Stone duality and quasi-orbit spaces for C^*-inclusions

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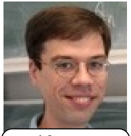
joint work with



Ralf Meyer

see arXive:1804.09387

Classical setting

Consider a continuous group action $G \to \mathrm{Homeo}(X)$ where

X - locally compact Hausdorff space, G - locally compact group

The **orbit space** $X/G = \{Gx : x \in X\}$ is usually badly non-Hausdorff.

- T_0 replacement for X/G is the **quasi-orbit space** $X/\sim$ where

$$x \sim y \iff \overline{Gx} = \overline{Gy}$$

- noncommutative replacement is the crossed product $C_0(X) \rtimes G$.

Effros-Hahn conjecture 1967

All primitive ideals in $C_0(X) \rtimes G$ are induced from stability groups of points in X.

Related Fact: Gootmann-Rosenberg 1979

If G is second countable and amenable and X is second countable, then there is a continuous map from the primitive ideal space of $C_0(X) \rtimes G$ onto $X/\sim$.

Classical setting II

Consider a continuous group action $\alpha : G \to \mathrm{Aut}(A)$ where

A - C^*-algebra, G - locally compact group

Let $\check{A}$ be the primitive ideal space of A with Jacobson topology, given by the lattice $\mathbb{I}(A)$ of ideals in A. The **quasi-orbit space** $\check{A}/\sim$ is defined using the induced action $\check{\alpha} : G \to \mathrm{Homeo}(\check{A})$:

$$x \sim y \qquad \Longleftrightarrow \qquad \overline{Gx} = \overline{Gy}$$

Fact: related to generalized Effros-Hahn conjecture

Let $B := A \rtimes G$ be the crossed product. If G is second countable, amenable and A is separable, then there is

$$\check{B} \twoheadrightarrow \check{A}/\sim$$

a continuous and open surjective map called **quasi-orbit map**.

Question

Can we construct the quasi-orbit map just from the inclusion $A \subseteq \mathcal{M}(B)$?

Stone duality

The category of **locales** has complete, distributive lattices as objects and maps that preserve arbitrary joins and finite meets as arrows.

If X is a topological space, then the family of open subsets $\mathbb{O}(X)$ is a locale. If $f\colon X \to Y$ is a continuous map, then $f^{-1}\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ is a locale morphism. Thus $\mathbb{O}$ is a contravariant functor from topological spaces to locales.

If L is a complete lattice, then the set of **prime elements**

$$\mathrm{Prime}(L) := \{p \in L\backslash\{1\} : l_1 \wedge l_2 \leqslant p \text{ for } l_1, l_2 \in L \text{ implies } l_1 \leqslant p \text{ or } l_2 \leqslant p\}$$

is a topological space where $L \ni l \to U_l := \{p \in \mathrm{Prime}(L) : l \nleqslant p\} \in \mathbb{O}(\mathrm{Prime}(L))$

Any locale morphism $G\colon L \to R$ induces continuous $G^*\colon \mathrm{Prime}(R) \to \mathrm{Prime}(L)$

Stone duality. The functors Prime and $\mathbb{O}$ are adjoint. They give equivalence between spatial locales and sober topological spaces.

Question: Which locale morphisms $G\colon L \to R$ induce continuous open surjective maps $G^*\colon \mathrm{Prime}(R) \to \mathrm{Prime}(L)$?

A locale morphism $G\colon L \to R$ is called **open** if it has a left adjoint $F\colon R \to L$:

$$\forall_{I\in R,\ J\in L} \qquad I \leqslant G(J) \iff F(I) \leqslant J \qquad\qquad (\textit{Galois connection})$$

such that the *Frobenius reciprocity condition* holds:

$$F(J \cap G(I)) = F(J) \cap I \text{ for all } I \in R,\ J \in L.$$

If $f\colon X \to Y$ is a continuous open map, then $f^{-1}\colon \mathbb{O}(Y) \to \mathbb{O}(X)$ is an open locale morphism with left adjoint given by $f\colon \mathbb{O}(X) \to \mathbb{O}(Y)$.

Thm. (B.K., R. Meyer)

Let $G\colon L \to \mathbb{O}(X)$ be a locale morphism and $\pi\colon X \to \mathrm{Prime}(L)$ a continuous map that are adjunct (X is a topological space and L a complete lattice). Assume also

- X is second countable, and all closed subspaces of X are Baire spaces.

π is an open surjection $\iff$ $G\colon L \to \mathbb{O}(X)$ is injective and open.

Cor. (Dixmier 1967) If a C^*-algebra A is separable, then the inclusion $\check{A} \subseteq \mathrm{Prime}(A) := \mathrm{Prime}(\mathbb{I}(A))$ is in fact equality.

We fix a general C^*-inclusion $A \subseteq \mathcal{M}(B)$.
The maps $r : \mathbb{I}(B) \to \mathbb{I}(A)$ and $i : \mathbb{I}(A) \to \mathbb{I}(B)$ where

$$r(J) := \{a \in A : aJ + Ja \subseteq J\}, \qquad i(I) := \overline{BIB}$$

form a Galois connection (adjunction): $\forall_{I\in\mathbb{I}(A), J\in\mathbb{I}(B)}\ \ I \subseteq r(J) \iff i(I) \subseteq J$

$\mathbb{I}^B(A) := r(\mathbb{I}(B))$ - **restricted ideals**, $\qquad \mathbb{I}^A(B) := i(\mathbb{I}(A))$ - **induced ideals**.

Consequences of Galois connection:

1. the map i preserves joins and r preserves meets, and give $\mathbb{I}^B(A) \cong \mathbb{I}^A(B)$
2. the inclusion $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(A)$ and $r \circ i \colon \mathbb{I}(A) \to \mathbb{I}^B(A)$ are adjunct
3. $i \circ r \colon \mathbb{I}(B) \to \mathbb{I}^A(B)$ and the inclusion $\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)$ are adjunct

Ex1. If $B := A \rtimes G$ for a locally compact group action $\alpha : G \to \mathrm{Aut}(A)$, then

$$\mathbb{I}^B(A) = \{I \in \mathbb{I}(A) : \alpha_g(I) = I \text{ for } g \in G\} \text{ - } \alpha\text{-invariant ideals in } A$$

$$\mathbb{I}^A(B) = \{I \rtimes G : I \text{ is an } \alpha\text{-invariant ideal in } A\}$$

Ex2. If $B := M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and $A := \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \right\} \cong \mathbb{C}^3$ then

$$\mathbb{I}^A(B) = \mathbb{I}(B) \text{ and } \mathbb{I}^B(A) = \{0, \{aE_{11} \oplus 0 : a \in \mathbb{C}\}, \{0 \oplus bE_{11} : b \in \mathbb{C}\}, A\}$$

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Rem. Putting $\mathrm{Prime}^B(A) := \mathrm{Prime}(\mathbb{I}^B(A))$ and $\mathrm{Prime}^A(B) := \mathrm{Prime}(\mathbb{I}^A(B))$:

$$\mathrm{Prime}^B(A) \cong \mathrm{Prime}^A(B)$$

and this topological space could be a candidate for the quasi-orbit space.

But we want to relate it to $\check{A}$ and $\check{B}$!

Quasi-orbit space

Consider the following lattice-theoretic conditions:

(JR) *joins of restricted ideals remain restricted*

(FR1) $I \cap r \circ i(J) = r \circ i(I \cap J)$ *for all* $I \in \mathbb{I}^B(A)$ *and* $J \in \mathbb{I}(A)$

Lem. The inclusion map $\mathbb{I}^B(A) \hookrightarrow \mathbb{I}(A)$ is a locale morphism if and only if (JR). It is an open locale morphism if and only if (JR) + (FR1).

Cor. If (JR), the induced continuous map $\pi\colon \operatorname{Prime}(A) \to \operatorname{Prime}^B(A)$ satisfies

$\pi(\mathfrak{p})$ is the largest restricted ideal in A that is contained in $\mathfrak{p}$.

If in addition (FR1) and $\check{A}$ is second countable, then

$\pi\colon \check{A} = \operatorname{Prime}(A) \to \operatorname{Prime}^B(A)$ is open and surjective.

Def.

Assuming (JR) the **quasi-orbit space** of the C^*-inclusion $A \subseteq \mathcal{M}(B)$ is the quotient space $\check{A}/{\sim}$ where $\mathfrak{p} \sim \mathfrak{q}$ if and only if $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$.

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Ex1. If $B := A \rtimes G$ for a locally compact group action $\alpha : G \to \mathrm{Aut}(A)$, then (JR), (FR1) hold, $\pi(\mathfrak{p}) = \bigcap_{g \in G} \alpha_g(\mathfrak{p})$ and $\check{A}/\sim$ is the usual quasi-orbit space.

Ex2. If $B := M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and $A := \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \right\}$ then (JR) fails.

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(MR_f) *finite meets of induced ideals are again induced*

(MR) *(arbitrary) meets of induced ideals are again induced*

(FR2) $I \cap F(J) = F(I \cap J)$ *for every* $I \in \mathbb{I}^A(B)$ *and* $J \in \mathbb{I}(B)$*, where* $F(J)$ *is the meet of all induced ideals that contain* J*.*

Lem. The inclusion map $\mathbb{I}^A(B) \hookrightarrow \mathbb{I}(B)$ is a locale morphism if and only if (MR_f). It is an open locale morphism if and only if (MR) + (FR2).

The induced continuous map is $i \circ r : \mathrm{Prime}(B) \to \mathrm{Prime}^A(B) \cong \mathrm{Prime}^B(A)$. Thus the map $r : \mathrm{Prime}(B) \to \mathrm{Prime}^B(A)$ is well defined and continuous.

Cor. Assume (JR), (FR1), (MR_f) and $\check{A}$ is second countable. We have a continuous map $\varrho\colon \check{B} \to \check{A}/{\sim}$ defined by $\pi_*^{-1} \circ r$, where $\pi_* : \check{A}/{\sim} \to \mathrm{Prime}^B(A)$ is a homeomorphism. The map ϱ is open surjective if and only if (MR) + (FR2)

Def. We call $\varrho\colon \check{B} \to \check{A}/{\sim}$ defined above the **quasi-orbit map** for $A \subseteq \mathcal{M}(B)$

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Def. We call $\varrho \colon \check{B} \to \check{A}/\sim$ defined above the **quasi-orbit map** for $A \subseteq \mathcal{M}(B)$

Thm0. Let $A \subseteq \mathcal{M}(B)$ be a C^*-inclusion with $\check{A}$ second countable. Reduction $r : \mathbb{I}(B) \to \mathbb{I}^B(A)$ induce homeomorphism $\check{B} \cong \check{A}/\sim$ if and only if r is injective and the conditions (JR), (FR1) hold.

Question: When the above lattice-theoretic conditions hold?

Applications (standing assumption: $\check{A}$ is second countable)

Thm1. Let $\alpha : G \to \mathrm{Aut}(A)$ be a locally compact group action and let B be the full $A \rtimes G$ or reduced crossed product $A \rtimes_{red} G$.

1. The quasi-orbit map $\varrho\colon \check{B} \to \check{A}/{\sim}$ exists. It is a homeomorphism if and only if r is injective and then $B = A \rtimes_{red} G$.
2. If the action α is exact and $B = A \rtimes_{red} G$ then the quasi-orbit map $\varrho\colon \check{B} \to \check{A}/{\sim}$ is open and surjective.
3. If G is discrete and not amenable, then there is an action such that the quasi-orbit map on the full crossed product $A \rtimes G$ is not open and surjective.

Thm2. Theorem 1 holds with group action replaced by a Fell bundle $\{B_g\}_{g\in G}$, with $A := B_e$ the unit fiber and B the cross sectional C^*-algebra (full or reduced).

Thm3. Statement (1) in Theorem 1 holds with group action replaced by an action of second countable groupoid $\mathcal{G}$ on a separable C^*-algebra A.

Applications (standing assumption: $\check{A}$ is second countable)

Thm4. Let S be an inverse semigroup with unit element $e \in S$. Let B be an S-graded C*-algebra, that is there is a family of closed linear subspaces $(B_t)_{t\in S}$ such that $B_g^* = B_{g^*}$, $B_g \cdot B_h \subseteq B_{gh}$ for all $g, h \in S$, and $\sum B_t$ is dense in B. The C^*-inclusion $A := B_e \subseteq B$ admits the quasi orbit map $\check{B} \to \check{A}/\sim$.

Thm5. Let $A \subseteq B$ be a regular C^*-inclusion, that is $AB = B$ and the normalizers

$$N_B(A) := \{b \in B : bAb^* \subseteq A, \quad b^*Ab \subseteq A\}$$

generates B as a C^*-algebra. The quasi-orbit map $\check{B} \to \check{A}/\sim$ exists.

Ex. Let X be the C*-correspondence over $A = \mathbb{C}^3$ built from the directed graph $\bullet \longleftarrow \bullet \longrightarrow \bullet$. Then $\mathcal{O}_X := M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and $A := \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \right\}$

Thm6. Let $B := \mathcal{O}(J, X)$ be the Cuntz-Pimsner algebra of a C^*-correspondence X over a C^*-algebra A relative to an ideal $J \subseteq J_X$. If $\check{B}$ is second countable, there is a continuous open and surjective map $r : \check{B} \to \mathrm{Prime}^{\gamma}(B) \cong \mathrm{Prime}(\mathrm{Pair}_J^X(A))$.

Let (C,Δ) be a C^*-**quantum group** generated by a manageable multiplicative unitary $W \in B(H \otimes H)$ (Woronowicz, Sołtan)

Let (A,α) be a **C^*-algebra with an action of** C. So $\alpha\colon A \to \mathcal{M}(A \otimes C)$ is an injective morphism, $\alpha(A)\cdot(1 \otimes C) = A \otimes C$ and $(\alpha \otimes \mathrm{Id}_C)\circ\alpha = (\mathrm{Id}_A \otimes \alpha)\circ\alpha$.

The **reduced crossed product** $B = A \rtimes_{\alpha,\mathrm{r}} C$ is defined as the C^*-subalgebra of $\mathcal{M}(A \otimes \mathbb{K}(H))$ generated by $\alpha(A)\cdot(1 \otimes \hat{C})$

Def. Let $I \in \mathbb{I}(A)$ be α-*invariant*, that is $\alpha(I)\cdot(1 \otimes C) \subseteq I \otimes C$. We say α *restricts to an action on* I if $\alpha(I)\cdot(1 \otimes C) = I \otimes C$. It *descends to an action on* A/I if the induced map $\dot{\alpha}\colon A/I \to \mathcal{M}(A/I \otimes C)$ is injective.

Thm7. Suppose C is exact as a C^*-algebra and that either

1. the quantum group $\hat{C}$ is discrete and α restricts to all invariant ideals, or
2. for any invariant ideal $I \lhd A$, α restricts to I and descends to A/I.

Then the quasi-orbit space $\check{A}/\sim$ exists and is a quotient of $\check{A}$ by an open relation.

Cor. For any coaction of a locally compact amenable group G the quasi-orbit space $\check{A}/\sim$ exists and is a quotient of $\check{A}$ by an open relation.