Second order difference equations solvable by factorization method

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The main purpose of this presentation is to apply the factorization method to difference equations. The factorization method offers the possibility of finding solutions of new classes of difference equations.
The factorization is a well-known method of solving differential equations, having roots in the works of G. Darboux, E. Schrödinger, P.A.M. Dirac, L. Infeld and T.E. Hull, W.Jr. Miller, B. Mielnik. Later this method (and some its modification) was applied to study of the most important eigenvalue problems in quantum mechanics for well-known solvable potentials like: harmonic oscillator, isotropic oscillator, Morse, Rosen-Morse, Eckart, Pöschl-Teller, etc. Additionally, the modified factorization method was application to the other topics as: supersymmetric quantum mechanics, shape–invariant potentials, inverse scattering method, coherent states, etc.

In our research we have restricted our attention to the analysis of the discrete version of this problem. We denote by $\ell_k(\mathbb{Z},\mathbb{R})$ and $\ell_k(\mathbb{Z},\mathbb{C})$, $k \in \mathbb{N} \cup \{0\}$, the sets of real-valued and complex-valued sequences $\{x(n)\}_{n\in\mathbb{Z}}$, respectively. We define the scalar product on $\ell_k(\mathbb{Z},\mathbb{C})$ as follows:

$$\langle x|y\rangle_k := \sum_{n=a}^{b} \overline{x(n)} y(n) \rho_k(n),$$

where $a, b \in \mathbb{Z},\ (a < b)$, and ρ_k is a weight function.

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$$\mathbf{\Delta}x(n) := (\mathbf{T}^+ - \mathbf{1})\,x(n) = x(n+1) - x(n)$$ **difference operator**

$$\mathbf{T}^{\pm}x(n) := x(n \pm 1)$$ **shift operators**

$$\langle x|y\rangle_k = \sum_{n=a}^{b} \overline{x(n)} y(n) \rho_k(n),$$

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We assume that the weight sequence satisfies the Pearson difference equation

$$\mathbf{\Delta}\left(b_k(n)\rho_k(n)\right) = \left(c_k(n) - b_k(n)\right)\rho_k(n),$$

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and the recursion relation

$$\rho_{k-1}(n) = c_k(n)\rho_k(n),$$

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where $\{b_k\}$ and $\{c_k\}$ are some real-valued sequences. Moreover, the function ρ_k fulfills the boundary conditions

$$b_k(a)\rho_k(a) = b_k(b+1)\rho_k(b+1) = 0.$$

We apply the factorization method to second order difference operators $\mathbf{H_k} : \ell_k(\mathbb{Z}, \mathbb{C}) \longrightarrow \ell_k(\mathbb{Z}, \mathbb{C})$ given by

$$\boxed{\mathsf{H}_k = z_k(n)\mathbf{T}^+ + w_k(n)\mathbf{T}^- + v_k(n),}$$

where $\{z_k\}, \{w_k\}, \{v_k\} \in \ell(\mathbb{Z}, \mathbb{R})$, $k \in \mathbb{Z}$.

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$$\mathsf{H}_k = \mathbf{A_k^+ A_k^-} + \alpha_k = \mathbf{A_{k+1}^- A_{k+1}^+} + \alpha_{k+1},$$

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$$\boxed{\mathsf{H}_k = \mathbf{A}^+_{\mathbf{k}}\mathbf{A}^-_{\mathbf{k}} + \alpha_k = \mathbf{A}^-_{\mathbf{k+1}}\mathbf{A}^+_{\mathbf{k+1}} + \alpha_{k+1},}$$

where $\alpha_k \in \mathbb{R}$. In this way, we obtain a chain of ladder operators ($\mathbf{A}^-_{\mathbf{k}}$ – lowering operators and $\mathbf{A}^+_{\mathbf{k}}$ – raising operators)

$$\ldots \ell_{k-1}(\mathbb{Z}, \mathbb{C})^{\circlearrowright \mathbf{H_{k-1}}} \underset{\mathbf{A}^-_{\mathbf{k}}}{\overset{\mathbf{A}^+_{\mathbf{k}}}{\rightleftarrows}} \ell_k(\mathbb{Z}, \mathbb{C})^{\circlearrowright \mathbf{H_{k}}} \underset{\mathbf{A}^-_{\mathbf{k+1}}}{\overset{\mathbf{A}^+_{\mathbf{k+1}}}{\rightleftarrows}} \ell_{k+1}(\mathbb{Z}, \mathbb{C})^{\circlearrowright \mathbf{H_{k+1}}} \ldots,$$

We will assume that these operators $\mathbf{A}^{\pm}_{\mathbf{k}}$ are realized as

$$\mathbf{A}^{-}_{\mathbf{k}} = \mathbf{T}^{+} + f_k(n) - 1 = \mathbf{\Delta} + f_k(n),$$

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The above condition $\mathbf{H}_{\mathbf{k}} = \mathbf{A}_{\mathbf{k}}^{+}\mathbf{A}_{\mathbf{k}}^{-} + \alpha_k = \mathbf{A}_{\mathbf{k+1}}^{-}\mathbf{A}_{\mathbf{k+1}}^{+} + \alpha_{k+1}$ is equivalent to three equations:

$$\begin{aligned}
1.\quad f_{k+1}(n) - 1 &= \frac{b_k(n)}{b_{k+1}(n)}\left(f_k(n-1) - 1\right),\\
2.\quad c_{k+1}(n) &= \frac{b_{k+1}(n)}{b_k(n)} c_k(n-1),\\
3.\ b_k(n) - b_{k+1}(n+1) &= \alpha_{k+1} - \alpha_k + \frac{b_k(n)}{b_{k+1}(n)}\left(f_k(n-1) - 1\right)^2 c_k(n-1)\\
&- \left(f_k(n) - 1\right)^2 c_k(n).
\end{aligned}$$

Basic assumption: the sequence $\{b_k\}$ does not depend on parameters k

We assume that

$$b_{k+1}(n) = b_k(n) =: b(n).$$

The conditions (1.) and (2.) give us the transformation formulas for the sequences $\{f_k\}$ and $\{c_k\}$

$$\begin{cases} f_{k+1}(n) = f_k(n-1) \\ c_{k+1}(n) = c_k(n-1) \end{cases} .$$

Under this hypothesis the condition (3.) yields a recurrence formula for the initial sequence $\{f_{k_0}\}$, because the left hand side of this equation does not depend on parameter k,

$$(f_{k_0}(n) - 1)^2 c_{k_0}(n) - (f_{k_0}(n-1) - 1)^2 c_{k_0}(n-1) =$$
$$= b(n+1) - b(n) + \alpha_{k_0+1} - \alpha_{k_0}.$$

Comparing the coefficients we find

$$b(n) = \beta_2 n^2 + \beta_1 n + \beta_2,$$

$$\left(f_{k_0}(n) - 1\right)^2 c_{k_0}(n) = \beta_2 n^2 + \left(2\beta_2 + \beta_1 + \alpha_{k_0+1} - \alpha_{k_0}\right) n + \gamma.$$

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Finally we obtain the expressions for f_{k_0}

$$f_{k_0}(n) = 1 \pm \sqrt{c_{k_0}^{-1}(n)}\sqrt{\beta_2 n^2 + (2\beta_2 + \beta_1 + \alpha_{k_0+1} - \alpha_{k_0})n + \gamma},$$

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Moreover, we obtain the following recurrence relation for constants a_k

$$\alpha_{k+1} = \alpha_k + \alpha_{k_0+1} - \alpha_{k_0} - 2\beta_2(k - k_0),$$

or, after the telescoping summation

$$\alpha_k = \alpha_{k_0+1} + (k - k_0 - 1)(\alpha_{k_0+1} - \alpha_{k_0}) - \beta_2(k - k_0)(k - k_0 - 1).$$

In conclusion if $k_0 = 0$ then explicit formulas for the raising and lowering operators satisfying the factorization condition are given by

$$\mathbf{A}_{\mathbf{k}}^{-} = \mathbf{T}^{+} \pm \sqrt{c_0^{-1}(n-k)}\sqrt{\beta_2(n-k)^2+(2\beta_2+\beta_1+\alpha_1-\alpha_0)(n-k)+\gamma},$$

$$\mathbf{A}_{\mathbf{k}}^{+} = \left(\beta_2 n^2+\beta_1 n+\beta_0\right)\mathbf{T}^{-}\pm$$

$$\pm\sqrt{c_0(n-k)}\sqrt{\beta_2(n-k)^2+(2\beta_2+\beta_1+\alpha_1-\alpha_0)(n-k)+\gamma}$$

and from this the explicit expression for $\mathbf{H}_{\mathbf{k}}$ can be derived. Above the parameters: β_2, β_1, β_0, γ, α_1, α_0 are arbitrary and $\{c_0\}$ is a free sequence.

Basic assumption: $b_k(n) = \lambda b_{k+1}(n), \quad \lambda \in \mathbb{R} \setminus \{0,1\}$

From now on, we assume that the sequence $\{c_k\}$ is defined by

$$c_k(n) := \frac{1}{f_k(n) - 1}.$$

The system of conditions is reduced now to

$$\begin{aligned} f_{k+1}(n) - 1 &= \frac{b_k(n)}{b_{k+1}(n)} \left(f_k(n-1) - 1\right), \\ b_k(n) - b_{k+1}(n+1) &= \alpha_{k+1} - \alpha_k + \frac{b_k(n)}{b_{k+1}(n)} \left(f_k(n-1) - 1\right) \\ &- \left(f_k(n) - 1\right). \end{aligned}$$

Comparing the coefficients we find

$$b_k(n) = \lambda^{-k} b_0(n) = \lambda^{-k}\left(\beta_2\lambda^{2n} + \beta_1\lambda^{n} + \beta_0\right),$$

$$f_k(n) - 1 = \lambda^{2n-k+1}\beta_2 + \lambda^{n}\left(f_0(0) - \lambda\beta_2 - \lambda^{-1}\beta_0 - 1 - \frac{\alpha_1 - \alpha_0}{1-\lambda}\right) + \lambda^{k}\left(\frac{\alpha_1 - \alpha_0}{1-\lambda} + \lambda^{-1}\beta_0\right),$$

$$\alpha_k = \lambda^{k}\frac{\alpha_0 - \alpha_1}{1-\lambda} - \lambda^{-k}\left(1 + \lambda^{2k-1}\right)\beta_0 + \frac{\alpha_1 - \lambda\alpha_0}{1-\lambda} + \left(1 + \lambda^{-1}\right)\beta_0.$$

Finally, we have the explicit form of creation and annihilation operators

$$\mathbf{A}_{\mathbf{k}}^{-} = \mathbf{T}^{+} + \lambda^{2n-k+1}\beta_2 + \lambda^{n}\left(f_0(0) - \lambda\beta_2 - \lambda^{-1}\beta_0 - 1 - \frac{\alpha_1 - \alpha_0}{1-\lambda}\right) + \lambda^{k}\left(\frac{\alpha_1 - \alpha_0}{1-\lambda} + \lambda^{-1}\beta_0\right),$$

$$\mathbf{A}_{\mathbf{k}}^{+} = \left(\beta_2\lambda^{2n-k} + \beta_1\lambda^{n-k} + \beta_0\lambda^{-k}\right)\mathbf{T}^{-} + 1.$$

These operators factorize the family of second order difference operators

$$\mathbf{H}_{\mathbf{k}} = \mathbf{T}^{+} + \lambda^{-k}\left(\beta_2\lambda^{2n} + \beta_1\lambda^{n} + \beta_0\right)\Bigg(\lambda^{2n-k-1}\beta_2 + \lambda^{n-1}\left(f_0(0) - \lambda\beta_2 - \lambda^{-1}\beta_0 - 1 - \frac{\alpha_1 - \alpha_0}{1-\lambda}\right) + \lambda^{k}\left(\frac{\alpha_1 - \alpha_0}{1-\lambda} + \lambda^{-1}\beta_0\right)\Bigg)\mathbf{T}^{-} + \lambda^{2n-k}(1+\lambda)\beta_2 + \lambda^{n}\left(f_0(0) + \beta_1\lambda^{-k} - \lambda\beta_2 - \lambda^{-1}\beta_0 - 1 - \frac{\alpha_1 - \alpha_0}{1-\lambda}\right) + \frac{\alpha_1 - \lambda\alpha_0}{1-\lambda} + \beta_0\left(1 + \lambda^{-1}\right).$$

If the operators $\mathbf{H_k}$ admit the factorization

$$\mathbf{H_k} = \mathbf{A_k^+ A_k^-} + \alpha_k = \mathbf{A_{k+1}^- A_{k+1}^+} + \alpha_{k+1},$$

then the eigenvalue problem

$$\mathbf{H_k} x_k^l(n) = \lambda_k^l x_k^l(n) \tag{1}$$

can be solved, $\{x_k^l\} \in \ell(\mathbb{Z}, \mathbb{R})$, $k, l \in \mathbb{Z}$, $\lambda_k^l \in \mathbb{R}$. For a large class of such homogeneous linear second order difference equations the factorization method enables us to find immediately the eigenvalues λ_k and eigenfunctions (solutions) $\{x_k\}$ by manufacturing process.

The underlying idea is to consider a pair of first order difference equations

$$\mathbf{A}^-_{\mathbf{k}} x^0_k(n) = 0 \qquad \text{or} \qquad \mathbf{A}^+_{\mathbf{k+1}} x^{-1}_k(n) = 0.$$

In the case $\lambda^0_k = \alpha_k$ or $\lambda^{-1}_k = \alpha_{k+1}$, any nonzero solution of above equation is also a solution of equation (1). Moreover

$$x^1_{k+1}(n) = \mathbf{A}^+_{\mathbf{k+1}} x^0_k(n) \qquad \text{or} \qquad x^{-2}_{k-1}(n) = \mathbf{A}^-_{\mathbf{k}} x^{-1}_k(n)$$

are solutions of equations (1) for $k+1$ and $\lambda^1_{k+1} = \alpha_k$ or for $k-1$ and $\lambda^{-2}_{k-1} = \alpha_{k+1}$, respectively. Repeating this procedure we may get a solution of (1)

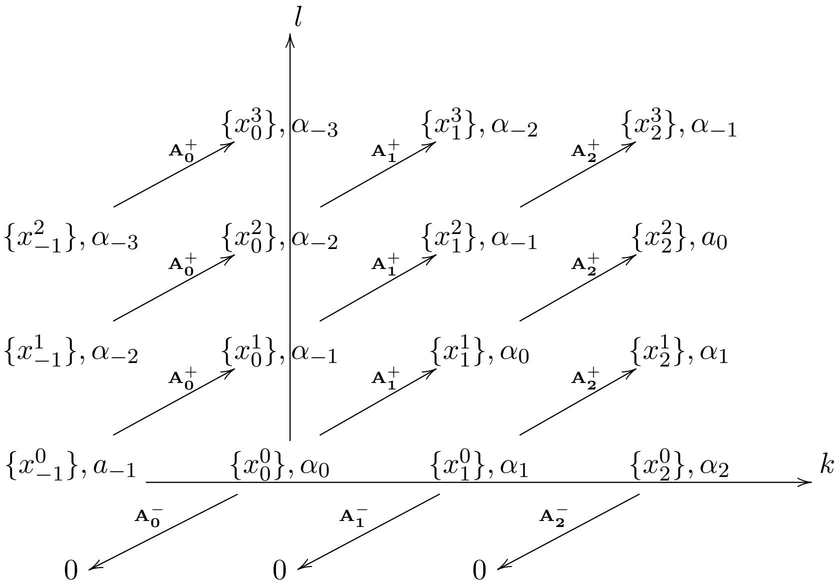
$$x^{p-k}_p(n) = \mathbf{A}^+_{\mathbf{p}} \mathbf{A}^+_{\mathbf{p-1}} \dots \mathbf{A}^+_{\mathbf{k+2}} \mathbf{A}^+_{\mathbf{k+1}} x^0_k(n)$$

for p and $\lambda^{p-k}_p = \alpha_k$ (after changing $k \to p$), $p > k$, or

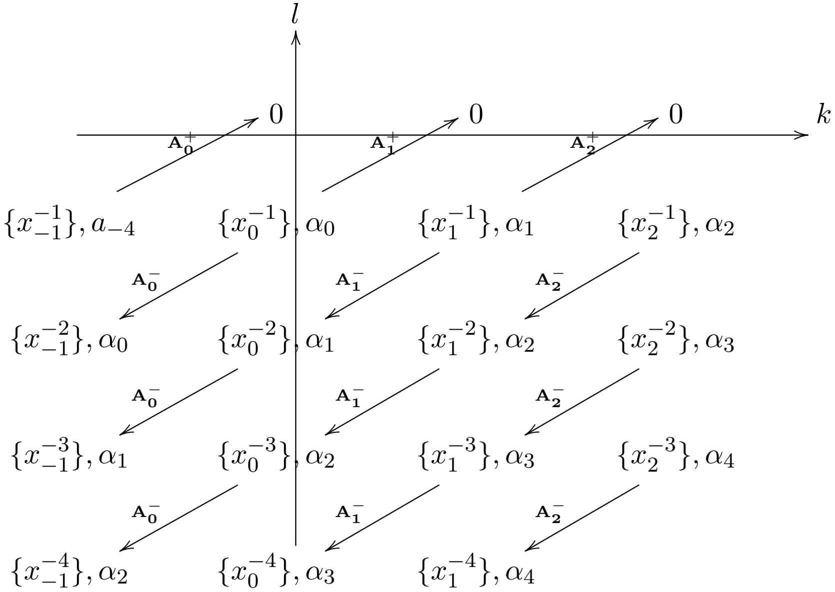
$$x^{r-k}_r(n) = \mathbf{A}^-_{\mathbf{r+1}} \mathbf{A}^-_{\mathbf{r+2}} \dots \mathbf{A}^-_{\mathbf{k-1}} \mathbf{A}^-_{\mathbf{k}} x^{-1}_k(n)$$

for r and $\lambda^{r-k}_r = \alpha_k$ (after changing $k \to r$), $r < k$.

Finally, we present the action of the raising operators on the diagram:



We present the action of the lowering operators on the diagram:



Example 1

Let us consider a case when $\beta_2 = \beta_1 = \gamma = 0$, $\beta_0 = 1$ and $f_k(n) = 0$ for $k \in \mathbb{Z}$, then $c_0(n) = (\alpha_1 - \alpha_0)n$. The raising and lowering operators are given by

$$\mathbf{A}_{\mathbf{k}}^{-} = \mathbf{T}^{+} - 1,$$
$$\mathbf{A}_{\mathbf{k}}^{+} = \mathbf{T}^{-} - (\alpha_1 - \alpha_0)(n - k).$$

The family of second order difference equations (1) parametrized by one parameter $l \in \mathbb{Z}$ can be rewritten now in the form

$$\big(-(\alpha_1 - \alpha_0)(n-k)\mathbf{T}^{+} - \mathbf{T}^{-} + (\alpha_1 - \alpha_0)(n-k+l) + 1\big)x_k^l(n) = 0.$$

The solutions for $l \in \mathbb{N} \cup \{0\}$ in this case are of the form

$l = 0 \quad x_k^0(n) = 1,$

$l = 1 \quad x_k^1(n) = -(\alpha_1 - \alpha_0)(n - k) + 1,$

$l = 2 \quad x_k^2(n) = (\alpha_1 - \alpha_0)(n - k)^2 - 2(\alpha_1 - \alpha_0)(n - k) + (\alpha_1 - \alpha_0)^2 + 1,$

$\dots$

and similarly for $l < 0$

$$l = -1 \quad x_k^{-1}(n) = \frac{\lambda}{(\alpha_1 - \alpha_0)^n (n-k-1)!},$$

$$l = -2 \quad x_k^{-2}(n) = \frac{\lambda}{(\alpha_1 - \alpha_0)^{n+1} (n-k)!} - \frac{\lambda}{(\alpha_1 - \alpha_0)^n (n-k-1)!},$$

$$\vdots$$

$$l = -m-1 \quad x_k^{-m-1}(n) = \sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i \lambda}{(\alpha_1 - \alpha_0)^{n+m} (n-k+m-1-i)!}.$$

where $\lambda \in \mathbb{R}$.

Example 2

Let us consider a case when $f_k(n) \equiv 0$. Then

$$\begin{aligned} & b_{k+1}(n) = b_k(n) =: b(n), \\ & c_k(n) = c_0(n-k), \\ & b(n) - b(n+1) = \alpha_{k+1} - \alpha_k + c_0(n-k-1) - c_0(n-k). \end{aligned}$$

Then, the relation $\mathbf{H_k} x_k^l(n) = \lambda_k^l x_k^l(n)$ is equivalent to the difference equation of hypergeometric type:

$$\Big(- b(n)\nabla + b(n) - c_0(n-k) \Big) \Delta x_k^l(n) = (\lambda_k^l - \alpha_k) x_k^l(n).$$

We transform the above equation into the standard form

$$\sigma(n)\Delta\nabla x_k^l(n) + \tau(n)\Delta x_k^l(n) + \lambda x_k^l(n) = 0,$$

where

$$\sigma(n) = -b(n),$$
$$\tau(n) = b(n) - c_0(n-k),$$
$$\lambda = \alpha_k - \lambda_k^l = \alpha_k - \alpha_{k-l} = -l\left(\tau'(n) + \frac{l-1}{2}\sigma''(n)\right).$$

It is well known that the above equation describes classical orthogonal polynomials of a discrete variable such as the Charlier, Meixner, Kravchuk, Hahn polynomials.

Example 3

Let us consider a case when $\lambda = \frac{1}{2}$, $\beta_2 = \beta_1 = 0$, $\beta_0 = 1$, $\alpha_1 = 1$ and $\alpha_0 = 2$. This gives us formulas for the creation and annihilation operators

$$\mathbf{A}_{\mathbf{k}}^{-} = \mathbf{T}^{+} + 2^{-n}(f_0(0) - 1),$$
$$\mathbf{A}_{\mathbf{k}}^{+} = 2^{k}\mathbf{T}^{-} + 1.$$

We obtain the chain of operators for $k \in \mathbb{Z}$ of the form

$$\mathbf{H}_{\mathbf{k}} = \mathbf{T}^{+} + 2^{k-n+1}(f_0(0) - 1)\mathbf{T}^{-} + 2^{-n}(f_0(0) - 1) + 3.$$

The eigenfunctions of these operators can be found in the form

$$\psi_k^l(n) = \left(2^k\mathbf{T}^{-} + 1\right) \ldots \left(2^{k-l+1}\mathbf{T}^{-} + 1\right) \psi_{k-l}^0(n)$$

for eigenvalue $\lambda_k^l = \alpha_{k-l} = 3 - 2^{k-l}$, where the ground states are given by

$$\psi_k^0(n) = (1 - f_0(0))^n 2^{-\frac{n(n-1)}{n}} \psi_k^0(0).$$