G with $\{\cdot,\cdot\} = 0$ is a Poisson Lie group

Coadjoint action in the linear case

Kirillov, 99, Merits and Demerits of the orbit method:

G. In the matrix group case $\mathfrak{g}$ is a subspace of $\mathrm{Mat}_n(\mathbb{R})$, the space of all $n \times n$ real matrices, and the adjoint representation is simply matrix conjugation:

$$Ad(g)X = g\,X g^{-1}, \ \ X \in \mathfrak{g}, \ \ g \in G.$$

Consider now the dual linear space to $\mathfrak{g}$ usually denoted by $\mathfrak{g}^*$. In the matrix case we can use the fact that $\mathrm{Mat}_n(\mathbb{R})$ has a bilinear form $\langle A, B\rangle = \mathrm{tr}(AB)$ which is invariant under conjugation. So, the space $\mathfrak{g}^*$ dual to the subspace $\mathfrak{g} \subset \mathrm{Mat}_n(\mathbb{R})$ can be identified with the quotient space $\mathrm{Mat}_n(\mathbb{R})/\mathfrak{g}^\perp$ where the sign $^\perp$ means the orthogonal complement with respect to $\langle\ ,\ \rangle$. In practice the latter space is often identified with some subspace $V \subset \mathrm{Mat}_n(\mathbb{R})$ which is transverse to $\mathfrak{g}^\perp$ and has the complementary dimension. Let p be the projection of $\mathrm{Mat}_n(\mathbb{R})$ on V parallel to $\mathfrak{g}^\perp$. Then the **coadjoint representation** K, which is dual to the adjoint representation defined above, can be written in the simple form

$$K(g) : F \mapsto p(gFg^{-1}). \qquad (1.1.1)$$

Korteweg-de Vries in Lax form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \qquad \Leftrightarrow \qquad \frac{\partial L}{\partial t} = [(L^{\frac{3}{2}})_+, L]$$

where

- $L = D^2 + u$ with $D = \frac{\partial}{\partial x}$
- For any pseudo-differential operator with Laurent series
  $R = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots,$
  $R_+ = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0.$
- $L^{\frac{1}{2}} = D + a_{-1}D^{-1} + a_{-2}D^{-2} + \ldots$ such that $(L^{\frac{1}{2}})^2 = L$

The n-th KdV hierarchy is the following hierarchy of equations indexed by $k \in \mathbb{N}$

$$\frac{\partial L}{\partial t} = [(L^{\frac{k}{n}})_+, L]$$

There is a 1-1 Correspondance between

1. connected and 1-connected finite-dimensional Poisson Lie groups
2. Lie bialgebras
3. Manin triples

Manin triple

Definition : A Manin triple is a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ and a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$
- $\langle \cdot, \cdot \rangle$ is $\mathfrak{g}$-invariant
- $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$
- $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic for $\langle \cdot, \cdot \rangle$

$\Rightarrow$ Given a Poisson Lie group G, there exists a dual Poisson Lie group with Lie algebra $\mathfrak{g}^*$.

Example 1

The dual Poisson Lie group of a Lie group G with $\{\cdot,\cdot\} = 0$ is $\mathfrak{g}^*$ with its Lie-Poisson structure and $[\cdot,\cdot]_{\mathfrak{g}^*} = 0$

Example 2

$M(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}^+(n)$ with $\langle A, B\rangle = \mathrm{Im}\ \mathrm{Tr} AB$ is a Manin triple.
Moreover

$$GL(n,\mathbb{C}) = U(n) \times B^+(n)$$

because of Iwasawa decomposition. This gives a dressing action

$$\varphi : B^+(n) \times U(n) \to U(n)$$

by $\varphi(b)(k) = k'$ where k' is the unique element of $U(n)$ such that $bk = k'b'$ with $b' \in B^+(n)$.

Bruhat-Poisson structure of finite-dimensional Grassmannians [Lu-Weinstein, 1990]

Proposition :

- $SU(n)$ and $SB(n, \mathbb{C})$ are dual Poisson-Lie groups
- the Grassmannians $Gr(p, n) = SU(n)/S(U(p) \times U(n-p))$ are Poisson homogeneous spaces
- $SB(n, \mathbb{C})$ acts on $Gr(p, n)$ by dressing transformations
- the symplectic leaves of $Gr(p, n)$ are the Schubert-Bruhat cells and coincide with the orbits under the action of $SB(n, \mathbb{C})$

The restricted Grassmannian

$H = L^2(\mathbb{S}^1, \mathbb{C})$
$H = H_+ \oplus H_-$
$H_+ = \{f \in H, f(z) = a_0 + a_1 z + a_2 z^2 + \dots\}$ where $z = e^{i\theta}$
$H_- = \{f \in H, f(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \dots\}$

$B \in GL(H_\pm, H_\pm)$ is Hilbert-Schmidt iff $\mathrm{Tr} B^* B < +\infty$

The restricted Grassmannian Gr_{res} **:** A closed subspace W of H belongs to the restricted Grassmannian Gr_{res} iff

1. $p_- \; : W \to H_-$ is Hilbert-Schmidt,
2. $p_+ \; : W \to H_+$ is Fredholm

The restricted Grassmannian

$$GL_{res} = \{\{ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \} \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$P_{res} = \{\{ \begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \} \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = GL_{res}/P_{res}$$

$$U_{res} = \{\{ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \} \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = U_{res}/(U(H_+) \times U(H_-))$$

Triangular group $B^+_{res} \subset GL_{res}$:

An invertible operator $g \in GL_{res}$ belongs to B^+_{res} if it is upper triangular with respect to the basis $\{z^{-n}, \dots, z^{-1}, 1, z, z^2, \dots\}$ of H, with strictly positive coefficents on the diagonal.

Remark : B^+_{res} acts on Gr_{res}

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

$\Gamma_+ = \{g = e^f, f \text{ holomorphic in } \mathbb{D}, f(0) = 0\}$
$\Rightarrow g = e^{t_1 z + t_2 z^2 + t_3 z^3 + \cdots} \in \Gamma_+$ acts on $L^2(\mathbb{S}^1, \mathbb{C})$ by multiplication and the corresponding operator is a Toeplitz upper triangular operator in B^+_{res}.

$\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(\mathscr{H}) \; : z^n W \subset W\}$.

Proposition 5.13 in [SW85] : The action of Γ_+ on $\mathrm{Gr}^{(n)}$ induces the flows of the KdV hierarchy. For $r \geq 1$, the flow $W \mapsto \exp(t_r z^r) W$ on $\mathrm{Gr}^{(n)}$ induces the r-th KdV flow.

Key Observation : $\Gamma_+ \subset \mathrm{B}^+_{\mathbf{res}}(\mathscr{H})$.
Key Difficulty :
$\mathrm{B}^+_{\mathbf{res}}(\mathscr{H})$ **is modelled on a non-reflexive Banach space.**

What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may by the zero function (Example by Michor and Mumford).
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (cf Queer Poisson Brackets)

Poisson manifold modelled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathscr{C}^1$ away from the origin)

(2) Leibniz rule does not imply existence of Poisson tensor (there exists derivation of order greater then 1)

(3) existence of Hamiltonian vector field is not automatic

Definition of a Banach Poisson manifold

Definition of a Poisson tensor :
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha,\beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition of a Generalized Poisson Manifold :
A Generalized Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Definition of Banach Poisson-Lie groups

Definition : A Banach Poisson-Lie group B is a Banach Lie group equipped with a Generalized Banach Poisson manifold structure such that the group multiplication $m : B \times B \to B$ is a Poisson map, where $B \times B$ is endowed with the product Poisson structure.

Proposition : Let B be a Banach Lie group and $(B, \mathbb{B}, \pi)$ a Banach Poisson structure on B. Then B is a Banach Poisson-Lie group if and only if

1. $\mathbb{B}$ is invariant under left and right multiplications by elements in B,
2. the subspace $\mathfrak{u} := \mathbb{B}_e \subset \mathfrak{b}^*$, where e is the unit element of B, is invariant under the coadjoint action of B on $\mathfrak{b}^*$ and the map

$$\begin{array}{rcccl} \pi_r & : & B & \to & \Lambda^2\mathfrak{u}^*(\mathfrak{u}) \\ & & g & \mapsto & R^{**}_{g^{-1}}\pi_g, \end{array}$$

is a 1-cocycle on B with respect to the coadjoint representation of B in $\Lambda^2\mathfrak{u}^*(\mathfrak{u})$.

Banach Lie bialgebras

Definition : Let $\mathfrak{b}$ be a Banach Lie algebra, and a duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{b},\mathfrak{u}}$ between $\mathfrak{b}$ and a normed vector space $\mathfrak{u}$. One says that $\mathfrak{b}$ is a Banach Lie bialgebra with respect to $\mathfrak{u}$ if

(1) $\mathfrak{b}$ acts continuously by coadjoint action on $\mathfrak{u}$.

(2) there is a 1-cocycle $\theta\ : \mathfrak{b} \to \Lambda^2\mathfrak{u}^*(\mathfrak{u})$ with respect to the adjoint representation of $\mathfrak{b}$ on $\Lambda^2\mathfrak{u}^*(\mathfrak{u})$, i.e. satisfying

$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = \ & \theta(y)(\mathrm{ad}^*_x\alpha,\beta)+\theta(y)(\alpha,\mathrm{ad}^*_x\beta)\\ & -\theta(x)(\mathrm{ad}^*_y\alpha,\beta)-\theta(x)(\alpha,\mathrm{ad}^*_y\beta)\end{aligned}$$

where $x,y\in\mathfrak{b}$ and $\alpha,\beta\in\mathfrak{u}$.

Banach Lie bialgebras versus Manin triple

Definition : [A. A. Odzijewicz, T. Ratiu, 2003]
We will say that $\mathfrak{b}$ is a Banach Lie-Poisson space with respect to $\mathfrak{u}$ if $\mathfrak{u}$ is in duality with $\mathfrak{b}$ and is a Banach Lie algebra $(\mathfrak{u}, [\cdot,\cdot]_{\mathfrak{u}})$ which acts continuously on $\mathfrak{b}$ by coadjoint action.

Theorem :
Consider two Banach Lie algebras $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ and $(\mathfrak{u}, [\cdot,\cdot]_{\mathfrak{u}})$ in duality. Denote by $\mathfrak{g}$ the Banach space $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$ with norm $\|\cdot\|_{\mathfrak{g}} = \|\cdot\|_{\mathfrak{b}} + \|\cdot\|_{\mathfrak{u}}$. The following assertions are equivalent.

(1) $\mathfrak{b}$ is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to $\mathfrak{u}$;

(2) $(\mathfrak{g}, \mathfrak{b}, \mathfrak{u})$ is a Manin triple for the natural non-degenerate symmetric bilinear map

$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{\mathfrak{g}} : & \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{K} \\ & (x,\alpha) \times (y,\beta) & \mapsto & \langle x,\beta\rangle_{\mathfrak{b},\mathfrak{u}} + \langle y,\alpha\rangle_{\mathfrak{b},\mathfrak{u}}. \end{array}$$

Duality pairing between $\mathfrak{b}_{\mathrm{res}}(\mathscr{H})$ and $\mathfrak{u}_{1,2}(\mathscr{H})$

$L_{\mathrm{res}}(\mathscr{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), B \text{ and } C \text{ Hilbert-Schmidt}\}$
$L_{1,2}(\mathscr{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A \text{ and } C \text{ Trace class}, B \text{ and } C \text{ Hilbert-Schmidt}\}$

Definition : [T. Goliński, A. Odzijewicz, 2010]
For $A = \begin{pmatrix} A_{++} & A_{-+} \\ A_{-+} & A_{--} \end{pmatrix} \in L_{1,2}(\mathscr{H})$, define the restricted trace of A by $\mathrm{Tr}_{\mathrm{res}} A = \mathrm{Tr} A_{++} + \mathrm{Tr} A_{--}$.

Proposition 2.1 in [GO10] : $\forall A \in L_{1,2}(\mathscr{H})$, $\forall B \in L_{\mathrm{res}}(\mathscr{H})$, $AB \in L_{1,2}(\mathscr{H})$, $BA \in L_{1,2}(\mathscr{H})$ and $\mathrm{Tr}_{\mathrm{res}} AB = \mathrm{Tr}_{\mathrm{res}} BA$.

Proposition : The following continuous bilinear map is a duality pairing between $\mathfrak{b}^+_{\mathrm{res}}(\mathscr{H})$ and $\mathfrak{u}_{1,2}(\mathscr{H})$

$$\begin{array}{rcccl} \langle \cdot, \cdot \rangle_{L_{\mathrm{res}}, L_{1,2}} & : & L_{\mathrm{res}}(\mathscr{H}) \times L_{1,2}(\mathscr{H}) & \longrightarrow & \mathbb{R} \\ & & (A, B) & \longmapsto & \mathrm{Im}\ \mathrm{Tr}_{\mathrm{res}}(AB). \end{array}$$

Example of bounded operator with unbounded triangular truncation

$$\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \frac{1}{n} & \ddots \\
\ddots & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \ddots \\
\ddots & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \ddots \\
\ddots & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ddots \\
\ddots & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \ddots \\
\ddots & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \ddots \\
\ddots & -\frac{1}{n} & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

the triangular truncation is unbounded on the Banach space of trace class operators but Does there exists a trace class operator whose triangular truncation is not trace class?

Consequence of previous slide : $\mathfrak{u}_{1,2}(\mathscr{H})$ is not preserved by the coadjoint action of $\mathfrak{b}^+_{\mathrm{res}}$. No Manin triple structure on $\mathfrak{b}^+_{\mathrm{res}}(\mathscr{H}) \oplus \mathfrak{u}_{1,2}(\mathscr{H})$! How to build a Banach Poisson-Lie group structure on $\mathrm{B}^+_{\mathrm{res}}(\mathscr{H})$?

Proposition : Consider the following map :

$$\begin{array}{cccc} F & : & L_{1,2}(\mathscr{H}) & \rightarrow & \mathfrak{b}^+_{\mathrm{res}}(\mathscr{H})^* \\ & & a & \mapsto & (b \mapsto \Im \mathrm{Tr}\, ab)\,. \end{array}$$

The kernel of F equals $\mathfrak{b}^+_{1,2}(\mathscr{H})$, therefore $L_{1,2}(\mathscr{H})/\mathfrak{b}^+_{1,2}(\mathscr{H})$ injects into the dual space $\mathfrak{b}^+_{\mathrm{res}}(\mathscr{H})^*$. Moreover $L_{1,2}(\mathscr{H})/\mathfrak{b}^+_{1,2}(\mathscr{H})$ is preserved by the coadjoint action of $\mathrm{B}^+_{\mathrm{res}}(\mathscr{H})$ and strictly contains $\mathfrak{u}_{1,2}(\mathscr{H})$ as a dense subspace.

Theorem : Consider the Banach Lie group $\mathrm{B}^{+}_{\mathrm{res}}(\mathscr{H})$, and

1. $\mathfrak{g}_{-} := \mathrm{L}_{1,2}(\mathscr{H})/\mathfrak{b}^{+}_{1,2}(\mathscr{H}) \subset \mathfrak{b}^{+}_{\mathrm{res}}(\mathscr{H})^{*}$,
2. $\mathbb{B} \subset T^{*}\,\mathrm{B}^{+}_{\mathrm{res}}(\mathscr{H})$, $\mathbb{B}_{b} := R^{*}_{b^{-1}}\mathfrak{g}_{-}$,
3. $\pi_{r} \; : \mathrm{B}^{+}_{\mathrm{res}}(\mathscr{H}) \to \Lambda^{2}\mathfrak{g}^{*}_{-}(\mathfrak{g}_{-})$ defined by

$$\pi_{r}(b)([x_1]_{\mathfrak{b}^{+}_{1,2}}, [x_2]_{\mathfrak{b}^{+}_{1,2}}) = \Im\mathrm{Tr}(b^{-1}\, p_{\mathfrak{u}_2}(x_1)\, b) \left[p_{\mathfrak{b}^{+}_{2}}(b^{-1}\, p_{\mathfrak{u}_2}(x_2)\, b)\right],$$

4. $\pi(b) = R^{**}_{b}\pi_{r}(b)$.

Then $\left(\mathrm{B}^{+}_{\mathrm{res}}(\mathscr{H}), \mathbb{B}, \pi\right)$ is a Banach Poisson-Lie group.

Theorem : Consider the Banach Lie group $\mathrm{U}_{\mathrm{res}}(\mathscr{H})$, and

1. $\mathfrak{g}_+ := \mathrm{L}_{1,2}(\mathscr{H})/\mathfrak{u}_{1,2}(\mathscr{H}) \subset \mathfrak{u}^*_{\mathrm{res}}(\mathscr{H})$,
2. $\mathbb{U} \subset T^*\,\mathrm{U}_{\mathrm{res}}(\mathscr{H})$, $\mathbb{U}_g = R^*_{g^{-1}}\mathfrak{g}_+$,
3. $\tilde{\pi}_r \; : \mathrm{U}_{\mathrm{res}}(\mathscr{H}) \to \Lambda^2\mathfrak{g}^*_+(\mathfrak{g}_+)$ defined by

$$\tilde{\pi}_r(g)([x_1]_{\mathfrak{u}_{1,2}(\mathscr{H})}, [x_2]_{\mathfrak{u}_{1,2}(\mathscr{H})}) = \Im\mathrm{Tr}(g^{-1}\, p_{\mathfrak{b}_2^+}(x_1)\, g)\left[p_{\mathfrak{u}_2}(g^{-1}\, p_{\mathfrak{b}_2^+}(x_2)\, g)\right],$$

4. $\tilde{\pi}(g) = R^{**}_g\tilde{\pi}_r(g)$.

Then $(\mathrm{U}_{\mathrm{res}}(\mathscr{H}), \mathbb{U}, \pi)$ is a Banach Poisson-Lie group.

Poisson geometry of the restricted Grassmannian and dressing action leading to the KdV equation

Theorem [A.B.T] : The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}) = \mathrm{U}_{\mathrm{res}}(\mathscr{H})/\,\mathrm{U}(\mathscr{H}_+) \times \mathrm{U}(\mathscr{H}_-) = \mathrm{GL}_{\mathrm{res}}(\mathscr{H})/\,\mathrm{P}_{\mathrm{res}}(\mathscr{H})$ carries a natural Poisson structure such that :

1. the projection $p\ :\ \mathrm{U}_{\mathrm{res}}(\mathscr{H}) \to \mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ is a Poisson map,
2. the natural action of $\mathrm{U}_{\mathrm{res}}(\mathscr{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ is a Poisson map,
3. the following right action of $\mathrm{B}^+_{\mathrm{res}}(\mathscr{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ is a Poisson map :

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}) \times \mathrm{B}^+_{\mathrm{res}}(\mathscr{H}) & \to & \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}) \\ (g\,\mathrm{P}_{\mathrm{res}}(\mathscr{H}), b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\mathrm{res}}(\mathscr{H}). \end{array}$$

4. the symplectic leaves of $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ are the Schubert cells and are the orbits of $\mathrm{B}^+_{\mathrm{res}}(\mathscr{H})$.

$\Rightarrow$ the action of $\Gamma^+ \subset \mathrm{B}^+_{\mathrm{res}}(\mathscr{H})$ is Poisson and generates the KdV hierarchy.

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