AREA FORMULA FOR GRAPH SURFACES IN SUB-LORENTZIAN GEOMETRY

Maria Karmanova

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Outline of the Talk

◇ Area for graph mappings: classical case

◇ Introduction to non-holonomic geometry (Carnot groups)
⇒ Sub-Riemannian differentiability
⇒ Graph mappings as example of Hölder mappings

◇ Metric properties of graph mappings
⇒ Polynomial sub-Riemannian differentiability (Carnot groups)
⇒ Intrinsic bases and corresponding measure

◇ Minkowski geometry

◇ General sub-Lorentzian structures

◇ Distance, measure and area for graphs

Euclidean Case: Graph Mappings

• $\varphi : \mathbb{R}^n \to \mathbb{R}$

$\Rightarrow$ *graph mapping* $\varphi_\Gamma : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $\varphi_\Gamma : x \mapsto (x, \varphi(x))$

◇ $\varphi \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}) \Rightarrow \varphi_\Gamma \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^{n+1})$

◇ φ_Γ is *differentiable* a. e.

◇ *the area formula*:

$$\int_D \sqrt{1 + \det\Big(D\varphi^*(x) D\varphi(x)\Big)}\, dx = \int_{\varphi(D)} d\mathcal{H}^n(y)$$

Non-Holonomic Geometry: Nilpotent Graded Groups

! Motivation: Hypoellipticity of $(X_1^2+\ldots+X_{n-1}^2-X_n)f=\varphi\in C^\infty$, $X_1,\ldots,X_n\in T\mathbb{R}^N$ (Hörmander, 1967, sufficient conditions)

- $\mathbb{G}$: a manifold with basis vector fields $X_1,\ldots,X_N$
- for Lie algebra V, we have $V=\bigoplus_{i=1}^{M} V_i$; $\quad X_i\in V_k \Rightarrow \deg X_i = k$

◇ $V_{i+1}\subset[V_1,V_i]$, $[V_i,V_M]=\{0\}$ $\Rightarrow$ nilpotent graded group

◇ $V_{i+1}=[V_1,V_i]$, $[V_1,V_M]=\{0\}$ $\Rightarrow$ Carnot group

◇ Group operation: $x=\exp\Big(\sum_{i=1}^{N}x_iX_i\Big)(u)$, $y=\exp\Big(\sum_{i=1}^{N}y_iX_i\Big)(u)$

$\Rightarrow x\cdot y=\exp\Big(\sum_{i=1}^{N}z_iX_i\Big)(u)$, where $z_i=x_i+y_i+\sum_{|\mu+\beta|_h=\deg X_i}F^i_{\mu,\beta}(u)x^\mu y^\beta$

- $|\lambda|_h=\sum_{j=1}^{N}\lambda_j\deg X_j$ for $\lambda=(\lambda_1,\ldots,\lambda_N)$

Applications of Non-Holonomic Structures

- subelliptic equations
- non-holonomic mechanics
- contact geometry
- physics
- robotechnics
- neurobiology
- astrodymanics

etc.

Distance Functions

- If $w = \exp\Big(\sum_{i=1}^{N} w_i X_i\Big)(v)$ then $d_\infty(v,w) = \max_{i=1,\ldots,N} \{|w_i|^{\frac{1}{\deg X_i}}\}$

- d_∞ is a quasimetric: $\exists C < \infty$ such that

$$d_\infty(v,w) \leq C(d_\infty(v,u) + d_\infty(u,w)) \quad \forall w,v,u \in \mathbb{G}$$

◇ Define $\theta_u : (v_1, \ldots, v_N) \mapsto \exp\Big(\sum_{i=1}^{N} v_i X_i\Big)(u) =: v$

$\Rightarrow$ $(v_1, \ldots, v_N)$ are normal coordinates of v w. r. t. u

Sub-Riemannian Differentiability ($\varphi : \mathbb{G} \to \widetilde{\mathbb{G}}$)

Definition [Pansu; see also Vodopyanov for general case].
A mapping $\varphi : D \to \widetilde{\mathbb{G}}$, $D \subset \mathbb{G}$, is *hc-differentiable* at $u \in D$, if there exists a horizontal homomorphism

$$L_u : \mathbb{G} \to \widetilde{\mathbb{G}}$$

such that

$$d_\infty(\varphi(w), L_u(w)) = o(d_\infty(u, w)), \;\; D \ni w \to u.$$

- hc-differential of φ at u is denoted by $\widehat{D}\varphi(u)$

Theorem (Pansu; see also Vodopyanov for general case).
Every intrinsically Lipschitz mapping $\varphi : D \to \widetilde{\mathbb{G}}$ is hc-differentiable almost everywhere.

◇ If $\varphi \in C^1_H(D, \widetilde{\mathbb{G}})$ then it is continuously hc-differentiable on D

Graphs as an Example of Hölder Mappings

- $\mathbb{G}, \widetilde{\mathbb{G}} \subset \mathbb{U}$, $\mathbf{0} = \mathbb{G} \cap \widetilde{\mathbb{G}}$, $\varphi : D \ni x \mapsto \exp\Big(\sum_{j=1}^{\widetilde{N}} \varphi_j(x) X_j\Big)(\mathbf{0}) \in \widetilde{\mathbb{G}}$
- *graph mapping* $\varphi_\Gamma : D \ni x \mapsto \exp\Big(\sum_{j=1}^{\widetilde{N}} \varphi_j(x) X_j\Big)(x) \in \mathbb{U}$

◇ $\varphi \in \mathrm{Lip}(D, \widetilde{\mathbb{G}}) \Rightarrow$ generally, $\varphi_\Gamma \notin \mathrm{Lip}(D, \widetilde{\mathbb{G}})$ (even if $\widetilde{\mathbb{G}} = \mathbb{R}$)

⊖ hc-differentiability: is not applicable!

Polynomial hc-Differentiability

- $\mathbb{G}$, $\widetilde{\mathbb{G}}$ are nilpotent graded groups, $D \subset \mathbb{G}$
- $\psi : D \to \widetilde{\mathbb{G}}$
- $\widetilde{\mathfrak{d}} : \psi(D) \times \widetilde{\mathbb{G}} \to \mathbb{R}_+$, $\widetilde{\mathfrak{d}}(x,x) = 0$, $\widetilde{\mathfrak{d}}(x,y) = \widetilde{\mathfrak{d}}(y,x)$

$\Rightarrow$ ψ is <u>polynomially hc-differentiable</u> at $x \in D$ w. r. t. $\widetilde{\mathfrak{d}}$ if

$\diamond$ there exists $\mathcal{L}_x : \mathbb{G} \to \widetilde{\mathbb{G}}$ such that

1) $\widetilde{\mathfrak{d}}(\psi(w), \mathcal{L}_x\langle w\rangle) = o(d_\infty(x,w))$, $D \ni w \to x$;

2) $\mathcal{L}_x(w) = \theta_{\psi(x)} \circ L_x \circ \theta_x^{-1}(w)$;

3) L_x is polynomial in $\{w_i\}_{i=1}^N$, where $\exp\Big(\sum\limits_{j=1}^N w_i X_i\Big)(x) = w$.

$\diamond$ If L_x is linear and block-diagonal then we have <u>hc-differentiability</u>

Polynomial hc-Differentiability of Graph Mappings

$\Rightarrow$ φ_Γ is polynomially hc-differentiable where φ is hc-differentiable
$\Rightarrow$ in coordinates w. r. t. x and $\varphi_\Gamma(x)$, $\widehat{D}_{\underset{\sim}{P}}\varphi_\Gamma(x)$ assigns to $y_1,\dots,y_N$ the element $(\Delta_1(x,y),\dots,\Delta_N(x,y),\widetilde{\Delta}_1(x,y),\dots,\widetilde{\Delta}_N(x,y))$,

$$\begin{cases} \Delta_i(x,y) &= y_i + \sum\limits_{\substack{|\kappa+\mu+\lambda|_h=\deg X_i,\\ \kappa,|\lambda|+|\mu|>0}} K^{X_i}_{\kappa,\mu,\lambda} y^\kappa(-\varphi(x))^\mu(\varphi(x))^\lambda \\ &\quad + \sum\limits_{\substack{|\alpha+\beta+\gamma+\tau|_h=\deg X_i,\\ \alpha,\tau>0}} L^{X_i}_{\alpha,\beta,\gamma,\tau}(-\varphi(x))^\beta(\varphi(x))^\gamma y^\alpha(\widehat{D}\varphi(x)\langle y\rangle)^\tau, \\ \widetilde{\Delta}_j(x,y) &= (\widehat{D}\varphi(x)\langle y\rangle)_j + \sum\limits_{\substack{|\kappa+\mu+\lambda|_h=\deg \widetilde{X}_j,\\ \kappa,|\lambda|+|\mu|>0}} K^{\widetilde{X}_j}_{\kappa,\mu,\lambda} y^\kappa(-\varphi(x))^\mu(\varphi(x))^\lambda \\ &\quad + \sum\limits_{\substack{|\alpha+\beta+\gamma+\tau|_h=\deg \widetilde{X}_j,\\ \alpha,\tau>0}} L^{\widetilde{X}_j}_{\alpha,\beta,\gamma,\tau}(-\varphi(x))^\beta(\varphi(x))^\gamma y^\alpha(\widehat{D}\varphi(x)\langle y\rangle)^\tau \end{cases}$$

The Intrinsic Basis

$\ominus$ Disadvantage: too big coefficients at X if $\deg X \geq 2$

$\diamond$ Idea: make ψ and $\widehat{D}_P\psi(x)\langle y\rangle$ «almost intrinsically Lipschitz»

$\Rightarrow$ if $\psi : \mathbb{G} \to \left(\widehat{\mathbb{G}}, \{{}^x\widehat{X}_k\}_{k=1}^{\widehat{N}}\right)$, and $\widehat{D}_P\psi(x)\langle y\rangle = (\kappa_1, \ldots, \kappa_{\widehat{N}})$ then

$$|\kappa_j| = O\Big(d_\infty(x,y)^{\deg \widehat{X}_j}\Big), \quad j = 1, \ldots, \widehat{N}$$

in some new basis $\{{}^x\widehat{X}_k\}_{k=1}^{\widehat{N}}$.

$\diamond$ The collection $\{{}^x\widehat{X}_k\}_{k=1}^{\widehat{N}}$ is adapted, or intrinsic basis at $\psi(x)$.

$\diamond$ If $y, z \in \widehat{\mathbb{G}}$, $y = \exp\Big(\sum_{i=1}^{\widehat{N}} y_k {}^x\widehat{X}_k\Big)(z)$, then $d_\infty^x(y,z) = \max\limits_{k=1,\ldots,\widehat{N}} \{|y_k|^{\frac{1}{\deg \widehat{X}_k}}\}$.

Differential of Polynomial Differential (Graph of Lipschitz Mappings, Initial Basis)

$$\begin{pmatrix}
E_{\dim V_1} & 0 & 0 & \dots & 0 & 0 \\
* & E_{\dim V_2} & 0 & \dots & 0 & 0 \\
* & * & E_{\dim V_3} & \dots & 0 & 0 \\
\dots & & & \cdots & 0 & 0 \\
* & * & * & \cdots & * & E_{\dim V_M} \\
(\widehat{D}\varphi)_{V_1,\widetilde{V}_1}(x) & 0 & 0 & \dots & 0 & 0 \\
* & (\widehat{D}\varphi)_{V_2,\widetilde{V}_2}(x) & 0 & \dots & 0 & 0 \\
* & * & (\widehat{D}\varphi)_{V_3,\widetilde{V}_3}(x) & \dots & 0 & 0 \\
* & * & * & \cdots & * & (\widehat{D}\varphi)_{V_M,\widetilde{V}_M}(x)
\end{pmatrix}$$

Differential of Polynomial Differential

(Graph of Smooth Lipschitz Mappings, Intrinsic Basis)

$$\begin{pmatrix}
E_{\dim V_1} & 0 & 0 & \dots & 0 & 0 \\
0 & E_{\dim V_2} & 0 & \dots & 0 & 0 \\
0 & 0 & E_{\dim V_3} & \dots & 0 & 0 \\
\dots & & & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & E_{\dim V_M} \\
(\widehat{D}\varphi)_{V_1,\widetilde{V}_1}(x) & 0 & 0 & \dots & 0 & 0 \\
0 & (\widehat{D}\varphi)_{V_2,\widetilde{V}_2}(x) & 0 & \dots & 0 & 0 \\
0 & 0 & (\widehat{D}\varphi)_{V_3,\widetilde{V}_3}(x) & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & (\widehat{D}\varphi)_{V_M,\widetilde{V}_M}(x)
\end{pmatrix}$$

Minkowski Geometry

- length of a vector in $\mathbb{R}^4$ equals

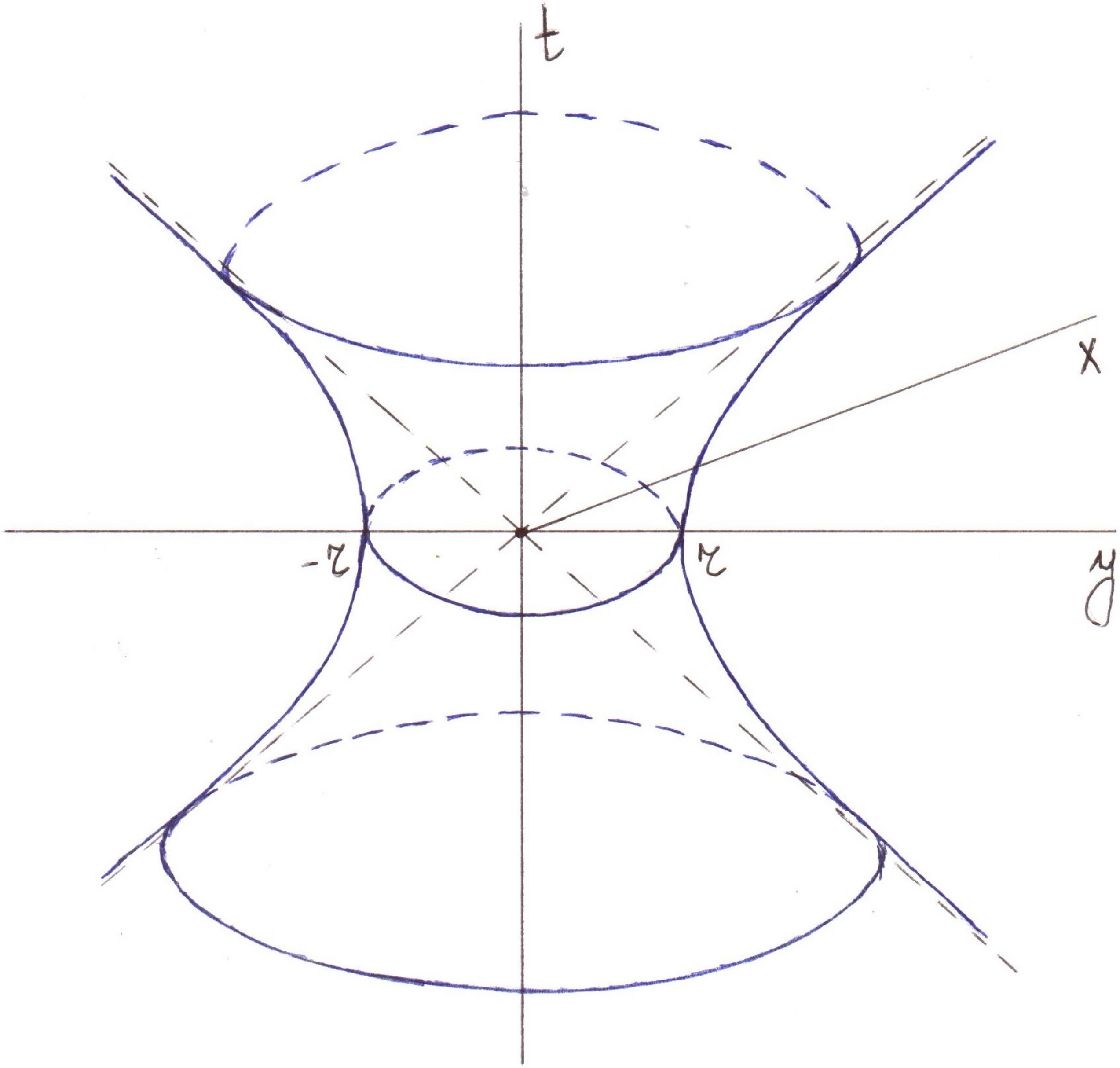
$$|v|_L = \begin{cases} \sqrt{x^2+y^2+z^2-t^2}, & x^2+y^2+z^2-t^2 \geq 0 \\ i\sqrt{|x^2+y^2+z^2-t^2|}, & x^2+y^2+z^2-t^2 < 0, \end{cases}$$

where $v = xe_1 + ye_2 + ze_3 + te_4$.

- if $a, b \in \mathbb{R}^4$ and $b = a + v$ then

$$d_L(a,b) = |v|_L.$$

- V. M. Miklyukov, A. A. Klyachin, V. A. Klyachin, 2011 (and references)



t
x
-r
r
y

Generalizations

- 5-dimensional sub-Lorentzian case: V. R. Krym, N. N. Petrov (including applications to physics)

- Multi-dimensional time: W. Craig, S. Weinstein, I. Bars, J. Terning, etc.

Sub-Lorentzian Structure with Multi-Dimensional Time

- $(\mathbb{G}, X_1, \dots, X_N), (\widetilde{\mathbb{G}}, \widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}) \subset \mathbb{U}$
- $\{X_1, \dots, X_N, \widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}\} = \{Y_1, \dots, Y_{\widehat{N}}\}$
- $\widehat{N} = N + \widetilde{N}$, $\widehat{M} = \max\{M, \widetilde{M}\}$, $\widehat{V} = \bigoplus_{k=1}^{\widehat{M}} \widehat{V}_k$, $[\widehat{V}_1, \widehat{V}_k] \subset \widehat{V}_{k+1}$
- $\widehat{V}_k = \mathrm{span}\{\widehat{V}_k^{+}, \widehat{V}_k^{-}\}$
- $\widehat{V}_k^{+} \cap \widehat{V}_k^{-} = \{0\}$
- $\widehat{V}_k^{-} \subseteq \widetilde{V}_k, \quad k = 1, \dots, \widehat{M}$

Sub-Lorentzian Squared Distance

- adapted basis: $\{{}^x X_1, \ldots, {}^x X_N, {}^x \widetilde{X}_1, \ldots, {}^x \widetilde{X}_{\widetilde{N}}\} = \{{}^x Y_1, \ldots, {}^x Y_{\widehat{N}}\}$

- If $w = \exp\Big(\sum_{j=1}^{\widehat{N}} w_j {}^x Y_j\Big)(v)$ then

$$
{}^x\mathfrak{d}_2^2(w,v) = \max_{k=1,\ldots,\widehat{M}} \Bigg\{ \operatorname{sgn}\Big(\sum_{j\colon Y_j \in \widehat{V}_k^+} w_j^2 - \sum_{j\colon Y_j \in \widehat{V}_k^-} w_j^2 \Big) \times \\
\times \Big| \sum_{j\colon Y_j \in \widehat{V}_k^+} w_j^2 - \sum_{j\colon Y_j \in \widehat{V}_k^-} w_j^2 \Big|^{1/k} \Bigg\}.
$$

Balls ans Light Cones

- ${}^{x}\,\mathsf{Box}_2^{\mathfrak{d}}(v,r) = \{w \in \mathbb{U} :\ {}^{x}\mathfrak{d}_2^2(v,w) < r^2\}$
- ${}^{x}\theta_v^{-1}({}^{x}\,\mathsf{Box}_2^{\mathfrak{d}}\, v, r))$ is a Cartesian product of B_k, where B_k is

$$\Bigg\{ y \in \mathbb{R}^{\dim \widehat{V}_k} :\ \mathrm{sgn}\Bigg(\sum_{j:\, Y_j \in \widehat{V}_k^+} y_j^2 - \sum_{j:\, Y_j \in \widehat{V}_k^-} y_j^2 \Bigg) \times \\ \times \Bigg| \sum_{j:\, Y_j \in \widehat{V}_k^+} y_j^2 - \sum_{j:\, Y_j \in \widehat{V}_k^-} y_j^2 \Bigg| < r^{2k} \Bigg\},\ k = 1, \ldots, \widehat{M}$$

- the <u>light cones</u> for $v \in \varphi_\Gamma(\Omega)$:

$$\left\{ \exp\Bigg(\sum_{j=1}^{\widehat{N}} y_j{}^{\varphi_\Gamma^{-1}(v)} Y_j \Bigg)(v) :\ \sum_{j:\, Y_j \in \widehat{V}_k^+} y_j^2 - \sum_{j:\, Y_j \in \widehat{V}_k^-} y_j^2 = 0,\ k = 1, \ldots, \widehat{M} \right\}$$

The Intrinsic Measure

- Assume that $S = \varphi_\Gamma(\Omega)$, $A \subset S$

◇ Define for $\nu = \sum_{j=1}^{M} j \dim V_j$ the measure

$$\mathcal{H}_\Gamma^\nu(A) = \prod_{j=1}^{M} \omega_{\dim V_j} \cdot \liminf_{\delta \to 0} \Big\{ \sum_{i \in \mathbb{N}} r_i^\nu : \bigcup_{i \in \mathbb{N}} {}^{\varphi_\Gamma^{-1}(x_i)}\mathrm{Box}_2^{\mathfrak{d}}(x_i, r_i) \supset A,\ x_i \in A,\ r_i < \delta \Big\}.$$

Requirements on $\widehat{D}\varphi$

$$D\widehat{D}_P\varphi_\Gamma(x) = \begin{pmatrix}
E_{\dim V_1} & 0 & \ldots & 0 & & \\
(\widehat{D}\varphi)^{+}_{\widetilde{V}_1,V_1}(x) & 0 & \ldots & 0 & & \\
(\widehat{D}\varphi)^{-}_{\widetilde{V}_1,V_1}(x) & 0 & \ldots & 0 & & \\
0 & E_{\dim V_2} & \ldots & 0 & & \\
0 & (\widehat{D}\varphi)^{+}_{\widetilde{V}_2,V_2}(x) & \ldots & 0 & & \\
0 & (\widehat{D}\varphi)^{-}_{\widetilde{V}_2,V_2}(x) & \ldots & 0 & & \\
\ldots & & & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & E_{\dim V_M} \\
0 & 0 & 0 & \cdots & 0 & (\widehat{D}\varphi)^{+}_{\widetilde{V}_M,V_M}(x) \\
0 & 0 & 0 & \cdots & 0 & (\widehat{D}\varphi)^{-}_{\widetilde{V}_M,V_M}(x)
\end{pmatrix}$$

◇ lengths of columns of $(\widehat{D}\varphi)^{-}_{\widetilde{V}_k,V_k}$ do not exceed $\frac{1}{\dim V_k} - c$, $c > 0$

The Area Formulas for Spacelike Surfaces

- $\varphi \in C^1_H$ (+some requirements on constants) or φ is smooth

◇ $\varphi_\Gamma(\Omega)$ is spacelike: it lies locally outside light cones
◇ the following area formula is valid for graph mappings φ_Γ:

$$\int\limits_{\Omega} {}^{SL}\mathcal{J}(\varphi, v)\, d\mathcal{H}^\nu(v) = \int\limits_{\varphi_\Gamma(\Omega)} d\ {}^{SL}\mathcal{H}^\nu_\Gamma(y),$$

where ${}^{SL}\mathcal{J}(\varphi, v)$ equals

$$\prod_{j=1}^{M} \sqrt{\det\Big(E_{\dim V_j} + \big(\widehat{\mathcal{D}}\varphi^+\big)^*_{\widetilde{V}_j, V_j}\big(\widehat{\mathcal{D}}\varphi^+\big)_{\widetilde{V}_j, V_j} - \big(\widehat{\mathcal{D}}\varphi^-\big)^*_{\widetilde{V}_j, V_j}\big(\widehat{\mathcal{D}}\varphi^-\big)_{\widetilde{V}_j, V_j}\Big)}.$$

$\varphi \notin C^1_H \Rightarrow \widehat{\mathcal{D}}\varphi = D\widehat{D}_P\varphi$, $\varphi \in C^1_H \Rightarrow \widehat{\mathcal{D}} = \widehat{D}$, $\big(\widehat{D}\varphi^+\big)_{\widetilde{V}_j, V_j} = 0$, $j > 2$

- Applications: maximal surfaces theory

Papers

1. Karmanova M. B. Area of Graph Surfaces on Carnot Groups with Sub-Lorentzian Structure // Dokl. Math., 2019. V. 99, No 2. P. 145–148.

2. Karmanova M. B. Sufficient Maximality Conditions for Surfaces on Two-Step Sub-Lorentzian Structures // Dokl. Math., 2019. V. 99, No 2. P. 214–217.

3. Karmanova M. B. Graphs of Nonsmooth Contact Mappings on Carnot Groups with sub-Lorentzian structure // Dokl. Math., 2019 (accepted)

4. Karmanova M. B. Two-Step sub-Lorentzian Structures and Graph Surfaces // Izv. Math., 2019 (accepted)

5. Karmanova M. B. Area of Graph Surfaces on Arbitrary Carnot Groups with Sub-Lorentzian Structure // Sib. Math. J., in print

THANK YOU!