Nonlinear coherent states associated with a measure on the positive real half line

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S. Twareque Ali

FIGURE : S. Twareque Ali in Bialowieża

This work is a development of project that Z. Mouayn during his visit to Concordia university on September 2014 has started with Professor S. Twareque Ali. Later, on January 2016, Professor S. Twareque Ali passed away. This work is dedicated to his memory.

Plan

I- Nonlinear coherent states

- The *canonical* coherent states CCS are written in terms of the so-called Fock basis $\{\varphi_n\}_{n=0}^{\infty}$

$$\vartheta_z = \left(e^{z\overline{z}}\right)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\overline{z}^n}{\sqrt{n!}} \varphi_n, \tag{1}$$

  for each fixed $z \in \mathbb{C}$ where $e^{z\overline{z}}$ is chosen so that to ensure the normalization condition $\langle \vartheta_z | \vartheta_z \rangle = 1$.
- The basis vectors $\{\varphi_n\}$ are orthonormal in the underlying quantum states Hilbert space $\mathcal{H}$, often termed a Fock space.

- The so-called *deformed coherent states* also known as *nonlinear coherent states* (NLCS) in the quantum optical literature are then defined by replacing the factorial $n!$ by $x_n! := x_1 x_2 ... x_n$, $x_0 = 0$, where $\{x_n\}_{n=0}^{\infty}$ is an infinite sequence of positive numbers and, by convention, $x_0! = 1$.
- Thus, for each $z \in \mathcal{D}$ some complex domain, one defines a generalized version of CCS as

$$\vartheta_z = (\mathcal{N}(z\bar{z}))^{-1/2} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{x_n!}} \varphi_n, \quad z \in \mathcal{D} \tag{2}$$

where again

$$\mathcal{N}(z\bar{z}) = \sum_{n=0}^{+\infty} \frac{(z\bar{z})^n}{x_n!} \tag{3}$$

is an appropriate normalizing constant.
- It is clear that the vectors ϑ_z are well defined for all z for which the sum (3) converges, i.e. $\mathcal{D} = \{z \in \mathbb{C}, |z| < R\}$ where $R^2 = \lim_{n\to+\infty} x_n$, with $R > 0$ could be finite or infinite, but not zero.

- As usual, we require that there exists a measure $d\eta$ on $\mathcal{D}$ for which the resolution of the identity condition,

$$\int_{\mathcal{D}} |\vartheta_z\rangle\langle\vartheta_z| \mathcal{N}(z\bar{z}) d\eta(z,\bar{z}) = \mathbf{1}_{\mathcal{H}} \tag{4}$$

holds. Here, $|\vartheta_z\rangle\langle\vartheta_z| \equiv T_z$ means the rank one operator $T_z : \mathcal{H} \longrightarrow \mathcal{H}$ defined by $T_z[\psi] = \langle\vartheta_z|\psi\rangle\vartheta_z,\ \psi \in \mathcal{H}$.

- In order for (4) to be satisfied, the measure $d\eta$ has to have the form

$$d\eta(z,\bar{z}) = \frac{d\theta}{2\pi} d\lambda(r), \quad z = re^{i\theta} \tag{5}$$

where the measure $d\lambda$ is a solution of the moment problem

$$\int_0^R r^{2n} d\lambda(r) = x_n!, \quad n = 0, 1, 2, ..., \tag{6}$$

provided that such a solution exists.

- In most of the cases that occur in practice, the support of the measure $d\eta$ is the whole domain $\mathcal{D}$, meaning that $d\lambda$ is supported on $(0, R)$.

II- NLCS associated with a measure

- We start from a family of measures of the form

$$d\mu_\beta(r) := r^\beta d\mu(r) \tag{7}$$

where $d\mu(r)$ is a positive measure, does not depend on β, supported by $(0, L)$, where L could be infinite.

- Assume that this measure have finite moments for all order and denote $\mathcal{D}_L = \{\xi \in \mathbb{C}, |\xi| < L\}$. Set

$$\mu_\beta := \int_0^L d\mu_\beta(r), \ \beta \geq 0. \tag{8}$$

- From these moments we consider the sequence of numbers :

$$x_n^\beta = \frac{\mu_{n+\beta}}{\mu_{n+\beta-1}}, \quad x_n^\beta! := x_n^\beta x_{n-1}^\beta \dots x_1^\beta = \frac{\mu_{n+\beta}}{\mu_\beta}, \quad x_0^\beta! \equiv 1, \quad n = 1, 2, 3, ..., \tag{9}$$

Definition

For $\beta \geq 0$, the vectors $\vartheta_{z,\beta} \equiv |z;\beta\rangle \in \mathcal{H}$ are defined through the superposition

$$\vartheta_{z,\beta} := \left(\mathcal{N}_\beta(z\overline{z})\mu_\beta\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\overline{z}^n}{\sqrt{x_n^\beta!}}\varphi_n, \quad (10)$$

for each z in $\mathcal{D}_\beta = \{\xi \in \mathbb{C}, |\xi| < R_\beta = (\lim_{n\to+\infty} x_n^\beta)^{\frac{1}{2}}\}$. For brevity, these states will be denoted $\mu_\beta-$NLCS.

- The normalization constant is given by

$$\mathcal{N}_\beta(z\bar{z}) = \sum_{n=0}^{+\infty} \frac{(z\overline{z})^n}{x_n^\beta!} \quad (11)$$

which converges for each $z \in \mathcal{D}_\beta$.

Proposition

Let $d\mu_\beta$ be a measure given in (7) and assuming that $\mathcal{D}_L \subseteq \mathcal{D}_\beta$, then the μ_β-NLCS in (10) satisfy the resolution of the identity operator of $\mathcal{H}$ as

$$\int_{\mathcal{D}_L} |\vartheta_{z,\beta}\rangle\langle\vartheta_{z,\beta}| \, d\eta_\beta(z,\bar{z}) = \mathbf{1}_{\mathcal{H}} \tag{12}$$

where $d\eta_\beta(z,\bar{z}) = (2\pi)^{-1} d\theta \mathcal{N}_\beta(z\bar{z}) d\mu_\beta(z\bar{z})$, $z \in \mathcal{D}_L$.

Proof. This can be proved by direct calculation and by using (8).

III- A polynomials realization of the basis $\{\varphi_n\}_{n=0}^{\infty}$

- On this Hilbert space $\mathcal{H}$ we may define the operators a_β and $a_\beta^\dagger$ by

$$a_\beta \varphi_n = \sqrt{x_n^\beta}\varphi_{n-1}, \quad a_\beta \varphi_0 = 0, \qquad a_\beta^\dagger \varphi_n = \sqrt{x_{n+1}^\beta}\varphi_{n+1}. \tag{13}$$

  Using these operators, it is now possible to identify the basis vectors $\{\varphi_n\}_{n=0}^{\infty}$ with another family of real orthogonal polynomials by following S. T. Ali & Ismail (2012).

- For this, we make the assumption :

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{x_n^\beta}} = \infty. \tag{14}$$

- Define the operators,

$$Q_\beta = \frac{1}{\sqrt{2}}\,[a_\beta + a_\beta^\dagger]\,, \qquad P_\beta = \frac{1}{i\sqrt{2}}\,[a_\beta - a_\beta^\dagger]\,, \tag{15}$$

analogues of the standard position and momentum operators. The operator Q_β acts on the basis vectors φ_n as

$$Q_\beta \varphi_n = \sqrt{\frac{x_n^\beta}{2}}\,\varphi_{n-1} + \sqrt{\frac{x_{n+1}^\beta}{2}}\,\varphi_{n+1}\,. \tag{16}$$

- If now the sum $\sum_{n=0}^{\infty} \dfrac{1}{\sqrt{x_n^\beta}}$ diverges, the operator Q_β is essentially self-adjoint and hence has a unique self-adjoint extension, which we again denote by Q_β. Thus there is a measure $d\omega_\beta(x)$ on $\mathbb{R}$ such that on the Hilbert space $L^2(\mathbb{R}, d\omega_\beta(x))$, Q^β is just the operator of multiplication by x.

- Consequently, on this space, the relation (16) takes the form

$$x\varphi_n = \sqrt{\frac{x_n^\beta}{2}}\varphi_{n-1} + \sqrt{\frac{x_{n+1}^\beta}{2}}\varphi_{n+1} \qquad (17)$$

which is a two-term recursion relation, familiar from the theory of orthogonal polynomials. Then, the φ_n may be realized as the polynomials obtained by orthonormalizing the sequence of monomials $1, x, x^2, x^3, \ldots$, with respect to this measure $d\omega_\beta$ (using a Gram-Schmidt procedure).

- Let us use the notation $p_n^\beta(x)$ to write the vectors φ_n, when they are so realized, as orthogonal polynomials in $L^2(\mathbb{R}, d\omega_\beta(x))$. Then, for any dw_β-measurable set $\Delta \subset \mathbb{R}$,

$$\langle \varphi_k, E(\Delta)\varphi_n \rangle = \int_\Delta p_k^\beta(x) p_n^\beta(x)\, d\omega_\beta(x), \qquad (18)$$

and

$$\langle \varphi_k, \varphi_n \rangle_{\mathcal{H}} = \int_{\mathbb{R}} p_k^\beta(x) p_n^\beta(x)\, \omega_\beta(x) dx = \delta_{k,n}\,. \qquad (19)$$

Example : The sequence $x_n^\beta := n + \beta$ with $\beta \geq 0$.

- A polynomials realization of the basis $\{\varphi_n\}_{n=0}^{\infty}$ is given by the associated Hermite polynomials

$$\varphi_n^\beta(x) = \frac{2^{-n/2}}{\sqrt{(\beta+1)_n}} H_n(x,\beta) \tag{20}$$

whose orthogonality measure is

$$d\omega_\beta(x) = \left(\sqrt{\pi}\Gamma(\beta+1)\right)^{-1} |D_{-\beta}(ix\sqrt{2})|^{-2} dx \tag{21}$$

where $D_a(\cdot)$ is the parabolic cylinder function.

- Comparing (17), where the sequence x_n^β is chosen to be $n + \beta$, with the three-terms recurrence relation in R. Askey & J. Wimp (1984) :

$$H_{n+1}(x,\beta) = 2xH_n(x,\beta) - 2(n+\beta)H_{n-1}(x,\beta), \tag{22}$$

with $H_{-1}(x,\beta) = 0,\ H_0(x,\beta) = 1$. Where $H_n(x,\beta)$ are the associated Hermite polynomials.

- Their explicit orthogonality relation

$$\int_{\mathbb{R}} \frac{H_n(x,\beta)H_k(x,\beta)}{|D_{-\beta}(ix\sqrt{2})|^2} dx = 2^n \sqrt{\pi}\Gamma(n+\beta+1)\delta_{n,k}, \tag{23}$$

where the parabolic cylinder function

$$D_\beta(z) = \exp\left(-\frac{z^2}{4}\right) \frac{2^{\beta/2}\sqrt{\pi}}{\Gamma(-\beta)} \sum_{k=0}^{+\infty} \frac{(-1)^k \Gamma(k-\beta)}{k!\Gamma\left(\frac{k-\beta+1}{2}\right)} \left(\frac{z}{\sqrt{2}}\right)^k. \tag{24}$$

IV- A class of 2D orthogonal polynomials

- Let $d\mu_\beta$ be the measure given in (7) and μ_β its moments with the normalization $\mu_0 = 1$. For each $\beta \geq 0$, let $\phi_n(r;\beta)$, $n = 0, 1, 2, ...$, be a family of real polynomials, orthogonal with respect to the measure $d\mu_\beta(r)$, that is

$$\int_0^L \phi_n(r;\beta)\, \phi_k(r;\beta)\, d\mu_\beta(r) = \zeta_n(\beta)\, \delta_{k,n} \tag{25}$$

  where $\zeta_n(\beta)$ is a positive sequence.
- The polynomial $\phi_n(r;\beta)$ may also be written as

$$\phi_n(r;\beta) = \sum_{j=0}^{n} c_j(n;\beta)\, r^{n-j}, \quad \phi_0 = 1, \tag{26}$$

  where the $c_j(n;\beta)$ are real coefficients.

- Using the above real polynomials, the authors Ismail & Zhang (2016) have constructed an orthogonal family of polynomials, $P_{n,m}(z,\overline{z};\beta),\ n,m=0,1,2,\ldots$, in the variables $z,\overline{z}\in\mathbb{C}$, by

$$P_{n,m}(z,\overline{z};\beta) = z^{n-m}\phi_m(z\overline{z};n-m+\beta), \quad n\geq m \tag{27}$$
$$= P_{m,n}(\overline{z},z;\beta), \quad m\geq n. \tag{28}$$

Note that (27)-(28) can together be put into a single expression as

$$P_{n,m}(z,\overline{z};\beta)=z^{n-(n\wedge m)}\overline{z}^{m-(n\wedge m)}\phi_{n\wedge m}(z\overline{z};|n-m|+\beta), \quad n,m\in\mathbb{Z}_+, \tag{29}$$

where $n\wedge m$ denotes the smaller of n and m.

- Ismail and Zhang proved that these polynomials form an orthogonal family, in the sense that

$$\frac{1}{2\pi}\int_0^{L^2}\int_0^{2\pi}P_{n,m}(z,\overline{z};\beta)\overline{P_{k,s}(z,\overline{z};\beta)}d\theta d\mu_\beta(r^2)=\zeta_{n\wedge m}\left(|n-m|+\beta\right)\delta_{n,k}\delta_{m,s} \tag{30}$$

where $\zeta_n(\beta)$ is as in (25).

- We note that the biorthogonal polynomials $P_{n,m}(z,\overline{z};\beta)$, $n,m = 0,1,2,...$, defined in (27), span the Hilbert space $L^{2,\beta}(\mathcal{D}_L)$. Its holomorphic and antiholomorphic subspaces, $\mathcal{H}^{\beta}_{\text{hol}}(\mathcal{D}_L)$ and $\mathcal{H}^{\beta}_{\text{a-hol}}(\mathcal{D}_L)$, are spanned by the monomials $P_{n,0}(z,\overline{z},\beta)$, and $P_{0,n}(z,\overline{z},\beta)$, $n = 0,1,2,...$, respectively.
- The above two monomial bases are in fact determinative of the entire set of polynomials $P_{n,m}(z,\overline{z},\beta)$ since they determine the measure $d\mu_\beta$, through the moment relation

$$\int_0^{L^2} |P_{n,0}(z,\overline{z},\beta)|^2 \, d\mu_\beta(r^2) = \int_0^{L^2} r^{2n} \, d\mu_\beta(r^2) = \mu_{n+\beta} \;, \tag{31}$$

and hence of the entire Hilbert space $L^{2,\beta}(\mathcal{D}_L)$.
- We denote the inner product defined on the Hilbert space $L^{2,\beta}(\mathcal{D}_L) := L^2(\mathcal{D}_L, (2\pi)^{-1} d\theta d\mu_\beta(r^2))$ by

$$\langle f, g\rangle_{L^{2,\beta}(\mathcal{D}_L)} = \frac{1}{2\pi}\int_{\mathcal{D}_L} f(z,\bar{z})\overline{g(z,\bar{z})}d\theta d\mu_\beta(z\bar{z}), \quad z = re^{i\theta}. \tag{32}$$

V- Generalized μ_β-NLCS

- From now on, we will use the notation

$$P^\beta_{n,m}(z,\overline{z}) := (\zeta_0(m+\beta))^{-1/2} P_{n,m}(z,\overline{z};\beta), \quad n,m \geq 0 \tag{33}$$

and we define the sequence $(x^\beta_{n,m})_{n\geq 0}$ by

$$x^\beta_{n,m} := \frac{\zeta_{n\wedge m}(|n-m|+\beta)}{\zeta_{(n-1)\wedge m}(|n-m-1|+\beta)}, \tag{34}$$

with the generalized factorial

$$x^\beta_{n,m}! := x^\beta_{n,m} x^\beta_{n-1,m} \dots x^\beta_{1,m} = \frac{\zeta_{n\wedge m}(|n-m|+\beta)}{\zeta_0(m+\beta)}, \quad x^\beta_{0,m}! \equiv 1. \quad m = 1,2,3,..., \tag{35}$$

and $\zeta_n(\alpha)$ is the coefficient in (25).

- By (33) and (35) the orthogonality relation (30) becomes

$$\frac{1}{2\pi}\int_0^{L^2}\int_0^{2\pi} P^\beta_{n,m}(z,\overline{z})\overline{P^\beta_{k,s}(z,\overline{z})} d\theta d\mu_\beta(r^2) = x^\beta_{n,m}!\, \delta_{n,k}\delta_{m,s}. \tag{36}$$

Definition

For fixed parameters $\beta \geq 0$ and $m \in \mathbb{Z}_+$, we define a set of generalized nonlinear coherent states (μ_β-GNLCS) as

$$\vartheta_{z,m,\beta} \equiv |z;\beta,m\rangle := \left(\mathcal{N}_{\beta,m}(z\overline{z})\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\overline{P^{\beta}_{n,m}(z,\overline{z})}}{\sqrt{x^{\beta}_{n,m}!}} \varphi_n \tag{37}$$

where $\mathcal{N}_{\beta,m}(z\overline{z})$ is a normalizing factor.

Proposition

The complex numbers z for which the μ_β-GNLCS (37) are defined belong to the disk $\mathcal{D}_{\beta,m} = \{\xi \in \mathbb{C},\ |\xi| < R_{\beta,m}\}$ with $R_{\beta,m} := \min_{0\leqslant i,j\leqslant m} R_{\beta,m,i,j}$ where

$$(R_{\beta,m,i,j})^2 = \lim_{n\to+\infty} \left| \frac{c_i(m;n-1+\beta)c_j(m;n-1+\beta)\zeta_m(n+\beta)}{c_i(m;n+\beta)c_j(m;n+\beta)\zeta_m(n-1+\beta)} \right|. \tag{38}$$

In particular, for $m=0$, we have $\mathcal{D}_{\beta,0} = \mathcal{D}_\beta$.

For $m=0$, the polynomials (33) reduce to $P_{n,0}^{\beta}(z,\bar{z}) = z^n/\sqrt{\mu_\beta}$, and $x_{n,0}^{\beta}! = \zeta_0(n+\beta)/\zeta_0(\beta) = \mu_{n+\beta}/\mu_\beta = x_n^\beta! \quad n=0,1,2,\ldots$. Consequently, the μ_β-GNLCS (37) reduce to the μ_β-NLCS in (10).

Proposition

Assuming that $\mathcal{D}_L \subseteq \mathcal{D}_{\beta,m}$, then the μ_β-GNLCS in (37) satisfy the resolution of the identity operator of $\mathcal{H}$ as

$$\int_{\mathcal{D}_L} |\vartheta_{z,m,\beta}\rangle\langle\vartheta_{z,m,\beta}| d\eta_{\beta,m}(z,\bar{z}) = \mathbf{1}_{\mathcal{H}}. \tag{39}$$

where $d\eta_{\beta,m}(z,\bar{z}) = (2\pi)^{-1} d\theta \mathcal{N}_{\beta,m}(z\bar{z}) d\mu_\beta(z\bar{z})$.

Proof. This can be proved directly with the help of the orthogonality relation (36).

VI- The coherent states transform and its range

Proposition

The generalized Bargmann transform associated with the coherent states $\vartheta_{z,m,\beta}$ is the isometric map $\mathcal{B}_{m,\beta} : \mathcal{H} \longrightarrow L^{2,\beta}(\mathcal{D}_L)$, defined by

$$\mathcal{B}_{m,\beta}[\phi](z) := \left(\mathcal{N}_{m,\beta}(z\bar{z})\right)^{\frac{1}{2}} \langle \phi, \vartheta_{z,m,\beta} \rangle_{\mathcal{H}}. \tag{40}$$

Thus, for $\phi, \ \psi \in \mathcal{H}$, we have

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \langle \mathcal{B}_{m,\beta}[\phi], \mathcal{B}_{m,\beta}[\psi] \rangle_{L^{2,\beta}(\mathcal{D}_L)}. \tag{41}$$

In particular,

$$\mathcal{B}_{m,\beta}[\varphi_n](\bar{z}) = \frac{P_{n,m}^{\beta}(z,\bar{z})}{\sqrt{x_{n,m}^{\beta}!}}, \quad n = 0, 1, 2, \cdots. \tag{42}$$

Note that the range of $\mathcal{B}_{m,\beta}$ is the span of polynomials $\left(P^\beta_{n,m}(z,\overline{z})/\sqrt{x^\beta_{n,m}!}\right)_{n\geq 0}$ which we denote by $\mathcal{A}^2_{\beta,m}(\mathcal{D}_L)$. This subspace of $L^{2,\beta}(\mathcal{D}_L)$ can be compared with the eigenspace of a specific operator that we construct in the following way.
We start by denoting

$$\widetilde{P}^\beta_{n,m}(z,\overline{z}) := \frac{P^\beta_{n,m}(z,\overline{z})}{\sqrt{x^\beta_{n,m}!}}, \tag{43}$$

which form an orthonormal basis in the full Hilbert space $L^{2,\beta}(\mathcal{D}_L)$. Next, we define two pairs of operators, A^β_i, $A^{\beta\dagger}_i$ $i=1,2$, and some related operators by using the orthonormalized polynomials (43) as follows.

$$\begin{aligned} A^\beta_1 \widetilde{P}^\beta_{n,m} &= \sqrt{x^\beta_{n,m}}\widetilde{P}^\beta_{n-1,m}, \quad n=1,2,3,\ldots, & A^\beta_1 \widetilde{P}^\beta_{0,m} &= 0, \\ A^\beta_2 \widetilde{P}^\beta_{n,m} &= \sqrt{x^\beta_{m,n}}\widetilde{P}^\beta_{n,m-1}, \quad m=1,2,3,\ldots, & A^\beta_2 \widetilde{P}^\beta_{n,0} &= 0, \end{aligned} \tag{44}$$

and their adjoints

$$A_1^{\beta\dagger}\widetilde{P}_{n,m}^{\beta}=\sqrt{x_{n+1,m}^{\beta}}\widetilde{P}_{n+1,m}^{\beta},\;\; A_2^{\beta\dagger}\widetilde{P}_{n,m}^{\beta}=\sqrt{x_{m+1,n}^{\beta}}\widetilde{P}_{n,m+1}^{\beta},\;\; n,m=0,1,2,\ldots. \tag{45}$$

Definition

Two operators are defined as

$$L_i^{\beta}:=\frac{1}{2}(A_i^{\beta\dagger}A_i^{\beta}+A_i^{\beta}A_i^{\beta\dagger}),\qquad i=1,2. \tag{46}$$

By $(44)-(45)$ we easily see that their action on the basis vectors $P_{m,n}^{\beta}$ is given by

$$L_1^{\beta}\widetilde{P}_{n,m}^{\beta}=\lambda_{m,n}^{\beta}\widetilde{P}_{n,m}^{\beta},\qquad L_2^{\beta}\widetilde{P}_{n,m}^{\beta}=\lambda_{n,m}^{\beta}\widetilde{P}_{n,m}^{\beta}, \tag{47}$$

where

$$\lambda_{m,n}^{\beta}=\frac{1}{2}(x_{m+1,n}^{\beta}+x_{m,n}^{\beta}). \tag{48}$$

VII- The example of the measure $r^\beta e^{-r} dr$

Case $m \neq 0$

The measure $r^\beta e^{-r} dr$ gives rise to the polynomials $\phi_n(r;\beta) = (-1)^n L_n^{(\beta)}(r)$, where $L_n^{(\beta)}$ is the Laguerre polynomial. The resulting complex polynomials from (27) by the above procedure are

$$H_{n,m}^{(\beta)}(z,\overline{z}) = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{(\beta+1)_n}{(\beta+1)_{n-k}} (-1)^k z^{n-k} \overline{z}^{m-k}, \qquad (49)$$

for $n \geq m$. In other words,

$$H_{n,m}^{(\beta)}(z,\overline{z}) = (-1)^m z^{n-m} L_m^{(\beta+n-m)}(z\overline{z}), \quad n \geq m. \qquad (50)$$

When $n < m$ they are defined by $H_{n,m}^{(\beta)}(z,\overline{z}) = H_{m,n}^{(\beta)}(\overline{z}, z)$.

In Ismail & Zhang (2016) these polynomials were denoted by $Z_{m,n}^{(\beta)}(z,\bar{z})$. These polynomials can be rewritten for all $n,m\in\mathbb{Z}_+$ as

$$H_{n,m}^{(\beta)}(z,\bar{z})=(-1)^{n\wedge m}|z|^{|n-m|}e^{i(n-m)\arg z}L_{n\wedge m}^{(|n-m|+\beta)}(z\bar{z}). \tag{51}$$

The function $\zeta_n(\beta)$ in (25) together with coefficients $c_j(n;\beta)$ in (26) were given by

$$\zeta_n(\beta)=\frac{\Gamma(\beta+n+1)}{n!},\quad c_j(n;\beta)=\frac{(-1)^{n-j}(\beta+1)_n}{(n-j)!j!(\beta+1)_{n-j}}. \tag{52}$$

The orthogonality relation satisfied by these polynomials is

$$\int_{\mathbb{C}}H_{j,n}^{(\beta)}(z,\bar{z})\overline{H_{l,m}^{(\beta)}(z,\bar{z})}(z\bar{z})^\beta e^{-z\bar{z}}rdrd\theta=\pi\frac{\Gamma(\beta+j\vee n+1)}{(j\wedge n)!}\delta_{j,l}\delta_{n,m}. \tag{53}$$

For $\beta=0$, the polynomials in (49) reduces to the complex Hermite polynomials :

$$H_{m,n}(z,\bar{z})=(m\wedge n)!H_{m,n}^{(0)}(z,\bar{z})=\sum_{k=0}^{m\wedge n}\binom{m}{k}\binom{n}{k}(-1)^k k!z^{m-k}\bar{z}^{n-k} \tag{54}$$

which were first introduced by Ito (1953).

In this case, the polynomials in (51) reads

$$P_{n,m}^{\beta}(z,\bar{z}) = (\Gamma(\beta+m+1))^{-\frac{1}{2}} H_{n,m}^{(\beta)}(z,\bar{z}), \tag{55}$$

the generalized factorial in (35) becomes

$$x_{n,m}^{\beta}! = \frac{\Gamma(\beta+n\vee m+1)}{\Gamma(\beta+m+1)\,(n\wedge m)!} \tag{56}$$

and the definition of GNLCS takes the following form.

Definition

Let $\beta \geq 0$ and $m = 0, 1, 2, ..., \infty$. The GNLCS can be defined throughout the following superposition

$$\vartheta_{z,m,\beta} := \left(\mathcal{N}_{\beta,m}(z\bar{z})\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\overline{H_{n,m}^{(\beta)}(z,\bar{z})}}{\sqrt{\frac{\Gamma(\beta+n\vee m+1)}{(n\wedge m)!}}} \varphi_n^{\beta} \tag{57}$$

where $\mathcal{N}_{\beta,m}(z\bar{z})$ is a normalization factor. Here $\{\varphi_n^{\beta}\}$ are given by (20).

Proposition

The overlap is given by

$$\langle \vartheta_{z,m,\beta}, \vartheta_{w,m,\beta} \rangle =$$

$$\left(\mathcal{N}_{\beta,m}(z\overline{z})\right)^{-\frac{1}{2}} \left(\mathcal{N}_{\beta,m}(w\overline{w})\right)^{-\frac{1}{2}} \left[\mathfrak{S}_{m,\beta}(L_j^\alpha)(z,w) + \frac{(\beta+1)_m}{m!\Gamma(\beta+1)} \mathfrak{S}_{m,\beta}({}_2F_2)(z,w)\right],$$

where

$$\mathfrak{S}_{m,\beta}(L_j^\alpha)(z,w) = \sum_{j=0}^{m-1} \frac{j!(\overline{z}w)^{m-j}}{\Gamma(\beta+m+1)} L_j^{(\beta+m-j)}(z\overline{z}) L_j^{(\beta+m-j)}(w\overline{w}) \quad (58)$$

$$\mathfrak{S}_{m,\beta}({}_2F_2)(z,w) = \sum_{k,l=0}^{m} \frac{(-m)_k(-m)_l (z\overline{z})^k (w\overline{w})^l}{k!l!(\beta+1)_k(\beta+1)_l} {}_2F_2\left(\begin{array}{c} 1, m+\beta+1 \\ k+\beta+1, l+\beta+1 \end{array} \Big| z\overline{w} \right) \quad (59)$$

in terms of Laguerre polynomials $L_j^{(\alpha)}$ and the ${}_2F_2$ hypergeometric series.

Proposition

The GNLCS (57) satisfy the following resolution of the identity

$$\int_{\mathbb{C}} |\vartheta_{z,m,\beta}\rangle\langle\vartheta_{z,m,\beta}| \, d\eta_{\beta,m}(z) = \mathbf{1}_{\mathcal{H}}, \tag{60}$$

where

$$d\eta_{\beta,m}(z) = (\Gamma(\beta+1))^{-1}\left[\sum_{j=0}^{m-1} \frac{j!(z\bar{z})^{m-j}}{\Gamma(\beta+m+1)}\left[L_j^{(\beta+m-j)}(z\bar{z})\right]^2 + \frac{(\beta+1)_m}{m!\Gamma(\beta+1)}\right.$$
$$\left.\times \sum_{k,l=0}^{m} \frac{(-m)_k(-m)_l \, (z\bar{z})^{k+l}}{k!l!(\beta+1)_k(\beta+1)_l} \, {}_2F_2\left(\begin{array}{c} 1, m+\beta+1 \\ k+\beta+1, l+\beta+1 \end{array} \Big| z\bar{z}\right)\right](z\bar{z})^{\beta} e^{-z\bar{z}} d\nu(z),$$

in terms of the ${}_2F_2-$series and the Lebesgue measure $d\nu$.

Theorem

The GNLCS (57) give rise to a generalized Bargmann transform through the isometric embedding $\mathcal{B}_{m,\beta} : L^2(\mathbb{R}, d\omega_\beta(x)) \longrightarrow \mathcal{A}^2_{\beta,m}(\mathbb{C})$ *defined by*

$$\mathcal{B}_{m,\beta}[\varphi](z) = \int_{\mathbb{R}} B_{\beta,m}(z,x)\varphi(x)d\omega_\beta(x), \tag{61}$$

where

$$B_{\beta,m}(z,x) =$$
$$\sum_{n=0}^{m-1} \frac{2^{n/2}}{\sqrt{(\beta+1)_n}} H_n(x,\beta) \left[\frac{(-1)^n z^{m-n}\sqrt{n!}}{\sqrt{\Gamma(\beta+m+1)}} L_n^{(m-n+\beta)}(z\bar{z}) - \frac{(-1)^m \bar{z}^{n-m}\sqrt{m!}}{\sqrt{\Gamma(\beta+n+1)}} L_m^{(n-m+\beta)}(z\bar{z}) \right]$$
$$+ \frac{z^m}{\sqrt{\Gamma(\beta+1)\, m!}} \sum_{k=0}^{m} \frac{(-m)_k(\beta+1-k)_k}{k!\,(z\bar{z})^k} F^{1:0;0;1}_{1:0;0;0}\left(\begin{matrix} [1:1,2,1]:-;-;[\beta:1] \\ [\beta-k+1:1,2,2]:-;-;- \end{matrix} \sqrt{2}x\bar{z}, -\bar{z}^2/2, -\bar{z}^2 \right).$$

The special function $F^{1:0;0;1}_{1:0;0;0}$ in the right hand side of the last equation is the generalized Lauricella function.

When $\beta = 0$, the measure $d\omega_0(x) = \pi^{-\frac{1}{2}} e^{-x^2} dx$ and $\mathcal{A}^2_{0,m}(\mathbb{C})$ turns out to be the space of true-m-polyanalytic functions that is the orthogonal difference $\mathfrak{F}^2_m(\mathbb{C}) \ominus \mathfrak{F}^2_{m-1}(\mathbb{C})$ between two consecutive m-analytic spaces

$$\mathfrak{F}^2_m(\mathbb{C}) = \{g : \mathbb{C} \to \mathbb{C}, \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^2} d\nu(z) < +\infty, \overline{\partial}^m g(z) = 0\} \qquad (64)$$

The space $\mathcal{A}^2_{0,m}(\mathbb{C})$ can also be realized as eigenspace in $L^2\left(\mathbb{C}, e^{-|z|^2} d\nu(z)\right)$ of the operator $\widetilde{\Delta}_0$ given by (79) and corresponding to the eigenvalue m.

Corollary

For $\beta = 0$, the integral transform (61) reduces to the true-polyanalytic Bargmann transform $\mathcal{B}_{m,0} : L^2(\mathbb{R}, \pi^{-\frac{1}{2}} e^{-x^2} dx) \longrightarrow \mathcal{A}^2_{0,m}(\mathbb{C})$ given by

$$\mathcal{B}_{m,0}[\varphi](z) = \pi^{-\frac{1}{2}} \int_{\mathbb{R}} B_{0,m}(z,x)\varphi(x) e^{-x^2} dx, \qquad (65)$$

where

$$B_{0,m}(z,x) = (-1)^m (2^m m!)^{-\frac{1}{2}} e^{\sqrt{2}x\bar{z} - \frac{1}{2}\bar{z}^2} H_m\left(x - \frac{z+\bar{z}}{\sqrt{2}}\right). \qquad (66)$$

The analytic case $m = 0$

In this case the moments $\mu_{n+\beta} = \Gamma(n+\beta+1)$, the sequence $x_n^\beta = n+\beta$, $\lim_{n\to\infty} x_n^\beta = +\infty$ and the resulting $\mu_\beta-$NLCS defined by (10) take the form

$$\vartheta_{z,\beta} = \left(\mathcal{N}_\beta(z\overline{z})\Gamma(\beta+1)\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\overline{z}^n}{\sqrt{(\beta+1)_n}}\varphi_n^\beta, \tag{67}$$

where the normalizing factor is given by

$$\mathcal{N}_\beta(z\overline{z}) = \frac{e^{z\overline{z}}}{\Gamma(\beta+1)}\,{}_1F_1(\beta,\beta+1;-z\overline{z}) \tag{68}$$

in terms of the confluent hypergeometric ${}_1F_1-$sum.

Proposition

Let $\beta \geq 0$. Then, the NLCS (67) satisfy the resolution of the identity

$$\int_{\mathbb{C}} |\vartheta_{z,\beta}\rangle\langle\vartheta_{z,\beta}|\, d\eta_\beta(z) = \mathbf{1}_{\mathcal{H}}, \tag{69}$$

where

$$d\eta_\beta(z) = (\Gamma(\beta+1))^{-1}\,{}_1F_1(\beta,\beta+1;-z\bar{z})(z\bar{z})^\beta d\nu(z). \tag{70}$$

Theorem

The NLCS (67) give rise to a generalized Bargmann transform through the unitary embedding $\mathcal{B}_\beta : L^2(\mathbb{R}, d\omega_\beta(x)) \to \mathcal{A}^2_\beta(\mathbb{C})$ *defined by*

$$\mathcal{B}_\beta[\varphi](z) = \int_{\mathbb{R}} B_\beta(z,x)\varphi(x)d\omega_\beta(x), \tag{71}$$

where

$$B_\beta(z,x) = F^{1:0;0;1}_{1:0;0;0}\left(\begin{matrix} [1:1,2,1] : -; -; [\beta:1] \\ [\beta+1:1,2,2] : -; -; - \end{matrix} \; \sqrt{2}x\bar{z}, -\bar{z}^2/2, -\bar{z}^2 \right). \tag{72}$$

In particular, when $\beta = 0$

$$B_0(z,x) = e^{-\frac{1}{2}\bar{z}^2 + \sqrt{2}x\bar{z}}, \quad z \in \mathbb{C}, \; x \in \mathbb{R}, \tag{73}$$

and $\mathcal{B}_0$ *is the well known classical Bargmann transform acting on function in* $L^2\left(\mathbb{R}, \pi^{-\frac{1}{2}} e^{-x^2} dx\right)$.

Proposition

For $\beta \geq 0$, the operator L_1^β in (46) can be expressed in a differential form as

$$L_1^\beta = -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}} - \frac{\beta}{z}\frac{\partial}{\partial \overline{z}} + \beta + \frac{1}{2}. \qquad (74)$$

Proof. Here the obtained class of 2D orthogonal polynomials obtained are $H_{n,m}^{(\beta)}(z,\overline{z})$ given in (49). According to Theorem 3.4 in Ismail & Zhang (2016), these polynomials satisfy the following second order partial differential equation

$$\left(\frac{\beta}{z} + \frac{\partial}{\partial z} - \overline{z}\right)\left(-\frac{\partial}{\partial \overline{z}}\right) H_{n,m}^{(\beta)}(z,\overline{z}) = m H_{n,m}^{(\beta)}(z,\overline{z}), \qquad n \geq m \qquad (75)$$

By comparing the actions of the operator in the left hand side of (75) , on the basis vectors $\widetilde{P}_{n,m}^{\beta} = H_{n,m}^{(\beta)}(z,\overline{z})$, with the actions of operator L_1^β as in (46), respectively, on the same basis, we deduce the expressions (74). □

Proposition

Let $\beta \geq 0$. Then, the subspace $\mathcal{A}^2_\beta(\mathbb{C})$ turns out to be the null space of the generalized Landau Hamiltonian

$$\tilde{\Delta}_\beta := -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}} - \frac{\beta}{z}\frac{\partial}{\partial \overline{z}} = L_1^\beta - \beta - \frac{1}{2}. \tag{76}$$

That is

$$\mathcal{A}^2_\beta(\mathbb{C}) = \left\{\phi \in L^{2,\beta}(\mathbb{C}), \quad \tilde{\Delta}_\beta\left[\phi\right] = 0\right\}. \tag{77}$$

Proof. Denote $\mathcal{E}_0^\beta := \left\{\phi \in L^{2,\beta}(\mathbb{C}), \quad \tilde{\Delta}_\beta\left[\phi\right] = 0\right\}$. Let $\phi \in \mathcal{A}^2_\beta(\mathbb{C})$. So $\phi \in L^{2,\beta}(\mathbb{C})$ and ϕ is entire. Then $\frac{\partial}{\partial \overline{z}}\left[\phi\right] = 0$. We apply to this equation the operator

$$\left(\frac{\partial}{\partial \overline{z}}\right)^* = -\frac{\partial}{\partial z} - \frac{\beta}{z} + \overline{z}$$

so we still have

$$\left(\frac{\partial}{\partial \overline{z}}\right)^* \frac{\partial}{\partial \overline{z}}\left[\phi\right] = 0.$$

This means

$$\tilde{\Delta}_\beta [\phi] = 0.$$

Therefore, $\phi \in \mathcal{E}_0^\beta$. We have proved that $\mathcal{A}_\beta^2(\mathbb{C}) \subset \mathcal{E}_0^\beta$. Conversely, let $\varphi \in \mathcal{E}_0^\beta$. Then, $\tilde{\Delta}_\beta [\varphi] = 0$ which means that

$$\left\langle \tilde{\Delta}_\beta [\varphi], \varphi \right\rangle = 0.$$

By Theorem 3.5 (in Ismail & Zhang (2016)) the operator $\tilde{\Delta}_\beta$ is positive in the sense that $\left\langle \tilde{\Delta}_\beta [\varphi], \varphi \right\rangle \geq 0$ with

$$\left\langle \tilde{\Delta}_\beta [\varphi], \varphi \right\rangle = 0 \quad \text{if and only if} \quad \frac{\partial}{\partial \bar{z}} \varphi = 0. \tag{78}$$

This implies that $\frac{\partial}{\partial \bar{z}} \varphi = 0$ which means that φ is an entire function. That is $\mathcal{E}_0^\beta \subset \mathcal{A}_\beta^2(\mathbb{C})$. □

Note that for $\beta = 0$ the operator (74) reduces to

$$\widetilde{\Delta}_0 := -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}} = L_1^0 - \frac{1}{2} \tag{79}$$

which can be obtained by intertwining (unitarily) the Hamiltonian describing the dynamics of a charged particle on the Euclidean xy-plane, while interacting with a perpendicular constant homogeneous magnetic field (in suitable unit system) :

$$H^L := \frac{1}{2}\left(\left(i\frac{\partial}{\partial x} - y\right)^2 + \left(i\frac{\partial}{\partial y} + x\right)^2\right) \tag{80}$$

acting on the Hilbert space $L^2\left(\mathbb{R}^2, dxdy\right)$ as follows

$$e^{\frac{1}{2}z\overline{z}}\left(\frac{1}{2}H^L - \frac{1}{2}\right)e^{-\frac{1}{2}z\overline{z}} = \widetilde{\Delta}_0, \quad z = x + iy. \tag{81}$$

The spectrum of the Hamiltonian $\frac{1}{2}H^L$ consists on $E_n := n + \frac{1}{2}$, $n = 0, 1, 2, ...$ known as Euclidean Landau levels with infinite degeneracy. The operator $\widetilde{\Delta}_0$ is acting on the Hilbert space $L^{2,0}(\mathbb{C}) := L^2\left(\mathbb{C}, e^{-z\overline{z}}d\nu\right)$ of Gaussian square integrable functions. Its spectrum in $L^{2,0}(\mathbb{C})$ consists on $\epsilon_n := n$, $n = 0, 1, 2, ...$.

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Thank you