Quantization of subgroups of the affine group

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Quantization of Poisson–Lie groups and Lie bialgebras

Recall that a Poisson–Lie group is a Lie group G with a Poisson bracket $\{\cdot, \cdot\}$ such that the mutliplication map $m\colon G \times G \to G$ is a Poisson map.

In the formal deformation setting, a quantization of G is a Hopf algebra structure on $C^\infty(G)[[h]]$ coinciding with the classical one modulo h and such that

$$m_h(f, g) - m_h(g, f) = \{f, g\}h \mod h^2.$$

Typically we assume that the coproduct, mapping a function f into $(g, h) \mapsto f(gh)$, does not deform. This is not a restriction for connected reductive Lie groups.

On the dual side, a Poisson–Lie structure on G corresponds to a Lie bialgebra structure on $\mathfrak{g}$, namely, a skew-symmetric map $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that δ is a 1-cocycle ($\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$) and $\delta^*\colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket.

A quantization of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a Hopf algebra structure on $U\mathfrak{g}[[h]]$ coinciding with the classical one modulo h and such that

$$\Delta_h(X) - \Delta_h^{\mathrm{op}}(X) = \delta(X)h \mod h^2.$$

Typically we assume that the coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X$ (for $X \in \mathfrak{g}$) does not change. Again, this is not a restriction if $\mathfrak{g}$ is reductive.

Etingof–Kazhdan quantization theorem

Theorem (Etingof–Kazhdan)

Any Lie bialgebra can be quantized.

There is nothing remotely close to this in the analytic setting. Some problems:

- Arguments/formulas are difficult to make sense of when h is not a formal parameter.

- There are real obstacles in the analytic setting.

Example: quantum $SU(1,1)$ group

The classical $SU(1,1)$ group consists of complex matrices $\begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$ of determinant one. The quantized algebra of functions is generated by two elements α, γ such that

$$\alpha\gamma = q\alpha\gamma, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$

$$\alpha^*\alpha - \gamma^*\gamma = 1, \quad \alpha\alpha^* - q^2\gamma^*\gamma = 1.$$

The coproduct is defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad \text{for} \quad (u_{ij})_{ij} = \begin{pmatrix} \alpha & q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

Woronowicz's no-go theorem

Theorem (Woronowicz)

Given two irreducible representation π_1 and π_2 of the relations for α and γ on Hilbert spaces H_1 and H_2, there is no way to define the tensor product representation, that is, a representation π such that $\pi(u_{ij})$ extends

$$\sum_k \pi_1(u_{ik}) \otimes \pi_2(u_{kj}).$$

As was realized by Korogodskii and later completed by Kustermans–Koelink, the right group to quantize in this case is $SU(1,1) \rtimes \mathbb{Z}/2\mathbb{Z}$, the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$

Coboundary Lie bialgebras

In many cases, given a Lie bialgebra $(\mathfrak{g}, \delta)$, the cobracket δ is a coboundary, that is,

$$\delta(X) = -X.r \quad \text{for some} \quad r \in \mathfrak{g} \otimes \mathfrak{g}.$$

The axioms for the cobracket are equivalent to $\mathfrak{g}$-invariance of $r + r_{21} \in g \otimes \mathfrak{g}$ and of

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}.$$

Twisting of coproduct

Theorem (Drinfeld)

Assume $r \in \mathfrak{g} \otimes \mathfrak{g}$ *is such that* $r_{21} = -r$ *and*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Then there exists an element $J = 1 + \frac{1}{2}rh + \cdots \in (U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ *such that*

$$(J \otimes 1)(\Delta \otimes \iota)(J) = (1 \otimes J)(\iota \otimes \Delta)(J).$$

A quantization of $(\mathfrak{g}, \delta)$ *can then be defined by letting*

$$\Delta_h = J\Delta(\cdot)J^{-1}.$$

Such a J is called a twist for $U\mathfrak{g}$ or a dual 2-cocycle on G.

As was shown by Belavin–Drinfeld, all r-matrices r as above are obtained in the following way, up to passing to a Lie subalgebra:

Assume $B(X,Y)$ is a nondegenerate (as a bilinear form) skew-symmetric 2-cocycle on $\mathfrak{g}$. Take a basis $(X_i)_i$ in $\mathfrak{g}$, put $B_{ij} = B(X_i, X_j)$, and then define

$$r = \sum_{i,j} (B^{-1})_{ij} X_i \otimes X_j.$$

When there exists B which is a coboundary, so that $B(X,Y) = f([X,Y])$ for some $f \in \mathfrak{g}^*$, then $\mathfrak{g}$ is called a Frobenius Lie algebra. Note that the assumption of nondegeneracy of B in this case is equivalent to openness of the coadjoint orbit of f.

Examples: $ax + b$ group

Consider the group of real invertible matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Its Lie algebra is spanned by two elements X, Y such that $[X, Y] = Y$. Consider the r-matrix

$$r = X \otimes Y - Y \otimes X.$$

As was shown by Coll–Gerstenhaber–Giaquinto and Ogievetsky the corresponding Lie bialgebra can be quantized using the twist

$$J = \exp\{X \otimes \log(1 + hY)\}.$$

An analogue of the above twist in the analytic setting was found by Stachura:

$$J = \exp\big\{X \otimes \log|1 + iY|\big\}\,\mathrm{Ch}\big(1 \otimes \mathrm{sgn}(1 + iY), I \otimes 1\big),$$

where $I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (acting in the regular representation) and $\mathrm{Ch} : \{-1, 1\} \times \{-1, 1\} \to \{-1, 1\}$ is the unique nontrivial bicharacter.

Dual cocycle in analytic setting

In operator algebras, a locally compact quantum group G is represented by a von Neumann algebra $M = L^\infty(G)$ together with a (strongly operator continuous and $*$-preserving) coproduct $\Delta\colon M \to M\bar{\otimes}M$ satisfying a number of axioms. We also have a group von Neumann algebra $\hat{M} = W^*(G)$ with coproduct $\hat{\Delta}$, and $L^\infty(\hat{G}) = \hat{M}$.

A dual unitary 2-cocycle on G is a unitary element $\Omega \in W^*(G)\bar{\otimes}W^*(G)$ such that

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

By a result of De Commer, we then have a new locally compact quantum G_Ω such that its group von Neumann algebra is $W^*(G)$ equipped with the new coproduct $\Omega\hat{\Delta}(\cdot)\Omega^*$.

Galois objects

Every dual unitary cocycle Ω gives rise to a G-Galois object $L^\infty(G)_\Omega$, a deformation of the function algebra $L^\infty(G)$ by Ω. It is equipped with an action of G (coaction of $L^\infty(G)$). This action is free and transitive in an appropriate sense.

Algebraically, given a finite dimensional Hopf algebra $\mathcal{H}$ and a twist J for the dual Hopf algebra $\mathcal{H}^*$, we can deform the algebra $\mathcal{H}$ by defining a new product m_J by

$$m_J(x \otimes y)(a) = m(\hat{\Delta}(a)J^{-1}).$$

The coproduct Δ defines a right coaction α of $\mathcal{H}$ n $\mathcal{H}_J$ with trivial coinvariants and such that the map

$$\mathcal{H}_J \otimes \mathcal{H}_J \to \mathcal{H}_J \otimes \mathcal{H}, \quad a \otimes b \mapsto \alpha(a)(b \otimes 1),$$

is a linear isomorphism.

Existence of dual cocycles

Theorem

Let G be a second countable locally compact group. For a unitary representation π of G on a Hilbert space H, TFAE:

- $(B(H), \mathrm{Ad}\,\pi)$ *is a G-Galois object;*
- π *is irreducible and the regular representation of G is a multiple of π.*

Furthermore, if these conditions are satisfied, then there exists a unique up to coboundary dual unitary 2-cocycle Ω on G such that

$$B(H) \cong L^\infty(G)_\Omega$$

as G-algebras. If G is nontrivial, the cocycle Ω is not a coboundary, moreover, the coproduct $\Omega\hat{\Delta}(\cdot)\Omega^*$ is not cocommutative.

Frobenius type subgroups of the affine group

From now on we consider semidirect products $Q \ltimes V$, where Q and V are locally compact second countable groups, V is abelian, and there exists an element $\xi_0 \in \hat{V}$ such that the map

$$\phi\colon Q \to \hat{V}, \quad q \mapsto q^\flat \xi_0,$$

is a measure class isomorphism, where $\flat$ denotes the dual action.

Theorem (Ooms)

Assume Q is a Lie group, ρ is a representation of Q on a vector space V of the same dimension as Q. Then the Lie algebra of $Q \ltimes V$ is Frobenius if and only if the action of Q defined by the contragredient representation ρ^c has an open orbit in V^*.

Some examples

1) Let $\mathbb{K}$ be a locally compact field, τ be an order-two ring automorphism of $\mathrm{Mat}_n(\mathbb{K})$. Consider the quaternionic type group $\mathbb{H}_n^{\pm}(\mathbb{K},\tau)$ given by the subgroup of $\mathrm{GL}_{2n}(\mathbb{K})$ of elements of the form

$$\begin{pmatrix} A & B \\ \pm\tau(B) & \tau(A) \end{pmatrix}, \quad A, B \in \mathrm{Mat}_n(\mathbb{K}).$$

Set

$$V = \mathrm{Mat}_n(\mathbb{K}) \oplus \mathrm{Mat}_n(\mathbb{K}) \quad \text{and} \quad Q = \mathbb{H}_n^{\pm}(\mathbb{K},\tau).$$

Here both (V, Q) and $(\hat{V}, Q)$ satisfy our assumptions.

2) For $n \geq 1$ and $m \geq 2$, let

$$\hat{V} = \underbrace{\mathsf{Mat}_n(\mathbb{K}) \oplus \cdots \oplus \mathsf{Mat}_n(\mathbb{K})}_{m} \quad \text{and} \quad Q = \begin{pmatrix} 1 & \cdots & 0 & \mathsf{Mat}_n(\mathbb{K}) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \mathsf{Mat}_n(\mathbb{K}) \\ 0 & \cdots & 0 & \mathrm{GL}_n(\mathbb{K}) \end{pmatrix}.$$

Then, the dual pair (V, Q) satisfies our assumptions but the pair $(\hat{V}, Q)$ does not.

3) Let $\mathbb{A}$ be a nondiscrete second countable locally compact ring such that the set $\mathbb{A}^\times$ of invertible elements is of full Haar measure. Then, the pair $(\hat{\mathbb{A}}, \mathbb{A}^\times)$ satisfies our assumptions. As a concrete example, choose a sequence $\{p_n\}_n$ of prime numbers such that

$$\sum_n \frac{1}{p_n} < \infty$$

Then we can take $\mathbb{A} = \prod'_n(\mathbb{Q}_{p_n}, \mathbb{Z}_{p_n})$.

Kohn–Nirenberg quantization

The Kohn–Nirenberg quantization is initially defined as the continuous injective linear map

$$\mathrm{Op}_{\mathrm{KN}} : S'(V \times \hat{V}) \longrightarrow L\big(S(V), S'(V)\big),$$

from tempered Bruhat distributions on $V \times \hat{V}$ to continuous linear operators from the Bruhat-Schwartz space on V to tempered Bruhat distributions on V by the formula

$$\mathrm{Op}_{\mathrm{KN}}(F)\varphi(v) := \int_{V\times\hat{V}} e^{i\langle \xi, v-v'\rangle}\, F(v',\xi)\,\varphi(v')\,dv'd\xi$$

for $F \in S'(V \times \hat{V})$, $\varphi \in S(V)$. The distributional kernel of the operator $\mathrm{Op}_{\mathrm{KN}}(F)$ is therefore given by

$$(v,v') \mapsto \big((1 \otimes \mathcal{F}_V^*)F\big)(v', v-v'),$$

where $\mathcal{F}_V$ denotes the partial Fourier transfrom in coordinate V.

From this we see that the Kohn–Nirenberg quantization map $\mathrm{Op}_{\mathrm{KN}}$ extends to a unitary isomorphism from $L^2(V \times \hat{V})$ onto $\mathrm{HS}(L^2(V))$. Hence, the Hilbert space $L^2(V \times \hat{V})$ can be endowed with an associative product

$$f_1 \star_0 f_2 := \mathrm{Op}^*_{\mathrm{KN}}\big(\mathrm{Op}_{\mathrm{KN}}(f_1)\,\mathrm{Op}_{\mathrm{KN}}(f_2)\big).$$

By suitably normalizing the Haar measure on $G = Q \ltimes V$ we get a unitary operator operator $U\colon L^2(V \times \hat{V}) \to L^2(G)$ defined by

$$(Uf)(q,v) = f\big(v, \phi(q)\big).$$

Using this unitary we can then transport the product $\star_0$ to a product $\star$ on $L^2(G)$. This product is equivariant with respect to the action of G on itself by left translations.

Theorem

The distributional kernel of $\star$ defines a unitary operator in $W^*(G)\bar{\otimes}W^*(G)$. It is a dual unitary 2-cocycle on G cohomologous to the dual cocycle defined in the previous theorem.
Explicitly, this cocycle equals

$$(\mathcal{F}_V^*\otimes 1)\, U_\Xi\, (\mathcal{F}_V\otimes 1),$$

where U_Ξ is the unitary defined by the transfromation

$$\Xi : Q\times\hat V\times G\to Q\times\hat V\times G,\quad (q,\xi,g)\mapsto \big(q,\xi,\phi^{-1}(\xi_0+\xi)g\big).$$

For $Q=\mathbb{R}^\times$, $V=\mathbb{R}$ and $\xi_0=-1$, the cohomologus cocycle $(\hat R\otimes\hat R)(\Omega_{21}^*)$ coincides with Stachura's cocycle.

Bicrossed products

A pair (G_1, G_2) of closed subgroups a locally compact second countable group G is called a matched pair if $G_1 \cap G_2 = \{e\}$ and $G_1 G_2$ is a subset of G of full measure.

Given such a pair, we have almost everywhere defined measurable left actions α of G_1 and β of G_2 on the measure spaces G_2 and G_1, resp., such that

$$gs^{-1} = \alpha_g(s)^{-1}\beta_s(g) \ \text{ for } \ g \in G_1,\ s \in G_2.$$

We can then define a bicrossed product a locally compact quantum group with the function algebra

$$G_1 \ltimes_\alpha L^\infty(G_2).$$

Twistings as bicrossed products

Theorem

With $G = Q \ltimes V$ and Ω as before, the quantum group $G_\Omega = (W^*(G), \Omega\hat{\Delta}(\cdot)\Omega^*)$ is isomorphic to the bicrossed product quantum group defined by the matched pair $(Q, \xi_0 Q \xi_0^{-1})$ of subgroups of $Q \ltimes \hat{V}$.