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Symmetries of the space of connections on a principal G-bundle and related symplectic structures

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The phase space of a typical Hamiltonian system is the cotangent bundle T^*P of its configurations space P equipped with the standard symplectic form. Usually one considers the case when Hamiltonian of this system is invariant with respect to the cotangent lift of an action of some group G on P. Therefore, it is sometimes reasonable to replace the standard symplectic form with another G-invariant symplectic form on T^*P which retains certain properties. Assuming that the action of G on P is free and the quotient space P/G is a manifold M one can consider P as the total space of the principal G-bundle $P(M, G)$.

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• The space $ConnP(M,G)$ of connections on the principal G-bundle $P(M,G)$.

• The space $CanT^*P$ of fibre-wise linear differential one-forms γ on the cotangent bundle T^*P, which annihilate the vectors tangent to the fibres of T^*P.

- G-principal bundle over a manifold M

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow{\scriptstyle \mu} \\ & & M \cong P/G \end{array}$$

where the free action of G we denote by

$$\kappa : P \times G \to P, \quad \kappa(p, g) := pg$$

and

$$\kappa_g : P \to P \qquad \kappa_g(p) := pg$$
$$\kappa_p : G \to P \qquad \kappa_p(g) := pg$$

Action of TG on TP

• TG is a Lie group with the product and the inverse defined by the tangent the product m and to the inverse ι in G:

$$Tm_{(g,h)}(X_g, Y_h) =: X_g \bullet Y_h = TL_g(h)Y_h + TR_h(g)X_g, \qquad (1)$$

$$T\iota_g(X_g) =: X_g^{-1} = -TL_{g^{-1}}(e) \circ TR_{g^{-1}}(g)X_g \qquad (2)$$

where $X_g \in T_gG$, $Y_h \in T_hG$ and $L_g(h) := gh$, $R_g(h) := hg$.

Action of TG on TP

The diffeomorphism $I : G \times T_eG \to TG$

$$I(g, X_e) = TR_g(e)X_e =: X_g \tag{3}$$

allows us to consider TG as the semidirect product $G \ltimes_{Ad_G} T_eG$ of G by the T_eG, where the group product of $(g, X_e), (h, Y_e) \in G \ltimes_{Ad_G} T_eG$ is given by

$$(g, X_e) \bullet (h, Y_e) = I^{-1}(I(g, X_e) \cdot I(h, Y_e)) = \tag{4}$$

$$= (gh, X_e + T(R_{g^{-1}} \circ L_g)(e)Y_e) = (gh, X_e + Ad_gY_e).$$

Action of TG on TP

Using the above isomorphisms we obtain the action of $G \ltimes_{Ad_G} T_eG$ on the tangent bundle TP as the tangent to κ

$$\Phi_{(g,X_e)}(v_p) := T\kappa_{g,p}((g, X_e), v_p) = T\kappa_g(p)v_p + T\kappa_p(g)TR_g(e)X_e \tag{5}$$

.

Applying the above action we obtain the following isomorphisms

$$TP/T^vP \cong TP/T_eG, \tag{6}$$

$$TP/TG \cong (TP/T_eG)/G \cong (TP/G)/T_eG, \tag{7}$$

$$TM = T(P/G) \cong TP/TG, \tag{8}$$

of vector bundles, where we write $T^vP := KerT\mu$ for the vertical subbundle of TP.

Groups of automorphisms of TP

• We consider the group $Aut_0(TP)$ of smooth automorphisms

$$\begin{array}{ccc} TP & \xrightarrow{A} & TP \\ \pi \downarrow & & \downarrow \pi \\ P & \xrightarrow{id} & P \end{array}$$

$A(p) : T_pP \to T_pP$ is an isomorphism
of the tangent space T_pP
depends smoothly on p

• $Aut_0(TP)$ is a normal subgroup of the group $Aut(TP)$ of all automorphisms of TP.

• The subgroup $Aut_{TG}(TP) \subset Aut_0(TP)$ consisting of those elements of $Aut_0(TP)$ whose action on TP commutes with the action (5) of $TG \cong G \ltimes_{Ad_g} T_eG$ on TP, i.e.

$$\begin{array}{ccc} TP & \xrightarrow{A} & TP \\ \Phi \downarrow & & \downarrow \Phi \\ TP & \xrightarrow{A} & TP \end{array} \qquad A(pg) \circ \Phi_{(g,X_e)} = \Phi_{(g,X_e)} \circ A(p)$$

• The group $Aut_{TG}(TP)$ acts also on vector bundles $TP/G \to M$ and $TM \to M$.

• We define the subgroup $Aut_N TP \subset Aut_{TG} TP$ consisting of $A \in Aut_{TG} TP$ such that $A(p) = id_p + B(p)$, where $B(p) : T_p P \to T^v_p P$.
From the definition of $B(p)$ one has $Im B(p) \subset T^v_p P \subset Ker B(p)$. Thus $B_1(p)B_2(p) = 0$ for any $id + B_1,\ id + B_2 \in Aut_N P$. So, one has

$$(id_p + B_1(p))(id_p + B_2(p)) = id_p + B_1(p) + B_2(p). \qquad (9)$$

This shows that $Aut_N TP$ is a commutative subgroup of $Aut_{TG} TP$.

Proposition

One has the following short exact sequence

$$\{0\} \to Aut_N TP \xrightarrow{\iota} Aut_{TG} TP \xrightarrow{\lambda} Aut_0 TM \to \{id\} \tag{10}$$

of the group morphisms,
where ι is the inclusion map
and λ is an epimorphism covering the identity map of M defined by

$$\lambda(A)(\mu(p))(T\mu(p))v_p := (T\mu(p) \circ A(p))v_p \tag{11}$$

for $v_p \in T_p P$.

Connection form

• A connection form on P is a T_eG-valued differential one-form α satisfying the conditions

$$\alpha_p \circ T\kappa_p(e) = id_{T_eG} \tag{12}$$

$$\alpha_{pg} \circ T\kappa_g(p) = Ad_{g^{-1}} \circ \alpha_p \tag{13}$$

valid for value α_p of α at $p \in P$ and $g \in G$.
Using α one defines the decomposition

$$T_pP = T_p^vP \oplus T_p^{\alpha,h}P \tag{14}$$

of T_pP on the vertical T_p^vP and the horizontal $T_p^{\alpha,h}P := Ker\alpha_p$ subspaces which also satisfy the G-equivariance properties

$$T\kappa_g(p)T_p^vP = T_{pg}^vP, \tag{15}$$

$$T\kappa_g(p)T_p^{\alpha,h}P = T_{pg}^{\alpha,h}P. \tag{16}$$

From the decomposition (14) for any $p \in P$ one obtains the vector spaces isomorphism

$$\Gamma_\alpha(p) : T_{\mu(p)}M \to T_p^{\alpha,h}P \tag{17}$$

such that

$$\Gamma_\alpha(pg) = T\kappa_g(p) \circ \Gamma_\alpha(p) \tag{18}$$

and

$$T\mu(p) \circ \Gamma_\alpha(p) = id_{\mu(p)}, \qquad \Gamma_\alpha(p) \circ T\mu(p) = \Pi_\alpha^h(p), \tag{19}$$

where $\Pi_\alpha^h(p)$ is defined by the decomposition

$$id_p = \Pi_\alpha^v(p) + \Pi_\alpha^h(p) \tag{20}$$

of the identity map of T_pP on the projections corresponding to (14).

Proposition

A fixed connection α defines the injection

$$\sigma_\alpha : Aut_0 TM \to Aut_{TG} TP$$

$$\sigma_\alpha(\tilde{A})(p) := \Pi^v_\alpha(p) + \Gamma_\alpha(p) \circ \tilde{A}(\mu(p)) \circ T\mu(p), \qquad (21)$$

where $\tilde{A} \in Aut_0 TM$, the surjection $\beta_\alpha : Aut_{TG} TP \to Aut_N TP$ by $\beta_\alpha(A) := A\sigma_\alpha(\lambda(A))^{-1}$, where $A \in Aut_{TG} TP$, which are arranged into the short exact sequence

$$\{\mathrm{id}_{TM}\} \to Aut_0 TM \xrightarrow{\sigma_\alpha} Aut_{TG} TP \xrightarrow{\beta_\alpha} Aut_N TP \to \{id_{TP}\}$$

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inverse to the sequence (10). The map σ_α is a monomorphism

$$\sigma_\alpha(\tilde{A}_1\tilde{A}_2) = \sigma_\alpha(\tilde{A}_1)\sigma_\alpha(\tilde{A}_2)$$

of the groups and β_α satisfies

$$\beta_\alpha(A_1A_2) = \beta_\alpha(A_1)\sigma_\alpha(\lambda(A_1))\beta_\alpha(A_2)\sigma_\alpha(\lambda(A_1))^{-1}.$$

Proposition

Using the decomposition

$$A(p) = (\mathrm{id}_p + B(p))\sigma_\alpha(\tilde{A})(p) \tag{22}$$

of $A \in Aut_{TG}TP$, where $\mathrm{id}_p + B(p) \in Aut_N TP$ and $\tilde{A} \in Aut_0 TM$,
one defines an isomorphism

$$Aut_{TG}TP \longrightarrow Aut_0 TM \ltimes_\alpha End_N TP,$$

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where the product of $(\tilde{A}_1, B_1), (\tilde{A}_2, B_2) \in Aut_0 TM \ltimes_\alpha End_N TP$ is given by

$$[(\tilde{A}_1, B_1) \cdot (\tilde{A}_2, B_2)](p) := \quad (23)$$

$$= (\tilde{A}_1(\mu(p))\tilde{A}_2(\mu(p)), B_1(p) + B_2(p) \circ \Gamma_\alpha(p) \circ \tilde{A}_1^{-1}(\mu(p)) \circ T\mu(p)).$$

Action on the space of connections

• Let $ConnP(M,G)$ be the space of all connections on $P(M,G)$. We define

$$\phi_A(\alpha)_p := \alpha_p \circ A(p)^{-1} \tag{24}$$

the left action $\phi_A : ConnP(M,G) \to ConnP(M,G)$ of $Aut_{TG}TP$ on $ConnP(M,G)$, i.e. ϕ satisfies $\phi_{A_1 A_2} = \phi_{A_1} \circ \phi_{A_1}$ for $A_1, A_2 \in Aut_{TG}TP$.

The following proposition shows that one can define the group $Aut_{TG}TP$ in terms of connections space $ConnP(M,G)$.

Proposition

If $A \in Aut_0(TP)$ and $\phi_A(ConnP(M,G)) \subset ConnP(M,G)$ then $A \in Aut_{TG}(TP)$.

• We recall that the standard symplectic form on T^*P is $\omega_0 = d\gamma_0$, where $\gamma_0 \in C^\infty T^*(T^*P)$ is the canonical one-form on T^*P defined at $\varphi \in T^*P$ by

$$\langle \gamma_{0\varphi}, \xi_\varphi \rangle := \langle \varphi, T\pi^*(\varphi)\xi_\varphi \rangle,$$

where $\pi^* : T^*P \to P$ is the projection of T^*P on the base and $\xi_\varphi \in T_\varphi(T^*P)$.

A linear vector field

• By definition a *linear vector field* on T^*P is a pair (ξ, χ) of vector fields $\xi \in C^\infty T(T^*P)$ and $\chi \in C^\infty TP$ such that

$$\begin{array}{ccc} T^*P & \xrightarrow{\ \xi\ } & T(T^*P) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle T\pi^*} \\ P & \xrightarrow{\ \chi\ } & TP \end{array}$$

defines a morphism of vector bundles. Note here that $T\pi^*(\varphi)\xi_\varphi = \chi_{\pi^*(\varphi)}$.

A linear vector field

• We will denote by $LinC^\infty T(T^*P)$ the Lie algebra of linear vector fields over the vector bundle $\pi^* : T^*P \to P$. The Lie bracket of (ξ_1, χ_1), $(\xi_2, \chi_2) \in LinC^\infty T(T^*P)$ is defined by

$$[(\xi_1, \chi_1), (\xi_2, \chi_2)] := ([\xi_1, \xi_2], [\chi_1, \chi_2])$$

and the vector space structure on $LinC^\infty T(T^*P)$ by

$$c_1(\xi_1, \chi_1) + c_2(\xi_2, \chi_2) := (c_1\xi_1 + c_2\xi_2, c_1\chi_1 + c_2\chi_2).$$

Let $LinC^\infty(T^*P)$ denote the vector space of smooth fibre-wise linear functions on T^*P. Spaces $LinC^\infty T(T^*P)$ and $LinC^\infty(T^*P)$ have structures of $C^\infty(P)$-modules defined by $f(\xi, \chi) := ((f \circ \pi^*)\xi, f\chi)$ and by $fl := (f \circ \pi^*)l$, respectively, where $f \in C^\infty(P)$ and $l \in LinC^\infty(T^*P)$.

Definition

• A differential one-form $\gamma \in C^\infty T^*(T^*P)$ is called a *generalized canonical form* on T^*P if:

(i) $\gamma_\varphi \neq 0$ for any $\varphi \in T^*P$,

(ii) $kerT\pi^*(\varphi) \subset ker\ \gamma_\varphi$

(iii) $\langle \gamma, \xi \rangle \in LinC^\infty(T^*P)$ for any $\xi \in LinC^\infty T(T^*P)$.

The space of generalized canonical forms on T^*P will be denoted by $CanT^*P$. Let us note here that $\gamma_0 \in CanT^*P$.

Proposition

(i) The map $\Theta : Aut_0 TP \to Can T^* P$ defined by

$$\langle \Theta(A)_\varphi, \xi_\varphi \rangle := \langle \varphi, A(\pi^*(\varphi)) T\pi^*(\varphi)) \xi_\varphi \rangle, \qquad (25)$$

where $\xi_\varphi \in T_\varphi(T^* P)$, is bijective.

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where $\xi_\varphi \in T_\varphi(T^*P)$, is bijective.

(ii) The natural left action $L^* : Aut_0TP \times CanT^*P \to CanT^*P$ of Aut_0TP on $CanT^*P$ defined by

$$\langle(L_A^*(\gamma))_\varphi, \xi_\varphi\rangle := \langle\gamma_{A^*(\varphi)}, TA^*(\varphi)\xi_\varphi\rangle, \tag{26}$$

where $A^* : T^*P \to T^*P$ is the dual of $A \in Aut_0TP$, is a transitive and free action.

The lift $\Phi_g^* : T^*P \to T^*P$ of the action $\kappa_g : P \to P$ to the cotangent bundle T^*P is defined by

$$\Phi_g^*(\varphi)(pg) = (T\kappa_g(p)^{-1})^*\varphi \quad (27)$$

where $p = \pi^*(\varphi)$.
For G-invariant symplectic form $\omega_A = d\gamma = d\Theta(A)$ one has the G-equivariant momentum map $J_A : T^*P \to T_e^*G$ given by $J_A = J_0 \circ A^*$.

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$$J_0(\varphi) = \varphi \circ T\kappa_{\pi^*(\varphi)}(e).$$

It is reasonable to define the space

$$Can_{TG}T^*P := \Theta(Aut_{TG}TP)$$

which is an $Aut_{TG}TP$-invariant subspace of the space $CanT^*P$.

Proposition

(i) The generalized canonical form $\Theta(A)$ belongs to $Can_{TG}T^*P$ if and only if

$$(\Phi_g^*)^*\Theta(A) = \Theta(A)$$

and $J_A = J_0 \circ A^* = J_0$.

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(ii) One can consider $Can_{TG}T^*P$ as the orbit of the subgroup $Aut_{TG}TP \subset Aut_0TP$ taken through γ_0 with respect to the free action L^* defined in (26).

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(ii) One can consider $Can_{TG}T^*P$ as the orbit of the subgroup $Aut_{TG}TP \subset Aut_0TP$ taken through γ_0 with respect to the free action L^* defined in (26).

(iii) If $A \in Aut_0TP$ and $L_A^*(Can_{TG}T^*P) \subset Can_{TG}T^*P$ then $A \in Aut_{TG}TP$.

Corrolary

Fixing a connection α one obtains an embedding

$$\iota_\alpha : ConnP(M,G) \hookrightarrow Can_{TG}T^*P$$

defined as follows

$$\iota_\alpha(\alpha') := \varphi \circ T\pi^*(\varphi) + \varphi \circ T\kappa_{\pi^*(\varphi)}(e) \circ (\alpha'_{\pi^*(\varphi)} - \alpha_{\pi^*(\varphi)}) \circ T\pi^*(\varphi). \tag{28}$$

- A G-equivariant diffeomorphism

$$T^*P \xrightarrow{\quad I_\alpha \quad} \overline{P} \times T_e^*G$$

- A G-equivariant diffeomorphism

$$T^*P \xrightarrow{\qquad I_\alpha \qquad} \overline{P} \times T_e^*G$$

dependent on a fixed connection α

$$I_\alpha(\varphi) := (\Gamma_\alpha^*(\pi^*(\varphi))(\varphi), \pi^*(\varphi), \varphi \circ T\kappa_{\pi^*(\varphi)})),$$

where

$$\overline{P} := \{(\tilde{\varphi}, p) \in T^*M \times P : \; \tilde{\pi}^*(\tilde{\varphi}) = \mu(p)\}$$

is the total space of the principal bundle $\overline{P}(T^*M, G)$ being the pullback of the principal bundle $P(M, G)$ to T^*M by the projection $\tilde{\pi}^* : T^*M \to M$ of T^*M on the base M.

$$T^*P \xrightarrow{I_\alpha} \overline{P} \times T_e^*G$$

$$T^*P \underset{I_\alpha^{-1}}{\overset{I_\alpha}{\rightleftarrows}} \overline{P} \times T_e^*G$$

$$I_\alpha^{-1}(\tilde{\varphi}, p, \chi) = \tilde{\varphi} \circ T\mu(p) + \chi \circ \alpha_p \tag{29}$$

is the inverse to I_α.
Because of the group $Aut_{TG}TP$ acts on TP, we also can define the natural right action of $Aut_{TG}TP$ on T^*P

$$(A^*\varphi)(\pi^*(\varphi)) := \varphi \circ A(\pi^*(\varphi))$$

for $A \in Aut_{TG}TP$,
and the action of G on T^*P

$$\Phi_g^*(\varphi)(pg) = (T\kappa_g(p)^{-1})^*\varphi.$$

Using I_α we transport above actions to $\overline{P} \times T_e^* G$:

$$\Lambda_\alpha(A)(\tilde{\varphi}, p, \chi) := (I_\alpha \circ A^* \circ I_\alpha^{-1})(\tilde{\varphi}, p, \chi) =$$

$$= ((\tilde{\varphi} \circ T\mu(p) + \chi \circ \alpha_p) \circ A(p) \circ \Gamma_\alpha(p), p, \chi) \tag{30}$$

and by

$$\psi_g^*(\tilde{\varphi}, p, \chi) := (I_\alpha \circ \Phi_g^* \circ I_\alpha^{-1})(\tilde{\varphi}, p, \chi) = (\tilde{\varphi}, pg, Ad^*_{g^{-1}} \chi), \tag{31}$$

respectively.

Using $I_\alpha^{-1} : \overline{P} \times T_e^* G \to T^* P$ we pull the generalized canonical form $\Theta(A)$ back to $\overline{P} \times T_e^* G$, where

$$\langle \Theta(A)_\varphi, \xi_\varphi \rangle := \langle \varphi, A(\pi^*(\varphi)) T\pi^*(\varphi)) \xi_\varphi \rangle, \tag{32}$$

for $\xi_\varphi \in T_\varphi(T^* P)$.

For $A = (\mathrm{id}_{TP} + B)\sigma_\alpha(\tilde{A})$ we have

$$(I_\alpha^{-1})^* \Theta(A)(\tilde{\varphi}, p, \chi) = \tag{33}$$

$$= \tilde{\varphi} \circ \tilde{A}(\mu(p)) \circ T(\tilde{\pi}^* \circ pr_1)(\tilde{\varphi}, p, \chi) + \chi \circ \alpha_p \circ A(p) \circ Tpr_2(\tilde{\varphi}, p, \chi) =$$

$$= pr_1^*(\tilde{\Theta}(\tilde{A})(\tilde{\varphi}, p, \chi) + \langle pr_3(\tilde{\varphi}, p, \chi), pr_2^*(\phi_{A^{-1}}(\alpha))(\tilde{\varphi}, p, \chi) \rangle,$$

where $pr_3(\tilde{\varphi}, p, \chi) := \chi$.

The symplectic form corresponding to (33) is given by

$$d((I_\alpha^{-1})^*\Theta(A)) = \tag{34}$$

$$= pr_1^*(d\tilde{\Theta}(\tilde{A})) + \langle d\ pr_3 \ \overset{\wedge}{,}\ pr_2^*(\phi_{A^{-1}}(\alpha))\rangle + \langle pr_3, pr_2^*(d\phi_{A^{-1}}(\alpha))\rangle.$$

Considering P as the configuration space of a physical system which has a symmetry described by G one consequently assumes that its Hamiltonian $H \in C^\infty(T^*P)$ is a G-invariant function on T^*P, i.e. $H \circ \Phi_g^* = H$ for $g \in G$. Hence it is natural to consider the class of Hamiltonian systems on G-symplectic manifold (T^*P, ω_A, J_0) with a G-invariant Hamiltonian H.

Using the isomorphism $(T^*P, \omega_A, J_0) \cong (\overline{P} \times T_e^*G, (I_\alpha^{-1})^*\omega_A, pr_3)$ of G-symplectic manifolds, where the symplectic form $(I_\alpha^{-1})^*\omega_A$ was presented above and the momentum map is $J_0 \circ I_\alpha = pr_3$, one defines the G-invariant Hamiltonian $H \in C^\infty(\overline{P} \times T_e^*G)$ as follows

$$H(\tilde{\varphi}, p, \chi) := (\tilde{H} \circ \overline{\mu})(\tilde{\varphi}, p, \chi) + (C \circ pr_3)(\tilde{\varphi}, p, \chi),$$

where $\overline{\mu} : \overline{P} \to T^*M$ is the projection of the total space $\overline{P}$ of the principal G-bundle $\overline{P}(T^*M, G)$ on the base T^*M, $\tilde{H} \in C^\infty(T^*M)$ and $C \in C^\infty(T_e^*G)$ is Casimir function with respect to the standard Lie Poisson structure of T_e^*G.

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$$H_\alpha(\varphi) := (H \circ I_\alpha)(\varphi) = (\tilde{H} \circ \Gamma_\alpha^*)(\varphi) + (C \circ J_0)(\varphi).$$

Using the isomorphism $(T^*P, \omega_A, J_0) \cong (\overline{P} \times T_e^*G, (I_\alpha^{-1})^*\omega_A, pr_3)$ of G-symplectic manifolds, where the symplectic form $(I_\alpha^{-1})^*\omega_A$ was presented above and the momentum map is $J_0 \circ I_\alpha = pr_3$, one defines the G-invariant Hamiltonian $H \in C^\infty(\overline{P} \times T_e^*G)$ as follows

$$H(\tilde{\varphi}, p, \chi) := (\tilde{H} \circ \overline{\mu})(\tilde{\varphi}, p, \chi) + (C \circ pr_3)(\tilde{\varphi}, p, \chi),$$

where $\overline{\mu} : \overline{P} \to T^*M$ is the projection of the total space $\overline{P}$ of the principal G-bundle $\overline{P}(T^*M, G)$ on the base T^*M, $\tilde{H} \in C^\infty(T^*M)$ and $C \in C^\infty(T_e^*G)$ is Casimir function with respect to the standard Lie Poisson structure of T_e^*G. Coming back to the phase space (T^*P, ω_A, J_0) one obtains the G-Hamiltonian system with the Hamiltonian

$$H_\alpha(\varphi) := (H \circ I_\alpha)(\varphi) = (\tilde{H} \circ \Gamma_\alpha^*)(\varphi) + (C \circ J_0)(\varphi).$$

Let us stress that in the case of $(T^*P, \omega_A, J_0, H_\alpha)$ only the Hamiltonian H_α of the system depends on $\alpha \in ConnP(M, G)$ and in the case of $(\overline{P} \times T_e^*G, (I_\alpha^{-1})^*\omega_A, pr_3, H)$ the only symplectic form $(I_\alpha^{-1})^*\omega_A$.

REFERENCES:

1. J. P. Dufour, N. T.Zung. *Poisson Structures and their normal forms*, 242, Birkhauser Verlag, 2000.
2. K. C. H. Mackenzie.*General theory of Lie groupoids and Lie algebroids*, 213, *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
3. K. C. H. Mackenzie, A. Odzijewicz, A.Sliżewska. *Poisson geometry related to Atiyah sequences*, SIGMA 14 (2018), 005
4. S.Sternberg. *Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field*. Proc.Natl.Acad.Sci.USA.,74(12):5253-5254, 1988.
5. A. Weinstein.*A universal phase space for particles in Yang-Mills fields*. Lett. Math. Phys., 2(5):417–420, 1977/78.

THANK YOU FOR ATTENTION