Imaginary time Hamiltonian flows and applications to Quantization, Kahler geometry and representation theory

José Mourão

CAMGSD, Mathematics Department, IST

XXXVIII Workshop on Geometry Methods in Physics

Bialowieza, June 30 – July 6

On work in collaboration with

T. Baier, J. Hilgert, O. Kaya & J. P. Nunes

Index

1. Kähler manifolds and space of Kähler metrics

Kähler manifolds (M, ω, J) are symplectic manifolds (M, ω) with a compatible complex structure J, *ie* such that the bilinear form $\gamma(X, Y) := \omega(X, JY)$ is a Riemannian metric, so that we get 3 structures, (M, ω, J, γ).

A symplectic manifold may not have compatible complex structures but if it has one it has an infinite dimensional space of them.

The symplectic form is automatically of type $(1,1)$ for any compatible complex structure and has a locally defined J-dependent Kähler potential k_J,

$$\omega = \frac{i}{2}\,\partial_J \overline{\partial_J}\, k_J$$

Example - $\mathbb{CP}^n$
The Fubini-Study Kähler form reads

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\, k_{FS} = \frac{i}{2}\partial\bar{\partial}\log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

On the other hand, fixing J, on a compact manifold M, two J–compatible closed 2–forms ω and ω' are in the same cohomology class iff their Kähler potentials k, k' can be chosen to differ by a global function

$$k' = k + \phi, \qquad \phi \in C^\infty(M)$$

Then, the space of Kähler forms compatible with J, in the given cohomology class, is naturally given by

$$\mathcal{H}(\omega, J) \cong \mathcal{H}_0(\omega, J)/\mathbb{R} := \left\{ \phi \in C^\infty(M) \;:\; \omega_\phi = \omega + \frac{i}{2} \partial_J \bar{\partial}_J \phi > 0 \right\} /\mathbb{R}$$

This (infinite dimensional) manifold (convex open nbd of 0 in $C^\infty(M)$) has a natural metric introduced by Mabuchi,

$$G_\phi(h_1, h_2) = \int_M h_1 \, h_2 \, \frac{\omega_\phi^n}{n!}, \qquad \text{where } \omega_\phi = \omega + \frac{i}{2} \partial_J \bar{\partial}_J \phi \tag{1}$$

Example - $\mathbb{CP}^n$

The space of Kähler potentials on $\mathbb{CP}^n$ with fixed cohomology class is then given by the following open convex subset of $C^\infty(\mathbb{CP}^n)$:

$$\begin{aligned}\mathcal{H}_0(\omega_{FS}, J) &= \left\{\phi \in C^\infty(\mathbb{CP}^n) \ : \ \omega_\phi = \frac{i}{2}\partial\bar{\partial}\left[\log(1 + |z_1|^2 + \cdots + |z_n|^2) + \phi\right] > 0\right\} \\ &\subset C^\infty(\mathbb{C}\mathcal{P}^n)\end{aligned} \quad (2)$$

So $\mathcal{H}_0(\omega, J)$ has trivial topology but a very interesting metric.

As showed by Donaldson, the Mabuchi metric is the metric associated with the realization of $\mathcal{H}(\omega, J)$ as the symmetric space $\mathrm{Ham}_{\mathbb{C}}(M, \omega)/\mathrm{Ham}(M, \omega)$.

2. Geometry on the space of Kähler metrics on M and HCMA

Let M be compact and simply connected.

Theorem 1 (Mabuchi/Semmes/Donaldson) *The geodesics for the metric (1) are the stationary points of the energy functional*

$$E(\phi) = \int_0^1 \int_M \dot{\phi}_t^2 \, dt \, \frac{\left(\omega + \frac{i}{2}\partial\bar{\partial}\phi_t\right)^n}{n!} \,.$$

Donaldson further shows that $\mathcal{H}$ with the Mabuchi metric is an infinite dimensional analogue of the symmetric spaces of non–compact type of the form

$$PSL(N, \mathbb{C})/PSU(N) \,,$$

with $PSL(N, \mathbb{C})$–invariant metric.

(I) First argument supporting $\mathcal{H}(\omega, J) \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:
$\mathcal{H}$ as a quotient
Let

$$Ham_{\mathbb{C}}(M,\omega) \;:=\; \left\{\psi \in Diff(M) \,:\, \left(\psi^{-1}\right)^{*}(\omega) \in \mathcal{H}\right\} \quad (3)$$

$$\underset{\subset}{\text{not a subgr}} \; \text{Diff}(M)$$

we obtain, from Moser theorem, that the following map is a bijection

$$\begin{aligned} Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega) \;&\cong\; \mathcal{H}(\omega, J) \\ [\psi] \;&\mapsto\; \left(\psi^{-1}\right)^{*}(\omega)\,. \end{aligned}$$

(II) Second argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M, \omega)/Ham(M, \omega)$:
Tangent space at a Kähler potential

We have $T_{\omega_\phi}\mathcal{H} \cong C^\infty(M)/\mathbb{R}$
and

$$\mathcal{L}_{JX_H^{\omega_\phi}}\left(\omega_\phi\right) = -\frac{i}{2}\partial\bar{\partial}H,$$

(III) Third argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:
Curvature formulas

Theorem 2 (Donaldson) *The curvature of the Mabuchi metric (1) and the sectional curvature read*

$$R_\phi(f_1,f_2)f_3 = -\frac{1}{4}\{\{f_1,f_2\}_\phi, f_3\}_\phi, \; K_\phi(f_1,f_2) = -\frac{1}{4}||\{f_1,f_2\}_\phi||^2_\phi .$$

for all $f_1, f_2, f_3 \in T_\phi\mathcal{H}$, *where*

$$T_\phi\mathcal{H} = \left\{ f \in C^\infty(M) \, : \, \int_M f\,\omega_\phi^n = 0 \right\} \cong Lie(Ham(M,\omega_\phi)) .$$

Remark

The above expressions are in full agreement with the formulas for the curvature of the finite dimensional symmetric spaces $K_{\mathbb{C}}/K$,

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$$

and

$$K(X,Y) = -\frac{1}{4}||[X,Y]||^2 .$$

for all $X, Y, Z \in T_0 K_{\mathbb{C}}/K \cong iLie(K) \cong Lie(K)$ and the Lie brackets are calculated in $Lie(K)$.

(IV) Fourth argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:

Limit of spaces of Bergman metrics

$$\mathcal{H} = \lim_{N\to\infty} PSL(N,\mathbb{C})/PSU(N)$$

Let $L \to M$ be a very ample holomorphic line bundle with $c_1(L) = \frac{1}{2\pi}[\omega]$ and $\dim H^0(M, L^p) = d_p + 1$. Every ordered basis $\underline{s} = (s_0, \dots, s_{d_p})$ defines an embedding $i_{\underline{s}} : M \to \mathbb{CP}^{d_p}$ and the p–th root of the pullback of the Fubini-Study hermitian structure defines an hermitian structure on $L \longrightarrow M$,

$$FS_p(\underline{s}) = \left(i^*_{\underline{s}}\, h_{FS}\right)^{1/k} = \frac{1}{(\sum_{j=0}^{d_p} |s_j(z)|^2)^{1/p}}$$

$$\mathcal{B}_p \;=\; \left\{k(\underline{s}) = -\log(FS_p(\underline{s})) : \underline{s} \text{ a basis of } H^0(M,L^p)\right\} \cong GL(d_p+1)/U(d_p+1)\,.$$

Every $k \in \mathcal{H}_0(\omega, J)$ defines an inner product on $H^0(M, L^p)$ via the Hermitean structure $h_p(k) = e^{-pk}$

$$\langle s, \tilde{s} \rangle_k = \int_M h_p(k)(s, \tilde{s}) \frac{\omega_k^n}{n!}$$

Let $\underline{s}_p(k)$ be an orthonormal basis for $\langle \cdot, \cdot \rangle_k$ and let

$$\begin{array}{rcl} \mathcal{H}_0(\omega, J) & \longrightarrow & \mathcal{B}_p \cong GL(d_p+1)/U(d_p+1) \\ k & \mapsto k_p & = -\log \left(FS_p(\underline{s}_p(k)) \right) . \end{array}$$

Then, we have:

Theorem 3 *(Tian, 1990)*

$$k = \lim_{p \to \infty} k_p \, .$$

(V) Fifth argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$: Geodesic equations on $\mathcal{H}$ and imaginary time Hamiltonian flows

The Homogeneuous Complex Monge–Ampère (HCMA) equation is the following nonlinear equation on a complex $(n+1)$–dimensional manifold N

$$MA(K) := \det\left(\frac{\partial^2 K}{\partial z_j \partial \bar{z}_l}\right) = 0,$$

or, equivalently,

$$\left(\partial\bar{\partial}K\right)^{n+1} = 0. \tag{4}$$

It is a very difficult equation with very few (genuinly complex) rank n solutions known.

Even for $n = 1$ the HCMA equation is very nontrivial.

Relation with geodesics on $\mathcal{H}$

Let us for simplicity consider the case $n = 1$.

Functions K on (open subsets of) $N = [0, T] \times S^1 \times M$, which are

(a) S^1–invariant and

(b) such that $g_{1\bar{1}} = \frac{\partial^2 K}{\partial z \partial \bar{z}}(t, z, \bar{z}) > 0$

so that $k_t = K(t, \cdot)$ is a path of Kähler potentials on M).

The HMA equation for these functions coincides with the geodesic equations for k_t.

$$\begin{aligned}\left(\partial_N \bar{\partial}_N K\right)^2 = 0 \quad &\Leftrightarrow \\ \frac{\partial^2 K}{\partial t^2}\frac{\partial^2 K}{\partial z \partial \bar{z}} - \left|\frac{\partial^2 K}{\partial t \partial \bar{z}}\right|^2 \quad &= \quad 0 \Leftrightarrow \\ \frac{\partial^2 K}{\partial t^2} = g^{1\bar{1}}\left|\frac{\partial^2 K}{\partial t \partial \bar{z}}\right|^2 \quad &\Leftrightarrow \\ \ddot{k}_t = \|\nabla \dot{k}_t\|^2_{k_t} & \qquad (5)\end{aligned}$$

Analogously in higher dimensions

Ellaborating on an idea of Semmes and Donaldson we will show how to reduce the Cauchy problem for the Mabuchi geodesics, with $k_t = k + \phi_t$.

$$\begin{cases} \ddot{k}_t &= ||\nabla \dot{k}_t||^2_{k_t} \\ k_0 &= k, \\ \dot{k}_0 &= -H, \end{cases} \qquad k_t \in C^\infty(U), H \in C^\infty(M). \tag{6}$$

to the problem of finding the integral curves of the Hamiltonian vector field X_H^ω, where $\omega = \frac{i}{2}\partial\bar{\partial}k$, followed by "rotating" t to the imaginary axis (in the complex t–plane)

$$\exp(sX_H^\omega) \rightsquigarrow \exp(\sqrt{-1}tX_H^\omega) \in Ham_{\mathbb{C}}(M,\omega) \overset{??}{\subset} Diff(M), \tag{7}$$

in a certain way.

To make sense of (7) we will be working on the symplectic picture (see section 3 below) in which ω is fixed and the complex structure J_t changes.

Then the imaginary time integral curves in (7) are solutions of the following coupled system

$$\begin{cases} \dot{x}_t = J_t X_H^\omega = \nabla^{\gamma_t} H \\ J_t = \left(\exp(\sqrt{-1} t X_H^\omega)\right)^* (J). \end{cases} \tag{8}$$

A solution of (6) is given formally by the Kähler potential ϕ_t of ω_t in

$$\omega_t = \left(\left(\exp(\sqrt{-1} t X_H^\omega)\right)^{-1}\right)^* (\omega)\,. \tag{9}$$

This is the so called Donaldson formal solution of the CHMA.

The problem is that to find the imaginary time flow $\exp(\sqrt{-1}\, t X^{\omega}_{H})$ with (8) is equivalent to solving a complicated system of PDE (see [Burns–Lupercio–Uribe, 2013]). So it is not clear what have we gaigned in going from the original HCMA (6) to the coupled system (8).

NO PDE needed!

3. Explicit "rotation" of hamiltonian flows to imaginary time

The missing step to transform Donaldson formal solution of the Cauchy problem (6) for the HCMA given by (9) into an actual solution is the rotation

$$\exp(sX_H^\omega) \rightsquigarrow \exp(\sqrt{-1}tX_H^\omega).$$

In the present section we will describe our solution to this problem obtained in [M-Nunes, IMNR2015]. One key technical tool to rotate the flow is the Gröbner theory of Lie series of vector fields (which is still very popular in numerical methods in astronomy – satelite motion, exoplanets, etc).

Theorem 4 (M-Nunes) *Let (M,J) be a compact complex manifold and $X \in \mathcal{X}(M)$ an analytic vector field. There exist local charts $((z_j),U)$ in neighourhoods of every point and $T > 0$ such that for all $\tau \in \mathbb{D}_T$ the functions*

$$z_j^\tau = e^{\tau X} z_j = u_j^\tau(x,y) + \sqrt{-1} v_j^\tau(x,y), \tag{10}$$

where $x_j = \Re(z_j), y_j = \Im(z_j), u_j^\tau(x,y) = \Re(z_j^\tau), v_j^\tau(x,y) = \Im(z_j^\tau)$, define on $V \subset U$ local J_τ–holomorphic charts for a unique complex structure J_τ and there exists a unique diffeomorphism $\varphi_\tau^{X,J}$ such that

$$J_\tau = \left(\varphi_\tau^{X,J}\right)^* (J) \text{ and } z_j^\tau = \left(\varphi_\tau^{X,J}\right)^* \left(z_j\right) .$$

The complex time flow is then given explicitly locally by

$$\varphi_\tau^{X,J}(x,y) = (u^\tau(x,y), v^\tau(x,y)), \tag{11}$$

We see that, as expected, if $\tau = t \in \mathbb{R}$ the complex time flow is J–independent and coincides with the real time flow

$$\varphi_t^{X,J} = \varphi_t^X \, .$$

Theorem 5 (M-Nunes) *Consider the Cauchy problem for the HCMA (6) on* $I \times M$ *(where we are already supressing the angular coordinate of the first factor in* $A \times M$*). Then by replacing* exp$(\sqrt{-1} t X_H^\omega)$*, in the formal solution (9), by* $\varphi_{it}^{X_H,J}$ *obtained as in Theorem 4 one obtains a solution of the HCMA.*

4. Infinite dimensional spaces of new solutions of the HCMA on an elliptic curve

Let us now illustrate the method of the previous section and obtain an infinite dimensional family of nonsymmetric solutions of the HCMA on an elliptic curve $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with J^ϵ defined by the holomorphic coordinate $z = x + \epsilon \sin(x) + iy$, where $|\epsilon| < 1$ and (x, y) are the standard periodic coordinates on $\mathbb{T}^2$. We choose $\omega = dx \wedge dy$, which corresponds to choosing an initial Kähler potential $k_0 = k$. Let $\dot{k}_0(x, y) = -H(y)$, a (periodic) function of y only.

Remark 6 The calculations remain simple if we consider the more general initial Kähler structure

$$z = u(x, y) + iv(x, y)$$

but we keep H as a function of y (or x) alone. ◊

To solve the HCMA with the given initial conditions let us FIRST find the real time hamiltonian flow of H. Since

$$X_H^\omega = H'(y)\,\frac{\partial}{\partial x}, \quad \text{we obtain} \quad \varphi_t^{X_H^\omega}(x,y) = \left(x + tH'(y), y\right).$$

Second we restrict this flow to J^ϵ–holomorphic coordinates, $z = x + \epsilon\,\mathrm{sin}(x) + iy$, and rotate it to the imaginary axis:

$$\begin{aligned} z^{it} &= \left(\varphi_s^{X_H}\right)^* (z)|_{s=\sqrt{-1}t} = \qquad (12) \\ &= x + \epsilon\,\mathrm{sin}(x)\,\mathrm{cosh}(tH'(y)) + i\left(y + tH'(y) + \epsilon\,\mathrm{cos}(x)\,\mathrm{sinh}(tH'(y))\right) \end{aligned}$$

We see that, as expected, though the evolution is linear in the geodesic (= imaginary hamiltonian) time t only in the symmetric (with respect to translations in x) case $\epsilon = 0$, the explicit expressions can be found also for $\epsilon \neq 0$ and for any function $H(y)$. From (12) we see that

$$\varphi_{it}^{X_H, J^\epsilon}(x+\epsilon\,\mathrm{sin}(x), y) = \left(x + \epsilon\,\mathrm{sin}(x)\,\mathrm{cosh}(tH'(y)), y + tH'(y) + \epsilon\,\mathrm{cos}(x)\,\mathrm{sinh}(tH'(y))\right).$$

5. Applications of special geodesics in the space of Kähler metrics

The main applications so far:

1. Donaldson–Tian theory of stability of Kähler manifolds
   Extend Kempf–Ness to the "action" of $Ham_{\mathbb{C}}(M, \omega)$ on $\mathcal{H}$.

2. Quantization and generalized Coherent State Transforms (gCST)

3. Representation theory

4. Hele–Shaw flow on Riemann surfaces

5. Geometry dependence of fractional quantum Hall trial states

We will concentrate on the applications 2 and 3. In fact they are intimately linked via geometric quantization.
Application 5 is work in progress with Gabriel Matos and João P. Nunes.

5.1 Geometric quantization and gCST

Geometric quantization is mathematically perhaps the best defined quantization

$$(M, \omega), \quad \frac{1}{2\pi\hbar}[\omega] \in H^2(M, \mathbb{Z})$$

Prequantum data: (L, ∇, h), $L \to M$, $F_\nabla = i\omega$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M, L) = \overline{\left\{ s \in \Gamma^\infty(M, L) : ||s||^2 = \int_M h(s, s) \, \frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\widehat{f}^{prQ} = Q(f) = -i\hbar \nabla_{X_f} + f$

This almost works! But the Hilbert space is too large, the representation is reducible. We need a smaller Hilbert space:
Prequantization $\Rightarrow$ Quantization

When we go from prequantization to quantization we have to add, to the classical geometric data, an additional piece of data, called a polarization. The problem is that the space of all polarizations is infinite dimensional and includes in particular $\mathcal{H}$ and also that the quantum theory does depend on this choice. So $\mathcal{H}(M,\omega)$ becomes like (part of the) space of quantizations of the classical physical system (M,ω) and we will use the geometry of $\mathcal{H}$ to relate different quantizations.

(uncorrected) Quantization: $\mathcal{H}^{prQ}$ is too large. Choose a polarization $\mathcal{P}$, $\mathcal{P}_m \subset T_m M^{\mathbb{C}}$ - Lagrangian and the distribution is integrable. The quantum Hilbert space is

$$\mathcal{H}^{\mathrm{Q}}_{\mathcal{P}} = \{\psi \in \mathcal{H}^{\mathrm{prQ}} \ : \ \nabla_X \psi = 0, \forall X \in \Gamma(\mathcal{P})\}$$

$$\hat{f} \text{ acts on } \mathcal{H}^{\mathrm{Q}}_{\mathcal{P}} \Leftrightarrow [X_f, \Gamma(\mathcal{P})] \subset \Gamma(\mathcal{P}) \Leftrightarrow f \in \mathcal{O}_{\mathcal{P}}$$

$\mathcal{O}_{\mathcal{P}}$–Poisson subalgebra of $\mathcal{P}$-quantizable observables.

Two extreme cases

- $\mathcal{P}$ is Kähler, $\mathcal{P} \cap \overline{\mathcal{P}} = \{0\}$, and is equivalent to a compatible complex structure, I. The pair (∇, I) defines on L the structure of an holomorphic line bundle $\mathcal{L}_I \to M$ and
$$\mathcal{H}^{\mathrm{Q}}_{\mathcal{P}} = \mathcal{H}^{\mathrm{Q}}_{I} \cong H^0(M, \mathcal{L}_I)$$

- $\mathcal{P}$ is real $\mathcal{P} = \overline{\mathcal{P}}$ (in this case we will allow polarizations with certain kinds of singularities) and defines a singular foliation of M by Lagrangian leaves.
If the leaves have noncontractible loops then polarized sections will be supported only on those leaves with trivial ∇-holonomy, called Bohr-Sommerfeld (BS) leaves.

Remarks

Importance of choice of polarization: Choosing a polarization is the same as choosing local maximal subalgebras of Poisson commuting real or complex observables

$$F_1, \ldots, F_n \Leftrightarrow \mathcal{P} =< X_{F_1}, \cdots, X_{F_1} > \quad on \quad U \subset M$$

which act diagonally. This is known to lead to inequivalent quantum theories (the same observables with different quantum spectra).

Once we choose the F_j we have two fundamental properties.

P$_1$ The quantum Hilbert space associated with this choice

$$\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}} = \Big\{ s \text{``} \in \text{''} \mathcal{H}^{\mathsf{prQ}} : \nabla_{X_{F_j}} s = 0 \,,\, j = 1, \dots n \Big\} \text{``} \subset \text{''} \mathcal{H}^{\mathsf{prQ}}$$

$$\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}} = \Big\{ s = \psi(F_1, \dots, F_n)\, e^{-k_{\mathcal{P}}}, ||s|| < \infty \Big\}$$

P$_2$ The observables F_j that define the polarization act diagonally on $\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}}$. Indeed, if $O = O(F_1, \dots, F_n)$, then

$$\widehat{O}^{\mathsf{prQ}}\, \psi(F_1, \dots, F_n)\, e^{-k_{\mathcal{P}}} = O(F_1, \dots, F_n)\, \psi(F_1, \dots, F_n)\, e^{-k_{\mathcal{P}}}$$

If $\mathcal{P}$ is Kähler than $\mathcal{P} \Leftrightarrow I$ and the local functions F_j defining $\mathcal{P}$ are in fact local $I-$holomorphic coordinates.

Then they define the Kähler metric and in fact the curvature of that metric is, in some sense, measuring the deviation from having the choice of picking $Re(F_j)$ and $Im(F_j)$ being canonically conjugate pairs. For example the polarization on $\mathbb{R}^2$ defined by

$$z = x + if(p)$$

is Kähler iff $f'(p) > 0\,, \forall p$ and, in that case, the scalar curvature of the Kähler metric is

$$Sc(\gamma) = -\left(\frac{1}{f'(p)}\right)'' .$$

Quantum Bundle

Let $\mathcal{T}$ be the space of polarizations. In $\mathcal{T}$ we have $\mathcal{H}$ and in its boundary real and mixed polarizations.

Geometric quantization gives us the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T}$$

and the tools to study the dependence of quantization on the choice of the complex structure or, more generaly, on the choice of polarization.

Integral transforms relating different quantizations

Step 1 Given two polarizations $\mathcal{P}_1$ and $\mathcal{P}_2$ we can hope to link them with a geodesic on $\mathcal{T}$, i.e. that there exists an Hamiltonian $H \in C^\omega(M)$ such that

$$\mathcal{P}_2 = e^{it\,\mathcal{L}_{X_H}}\big|_{t=1}\,\mathcal{P}_1$$

Step 2 Then geometric quantization gives us a way of lifting the geodesics to the quantum bundle and thus construct construct an integral transform

$$C_{\mathcal{P}_1\mathcal{P}_2} : \mathcal{H}^Q_{\mathcal{P}_1} \longrightarrow \mathcal{H}^Q_{\mathcal{P}_2}$$

Case 1 If the transform in Step 2 is unitary, as for example if $M = T^*K$, K a Lie group of compact type, $\mathcal{P}_1$ the vertical or Schrödinger (real) polarization and $\mathcal{P}_2$ the standard Kähler polarization (called adapted) for the bi-invariant metric on K and H is the norm square of the $K-$moment map, then we have established the equivalence of the two quantizations $\mathcal{H}^Q_{\mathcal{P}_1}$ and $\mathcal{H}^Q_{\mathcal{P}_2}$.

Case 2 If not then we may still use the transform to study the difference of the two quantizations. In cases in which we have "preferred polarizations" (i.e. preferred quantizations) we may use the transforms in step 2 to "correct" other, nonpreferred, quantizations.

Some terminology – CST versus KSH

If the starting polarization $\mathcal{P}_1$ in step 2 above is real and $\mathcal{P}_2$ is Kähler then the integral transform is called a Coherent State Transform (CST) and H is called a Thiemann complexifier. The name CST comes from the fact that they generalize the Segal–Bargmann CST for $M = \mathbb{R}^{2n}$

$$C_{\mathcal{P}_{\mathsf{Sch}}\mathcal{P}_{Fock}} : L^2(\mathbb{R}^n, dx) \longrightarrow \mathcal{H}L^2(\mathbb{C}^n, e^{-|z|^2} dxdy)\,.$$

In general the transforms $C_{\mathcal{P}_1\mathcal{P}_2}$ are called Kostant–Souriau–Heisenberg (KSH) transforms or generalized coherent state transforms.

5.2 Links with representation theory

The two best known (real) polarizations on $T^*\mathbb{R}^n \cong T\mathbb{R}^n$ are the **vertical (or Schrödinger)**

$$\begin{aligned} \mathcal{P}_{\mathsf{Sch}} &= \langle X_{x_j} = -\frac{\partial}{\partial p_j} \quad j = 1, \ldots, n \rangle \\ \mathcal{H}^Q_{\mathcal{P}_{\mathsf{Sch}}} &= \{\psi(x_1, \ldots, x_n),\ ||\psi|| < \infty\} = \\ &= L^2\left(\mathbb{R}^n, dx\right) \end{aligned}$$

and the **momentum polarizations**

$$\begin{aligned} \mathcal{P}_{\mathsf{mom}} &= \langle X_{p_j} = \frac{\partial}{\partial x_j} \quad j = 1, \ldots, n \rangle \\ \mathcal{H}^Q_{\mathcal{P}_{\mathsf{mom}}} &= \left\{\tilde{\psi}(p)\, e^{ip\cdot x}, \quad p \in \mathbb{R}^n,\ ||\tilde{\psi}|| < \infty\right\} \cong \\ &\cong L^2(\mathbb{R}^n) \end{aligned}$$

$t = \pi/2$

There are of course many different ways of getting from $\mathcal{P}_{\text{Sch}}$ to $\mathcal{P}_{\text{mom}}$ and we don't need to go through Kähler polarizations as we can use a simple (real) canonical transformation generated by H$= 1/2(||x||^2 + ||p||^2)$ (at time $t = \pi/2$) to achieve that (and define fractional Fourier Transform on the way).

$t = i\infty$

Alternatively (to get the Fourier transform) we can go into the Kähler world by using a one–parameter "group" of imaginary time canonical transformations taking us from $\mathcal{P}_{\text{Sch}}$ to $\mathcal{P}_{\text{mom}}$ in infinite imaginary time $t = i\infty$ generated by $H = ||\mu||^2/2 = ||p||^2/2$. For $n = 1$

$$e^{it\,\mathcal{L}_{X_H}}\,\langle X_x\rangle = \langle X_{x+itp}\rangle = \langle X_x + itX_p\rangle \stackrel{\text{t}\to\infty}{\longrightarrow} \langle X_p = X_\mu = \frac{\partial}{\partial x}\rangle$$

and this has a much wider range of applicability

Let (M, ω, T^n, μ) be an integrable system with an effective hamiltonian action of the n–dimensional torus T^n with moment map μ.

A$_1$ For all starting real, mixed or Kähler polarizations $\mathcal{P}_1$ for which the limit

$$\lim_{t \to \infty} e^{it\, \mathcal{L}_{X_H}} \mathcal{P}_1$$

exists, it is equal to the momentum polarization, $\mathcal{P}_{\mathrm{mom}} = < X_\mu >$. This includes cases for which the Schrödinger polarization does not exist (as is the case of toric manifolds).

A$_2$ Taking $H = ||\mu||^2$ and $t = i\infty$ can also be extended to (compact) nonabelian groups leading to a natural mixed polarization $\mathcal{P}_{KW}$ and a corresponding result for $\mathcal{H}^{\mathbb{Q}}_{\mathcal{P}_{KW}}$ in those cases.

In particular for $M = T^*K$ we get [Kirwin–Wu (unpublished)] and [Baier–Hilgert-Kaya-M-Nunes (reproved and extended to symmetric spaces and soon to all $K_{\mathbb{C}}$–manifolds with invariant Kähler structure)] that $\mathcal{P}_{KW}$ is a mixed polarization generated by Casimir functions of μ and complex valued functions on $K \times \mathcal{O}_\xi$ that are pullbacks of meromorphic functions on $\mathcal{O}_\xi \times \mathcal{O}_\xi$. For $\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}_{KW}}$ we get

$$\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}_{KW}} = \sum_{\lambda \in \Lambda^+_{\mathbb{Z}}} \delta(\mu^{\mathsf{Kir}}(g) - \lambda - \rho)\, H^0(L_{\lambda+\rho} \boxtimes L_{\lambda^*+\rho}),$$

where $\mu^{\mathsf{Kir}}(g)$ means the image of $g \in K_{\mathbb{C}}$ under the Kirwan moment map.

Thank you!