Riesz products and spectral decompositions for generic measure preserving transformations.

M – measure space (with finite or σ-finite Lebesgue non-atomic measure)

$\mathrm{Aut}(M)$ – the group of measure preserving transformations.

Convergence: $g_j \to g$ if for any measurable A, B we have

$$\mu(g_j(A) \cap B) \to g(A) \cap B$$

Baire category is well-defined for this group. Also a 'generic' in the sense of Baire category makes sense.

Preiminary construction.

Fix a sequence $m_1, m_2, \dots$, where $m_j \geqslant 2$.

Consider the set $\mathbb{O} = \mathbb{O}(m_1, m_2, \dots)$ of all 'integers'

$$\dots a_3 a_2 a_1;$$

in the mixed digital system infinite to the left; we have

$$0 \leqslant a_j \leqslant m_j - 1$$

we can add such numbers and multiply such numbers.

Formally, we consider the inverse limit of residue rings

$$\dots \longleftarrow \mathbb{Z}/m_1 m_2 m_3 \mathbb{Z} \longleftarrow \mathbb{Z}/m_1 m_2 \mathbb{Z} \longleftarrow \mathbb{Z}/m_1 \mathbb{Z}$$

Below we need only in the operation $A \mapsto A + 1$

Notation: the ball of level k is the set of all points having the fixed end:

$$\dots z_{k+2} z_{k+1} a_k \dots a_1.$$

The number of balls of level k is $m_1 \dots m_k$.

The *measure* of a ball of level k is $(m_1 \dots m_k)^{-1}$. thus the measure of $\mathbb{O}$ is 1.

9 in each position denotes the largest possible digit

The function

$$nine(u) = nine(\dots u_2 u_1)$$

is number of 9-s at the end of $\dots u_2 u_1$.

The function

$$prenine(u)$$

is the last non-nine digit in $\dots u_k \dots u_1$ (if $\dots u_2 u_1 \neq \dots 999$)

Construction of transformations. Fix m_1, m_2, …and additionally parameters $\gamma_j^i \geqslant 0$, where $j \in \mathbb{N}$, and $0 \leqslant i \leqslant m_j - 2$.

We allow one digit after the comma

$$\dots u_2 u_1, w$$

where $w \leqslant a_{nine(u)}^{prenine(u)}$.

We get a (locally compact totally disconnected) space $\mathbb{V} = \mathbb{V}(m, a)$

A **ball** $B_k[u_k \dots u_1, w]$ of level k is the set of all numbers

$$x_{k+2} x_{k+1} u_k \dots u_k, w$$

with fixed tail $u_k \dots u_k, w$. Its measure is $(m_1 \dots m_k)^{-1}$. Denote by h_k the number of balls of level k,

$$h_k = m_k h_{k-1} + \sum_{i=0}^{m_k-2} a_k^i, \qquad h_0 = 1.$$

Thus we get a measure on $\mathbb{V}$. (measure can be finite or infinite)

The measure preserving transformation is $Q : z \mapsto z + 0,1$.

For each ball $B_k[u_k \dots u_1, w]$ we define

$$\Upsilon_k[u_k \dots u_1, w]$$

being the number of additions

$$0 + 0,1 + \cdots + 0,1 = \dots 0u_k \dots u_1, w$$

$$\Theta_k(z) := \sum_{p=0}^{m_k-1} z^{ph_{k-1}+\sum_{i<p} a^i_{k-1}}.$$

Consider the sequence of measures

$$d\varkappa_n := \prod_{k=1}^{n} \frac{1}{m_k}\Theta_k(z)\overline{\Theta_k(z)}\ d\dot{z}$$

and the weak limit (*Riesz product*)

$$d\varkappa(z) = \prod_{k=1}^{\infty} \frac{1}{m_k}\Theta_k(z)\overline{\Theta_k(z)}\ d\dot{z} := \lim_{n\to\infty} d\varkappa_n.$$

Denote by $\mathcal{F} = \mathcal{F}(\mathbb{V})$ the space of locally constant compactly supported functions on $\mathbb{V}$. Consider the map $\mathcal{R}$ from $\mathcal{F}$ to $C^\infty(S^1)$ that sends each indicator function $I_k[\boldsymbol{\sigma}]$ of a ball σ of level k to a function $\Phi_k[\boldsymbol{\sigma}]$ on S^1 given by

$$\Phi_k[\boldsymbol{\sigma}] := \frac{z^{\Upsilon_k(\boldsymbol{\sigma})}}{\prod_{j=1}^{k} \Theta_k(z)}. \qquad (0.1)$$

In particular, $\Psi_0[\mathbf{0}] = 1$.

Theorem. If functions $\Theta_k(z)$ have no zeros on the circle, then the map $\mathcal{R}$ extends to a unitary operator

$$L^2(\mathbb{V}) \to L^2(S^1, d\varkappa)$$

and

$$\mathcal{R}(Q^{-1}\mathbf{s}) = z\,\mathcal{R}(\mathbf{s}).$$