On a General Formulation of Classical Nonlinear Electrodynamics with Conformal Symmetry

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A brief overview of the talk

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3. Nonlinear constitutive equations
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References

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1. Conformal symmetry and Maxwell's equations

The Minkowski $3+1$-dimensional spacetime $M^{(4)}$ is described by coordinates $x = (x^\mu), \mu = 0, 1, 2, 3$, or $x = (ct, x^k), k = 1, 2, 3$. The flat metric tensor is $\eta_{\mu\nu} = diag\,(1, -1, -1, -1)$.

We shall also write $(\partial_\mu) = (\partial/\partial x^\mu) \,=\, \left(c^{-1}\partial/\partial t, \nabla\right)$.

As usual $x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu$, summing over repeated indices. Sometimes for brevity we will set $x^2 = x_\mu x^\mu$. The light cone is defined by $\{x | x_\mu x^\mu = 0\}$.

The conformal group for $M^{(4)}$ includes spacetime translations, spatial rotations, Lorentz boosts, and dilations. It also contains the special conformal transformations. These may be obtained by conjugating translation $x'^{\,\mu} = x^{\,\mu} - b^{\,\mu}$ with a certain discrete transformation, conformal inversion (which is not usually taken to be an element of the conformal group).

Conformal inversion is given by:

$$x'^{\,\mu} = \frac{x^{\,\mu}}{x_\nu x^\nu}\,, \qquad (1.1)$$

which is singular on the light cone. From (1.1) one has locally in $M^{(4)}$, $\eta_{\mu\nu} dx'^{\,\mu} dx'^{\,\nu} = [1/(x_\sigma x^\sigma)^2]\eta_{\mu\nu} dx^\mu dx^\nu$, but globally, the causal structure is not preserved.

The special conformal transformations, obtained by inverting, translating, and inverting again, are,

$$x'^{\,\mu} = \frac{(x^{\,\mu} - b^{\,\mu} x_\nu x^{\,\nu})}{(1 - 2b_\nu x^{\,\nu} + b_\nu b^{\,\nu} x_\mu x^{\,\mu})}\ . \qquad (1.2)$$

We next write the well-known conformal symmetry of electromagnetism with respect to the transformation (1.1).

Recall Maxwell's equations (in SI units):

$$\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \mathrm{div}\,\mathbf{B} = 0,$$
$$\mathrm{curl}\,\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \quad \mathrm{div}\,\mathbf{D} = \rho, \tag{1.3}$$

where $\mathbf{j}$ and ρ are current and charge densities respectively.

The electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ are physically observable; their strength can be inferred from measurement of the force $\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ on a test charge q moving with velocity $\mathbf{v}$.

The constitutive equations relating $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$ describe properties of the medium (or possibly, of spacetime itself). Taken together with Maxwell's equations (1.3), the constitutive equations determine the symmetry. The usual linear theory is obtained from linear constitutive equations.

In covariant notation, the standard electromagnetic tensor fields are:

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}\,E_x & \frac{1}{c}\,E_y & \frac{1}{c}\,E_z \\ -\frac{1}{c}\,E_x & 0 & -B_z & B_y \\ -\frac{1}{c}\,E_y & B_z & 0 & -B_x \\ -\frac{1}{c}\,E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{1.4}$$

$$G_{\mu\nu} = \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & -H_z & H_y \\ -cD_y & H_z & 0 & -H_x \\ -cD_z & -H_y & H_x & 0 \end{pmatrix}. \tag{1.5}$$

The Hodge dual tensors are $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $\tilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$.

Here $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor with $\varepsilon^{0123} = 1$.

Maxwell's equations become:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \qquad \partial_\mu G^{\mu\nu} = j^\nu, \tag{1.6}$$

where $j^\mu = (c\rho, \mathbf{j})$ is the 4-current.

The first equations in (1.6) imply that one can set $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is an abelian gauge field. In general, there is no such representation for $G_{\mu\nu}$.

The field strength tensors $F_{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ are physically observable. The relation of $G_{\mu\nu}$ to the observable fields $F_{\mu\nu}$ is determined by the properties of the medium, and expressed through the constitutive equations.

Under the inversion (1.1) we obtain the coordinate transformations:

$$\partial'_\mu := \frac{\partial}{\partial x'^\mu} = x^2 \partial_\mu - 2x_\mu (x \cdot \partial)\,, \tag{1.7}$$

$$\Box' := \partial'_\mu \partial'^\mu = (x^2)^2 \Box - 4x^2 (x \cdot \partial)\,. \tag{1.8}$$

The transformations of the fields that respect the symmetry are:

$$A'_\mu(x') = x^2 A_\mu(x) - 2x_\mu (x^\alpha A_\alpha(x))\,, \tag{1.9}$$

$$F'_{\mu\nu}(x') = (x^2)^2 F_{\mu\nu}(x) - 2x^2 x^\alpha (x_\mu F_{\alpha\nu}(x) + x_\nu F_{\mu\alpha}(x)), \tag{1.10}$$

where $x^2 = x_\mu x^\mu$ and $(x \cdot \partial) = x^\alpha \partial_\alpha$.

Combining this symmetry with that of the Poincaré transformations and dilations, we have the well-known symmetry with respect to the conformal group.

2. Invariant functionals

To study nonlinear Maxwell equations with conformal symmetry, we write constitutive equations that depend only on conformal invariants.

In $M^{(4)}$ we have two fundamental Poincaré-invariant functionals:

$$I_1 = \frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) , \quad I_2 = -\frac{c}{4} F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x) . \tag{2.1}$$

Equivalently:

$$I_1 = \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2, \qquad I_2 = \mathbf{B} \cdot \mathbf{E}. \tag{2.2}$$

Any other Poincaré-invariant functional can be expressed in terms of I_1 and I_2. Thus, basing constitutive equations for relativistic nonlinear electrodynamics on I_1 and I_2 should achieve maximum generality.

I_2 may be called a pseudoinvariant, as it changes sign under spatial reflection (parity).

Under conformal inversion, I_1 and I_2 are no longer invariant. They transform by:

$$I_1'(x') = \frac{1}{2} F'_{\mu\nu}(x')(F')^{\mu\nu}(x') = (x^2)^4 I_1(x)\,, \tag{2.3}$$

$$I_2'(x') = -\frac{c}{4} F'_{\mu\nu}(x')(\tilde{F}')^{\mu\nu}(x') = -(x^2)^4 I_2(x)\,. \tag{2.4}$$

So it is the ratio $I_2(x)/I_1(x)$ that we should consider. This is a pseudoinvariant under conformal inversion:

$$\frac{I_2'(x')}{I_1'(x')} = -\frac{I_2(x)}{I_1(x)}. \tag{2.5}$$

Because the special conformal transformations involve *two* inversions, the functional $I_2(x)/I_1(x)$ is a true invariant under the special conformal group.

3. Nonlinear constitutive equations

General nonlinear constitutive equations with Poincaré symmetry take the form:

$$\mathbf{D} = M(I_1, I_2)\,\mathbf{B} + \frac{1}{c^2} N(I_1, I_2)\,\mathbf{E},$$
$$\mathbf{H} = N(I_1, I_2)\,\mathbf{B} - M(I_1, I_2)\,\mathbf{E}, \tag{3.1}$$

where $M(I_1, I_2)$ and $N(I_1, I_2)$ are smooth scalar functions of the invariants.

The linear constitutive equations in vacuum are $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$, where ε_0 and μ_0 are respectively the permittivity and permeability of empty space, with $\varepsilon_0 \mu_0 = c^{-2}$; (so that $M = 0$, and $N = 1/\mu_0 = \varepsilon_0 c^2$).

In covariant form, these constitutive equations become

$$G_{\mu\nu} = N(I_1, I_2)\,F_{\mu\nu} + cM(I_1, I_2)\,\tilde{F}_{\mu\nu}. \tag{3.2}$$

Now the constitutive functionals M and N in a system with conformal symmetry can depend *only* on the ratio $I_2(x)/I_1(x)$. Then let us write $M(I_1, I_2) = \mathcal{M}(u)$ and $N(I_1, I_2) = \mathcal{N}(u)$, where $u = I_2/cI_1$ is dimensionless.

For convenience, denote

$$I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \equiv X, \qquad I_2 = -\frac{c}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \equiv Y. \tag{3.3}$$

Then we rewrite the constitutive equations in covariant form as:

$$G_{\mu\nu}^{(\mathrm{conformal})} \equiv G_{\mu\nu} = \mathcal{N}(u) F_{\mu\nu} + c\mathcal{M}(u) \tilde{F}_{\mu\nu} \tag{3.4}$$

where $u = Y/cX$ is the dimensionless conformal-invariant functional.

4. Lagrangian and non-Lagrangian theories

When the equations of motion derive from a Lagrangian density $L(X, Y)$, the constitutive equations are:

$$G_{\mu\nu} = -\frac{\partial L}{\partial F^{\mu\nu}} = -2\frac{\partial L(X,Y)}{\partial X} F_{\mu\nu} + c\frac{\partial L(X,Y)}{\partial Y}\tilde{F}_{\mu\nu}. \tag{4.1}$$

The functions M and N may then be written

$$N(X,Y) = \frac{1}{\mu} + N_L(X,Y) = -2\frac{\partial L(X,Y)}{\partial X}, \tag{4.2}$$

$$M(X,Y) = M_L(X,Y) = \frac{\partial L(X,Y)}{\partial Y}. \tag{4.3}$$

We singled out the constant $1/\mu$ in (4.2) so that the choice $N_L(X,Y) = M_L(X,Y) = 0$ and $\mu = \mu_0$ corresponds to the standard Lagrangian of linear electrodynamics in vacuo: $L_0(X,Y) = -(1/2\mu_0)X$.

Then a necessary and sufficient condition for the theory to be Lagrangian is that the cross-derivatives be equal; i.e.,

$$-\frac{1}{2}\frac{\partial N(X,Y)}{\partial Y} = \frac{\partial M(X,Y)}{\partial X}\,, \qquad (4.4)$$

since both sides must equal $\partial^2 L(X,Y)/\partial X \partial Y$.

If M and N violate (4.4), then the theory cannot be Lagrangian.

5. Conformal-invariant Lagrangian nonlinear electrodynamics

Now let us consider conformal-invariant nonlinear electrodynamics, with the constitutive equations above: $G_{\mu\nu} = \mathcal{N}(u)\, F_{\mu\nu} + c\mathcal{M}(u)\, \tilde{F}_{\mu\nu}$.

If the theory is Lagrangian, we use the subscript L and write $M_L(X, Y) = \mathcal{M}_L(u)$ and $N_L(X, Y) = \mathcal{N}_L(u)$ (so that in the Lagrangian case $\mathcal{N} = [1/\mu] + \mathcal{N}_L$ and $\mathcal{M} = \mathcal{M}_L$), to remind us of the dependence of these functionals on the specific choice of Lagrangian L.

Then the condition that the system be Lagrangian takes the form:

$$\frac{d\mathcal{N}_L(u)}{du} - 2cu\frac{d\mathcal{M}_L(u)}{du} = 0, \qquad u = \frac{Y}{cX}. \tag{5.1}$$

Integrating (5.1), we can express $\mathcal{N}_L(u)$ as

$$\mathcal{N}_L(u) = 2c\int u\frac{d\mathcal{M}_L(u)}{du}du = 2cu\mathcal{M}_L(u) - 2c\int \mathcal{M}_L(u)\,du, \tag{5.2}$$

with the constant of integration subsumed into the term $1/\mu$.

Thus we obtain one form of the general constitutive equations respecting conformal symmetry,

$$G_{\mu\nu} = \left(\frac{1}{\mu} + 2uc\mathcal{M}_L(u) - 2c \int \mathcal{M}_L(u)\, du \right) F_{\mu\nu} + c\mathcal{M}_L(u)\, \tilde{F}_{\mu\nu} \,. \qquad (5.3)$$

The constitutive equations as represented by (5.3) depend on one arbitrary functional $\mathcal{M}_L(u)$, a function of the ratio of relativistic invariants $u = Y/cX$.

Remark: There may be a constant term in $\mathcal{M}_L$. But adding a constant κ to $\mathcal{M}_L$ does not change the observable physics. The result is just to add a term $\kappa\mathbf{B}$ to $\mathbf{D}$. But since $\mathrm{div}\,\mathbf{B} = 0$, the value of $\rho = \mathrm{div}\,\mathbf{D}$ is unchanged. Likewise, a term $-\kappa\mathbf{E}$ is added to $\mathbf{H}$. But the resulting term in the equation for $\mathbf{j}$ is offset by the term that was added to $\mathbf{D}$. Hence the system $\mathbf{E}$, $\mathbf{B}$, ρ, $\mathbf{j}$ is unaffected by κ; but it is these fields which describe all the observable forces produced by and acting on electric charges and currents.

Equivalent to the above, one may write

$$\mathcal{M}_L = \frac{1}{2c} \int \frac{1}{u} \frac{d\mathcal{N}_L}{du} \, du \tag{5.4}$$

and the constitutive equations become

$$G_{\mu\nu} = \left(\frac{1}{\mu} + \mathcal{N}_L(u) \right) F_{\mu\nu} + \left(\frac{1}{2} \int \frac{1}{u} \frac{d\mathcal{N}_L}{du} \, du \right) \tilde{F}_{\mu\nu} \, . \tag{5.5}$$

Here $\mathcal{N}_L(u)$ is taken to be the arbitrary function of u.

The general Lagrangian density L for conformal-invariant nonlinear electrodynamics can now be written in several equivalent forms.

In terms of the arbitrary functional $\mathcal{M}_L(u)$, we have:

$$L = L_0 + cX \int \mathcal{M}_L(u)du = L_0 + Y\left(\frac{1}{u}\int \mathcal{M}_L(u)du\right). \quad (5.6)$$

Alternatively, in terms of the arbitrary functional $\mathcal{N}_L(u)$, we have:

$$L = L_0 + Y\left(\frac{1}{2c}\int \frac{1}{u^2}\mathcal{N}_L(u)du\right). \quad (5.7)$$

Here $L_0 = -(1/2\mu)X$ describes standard linear electrodynamics, with $\mu = \mu_0$ and $G_{\mu\nu} = (1/\mu_0)F_{\mu\nu}$.

As remarked, if $\mathcal{M}_L(u) = \kappa$ (a constant), then

$$L_{\mathrm{lin}} = -\frac{1}{2\mu}X + \kappa Y$$

describes linear electrodynamics physically equivalent to that described by L_0. Otherwise, the general conformal-invariant electrodynamics described by (5.6) or (5.7) is nonlinear.

6. Examples

The well-known Born-Infeld Lagrangian density does not respect conformal symmetry:

$$L_{BI} = \frac{b^2}{\mu_0 c}\left(1 - \sqrt{1 + \frac{c^2}{b^2} I_1 - \frac{c^2}{b^4} I_2}\right), \tag{6.1}$$

which does not depend just on the ratio I_2/I_1. But let us illustrate some of the many possibilities for nonlinear electromagnetism with conformal symmetry in flat spacetime.

Example

1. Let $\mathcal{M}_L(u) = \lambda_1 u$, where the coefficient λ_1 (with the dimensionality of $\epsilon_0 c$) controls the magnitude of the nonlinearity. Then $\mathcal{N}_L(u) = \lambda_1 c u^2$, and

$$L = -\frac{1}{2\mu} X + \lambda_1 \frac{Y^2}{2cX}, \tag{6.2}$$

$$G_{\mu\nu} = \left(\frac{1}{\mu} + \lambda_1 \frac{Y^2}{cX^2}\right) F_{\mu\nu} + \lambda_1 \frac{Y}{X} \tilde{F}_{\mu\nu}. \tag{6.3}$$

Example

2. More generally, let $\mathcal{M}_L(u) = \lambda_n u^n$, $n \neq -1$ (where again, λ_n has the dimensionality of $\epsilon_0 c$). Then $\mathcal{N}_L(u) = \lambda_n [2cn/(n+1)]\, u^{n+1}$; and

$$L = -\frac{1}{2\mu} X + \lambda_n \frac{1}{(n+1)} \frac{Y^{n+1}}{c^n X^n}, \tag{6.4}$$

$$G_{\mu\nu} = \left(\frac{1}{\mu} + \lambda_n \frac{2n}{(n+1)} \frac{Y^{n+1}}{c^n X^{n+1}} \right) F_{\mu\nu} + \lambda_n \frac{Y^n}{c^{n-1} X^n} \tilde{F}_{\mu\nu}. \tag{6.5}$$

Note that if $n < 0$, the model is singular when $Y = 0$; i.e., when $\mathbf{B} \cdot \mathbf{E} = 0$.

Example

3. Let $\mathcal{N}_L(u) = \alpha u$ (where α has the dimensionality of $1/\mu_0$ or $\epsilon_0 c^2$). Then $\mathcal{M}_L(u) = (\alpha/2c) \ln |u|$, and we obtain

$$L = -\frac{1}{2\mu} X + \alpha \frac{Y}{2c} \ln \left| \frac{Y}{cX} \right| , \tag{6.6}$$

$$G_{\mu\nu} = \left(\frac{1}{\mu} + \alpha \frac{Y}{cX} \right) F_{\mu\nu} + \frac{\alpha}{2} \ln \left| \frac{Y}{cX} \right| \tilde{F}_{\mu\nu} \, . \tag{6.7}$$

This model also is singular when $u = 0$; i.e., when $\mathbf{B} \cdot \mathbf{E} = 0$.

Example

4. We may let $\mathcal{M}_L(u) = \lambda \sin bu$, where $b \neq 0$ is an additional dimensionless parameter. Then $\mathcal{N}_L(u) = 2\lambda c\,(u \sin bu + b^{-1}\cos bu)$; and now,

$$L = -\frac{1}{2\mu}X - \lambda\,\frac{c}{b}\,X\cos\left(\frac{b\,Y}{cX}\right). \tag{6.8}$$

$$G_{\mu\nu} = \left[\frac{1}{\mu} + 2\lambda\left(\frac{Y}{X}\right)\sin\left(\frac{b\,Y}{cX}\right) + \frac{c}{b}\cos\left(\frac{b\,Y}{cX}\right)\right]F_{\mu\nu} + \lambda c \sin\left(\frac{b\,Y}{cX}\right)\,\tilde{F}_{\mu\nu}\,. \tag{6.9}$$

One can obtain similar equations for the examples $\mathcal{M}_L(u) = \lambda \cos bu$, $\mathcal{M}_L(u) = \lambda \sinh bu$, and $\mathcal{M}_L(u) = \lambda \cosh bu$.

7. Summary and final remarks

We have described an approach to nonlinear classical Maxwell electrodynamics with conformal symmetry, based on generalized constitutive equations. These are expressed in terms of constitutive tensors that depend on a conformal-invariant functional of the field strengths. Our general description includes both Lagrangian and non-Lagrangian theories. The latter would of course include nonconservative (dissipative) systems.

A straightforward criterion characterizes the Lagrangian case, which leads to a general formula for the Lagrangian density. We present several examples, illustrating a variety of possibilities. These may occur as nonlinear perturbations of the usual, linear Maxwell theory.

Conformal symmetry distinguishes a particular class of nonlinear Lagrangian theories. Some choices may be candidates for the phenomenological description of conservative electrodynamics in the presence of matter, or even for describing possible fundamental properties of spacetime.

Elsewhere we have suggested further generalization based on the well-known identification of conformally compactified Minkowski spacetime with the projective light cone in a $(4+2)$-dimensional spacetime. The conformal symmetry is there expressed through (ordinary and hyperbolic) rotations. Conformal inversion is just a reflection. Writing the 6-dimensional analog of Maxwell's equations, there are now two independent conformal-invariant functionals of the field strengths. Dimensional reduction then permits one to recover additional possibilities in Minkowski spacetime. This is a topic of our ongoing research.

Thank you for your kind attention.