Quiver varieties and integrable systems

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Plan

1. Double derivations and double brackets.
2. Noncommutative (quasi-)Poisson geometry and (quasi-)Hamiltonian reduction.
3. From noncommutative to commutative: representation spaces.
4. Quiver varieties.
5. One-loop quiver, Calogero–Moser and Ruijsenaars–Schneider systems.
6. Integrable systems from cycilc quivers.
7. Other interesting examples.

Based on:

M. Van den Bergh, Trans. AMS (2008)
O. C., A. Silantyev, J. Math. Phys. (2017)
O. C., M. Fairon, J. Geom. and Physics (2017)
O. C., M. Fairon, arXiv:1811.08727
O. C., M. Fairon, in preparation

Noncommutative Poisson geometry

Let us describe an approach to noncommutative Poisson geometry due to Van den Bergh.

Basic ingredients of the usual Poisson geometry: functions, vector fields (derivations), Poisson bivectors and brackets.

Noncommutiative setup: an associative algebra A, double derivations, double Poisson bivectors and double brackets.

Double derivations (after Crawley–Boevey)

Double derivations are elements of $\mathrm{Der}(A, A \otimes A)$, i.e. linear maps $\alpha : A \to A \otimes A$ satisfying the Leibniz identity $\alpha(ab) = a\alpha(b) + \alpha(a)b$ for all $a, b \in A$.

One can make $\mathrm{Der}(A, A \otimes A)$ into an A-bimodule, by using the "inner" bimodule structure on $A \otimes A$:

$$c * (a \otimes b) = a \otimes cb\,, \quad (a \otimes b) * c = ac \otimes b\,.$$

Below we will use a distinguished element $E \in \mathrm{Der}(A, A \otimes A)$ defined by $E(a) = a \otimes 1 - 1 \otimes a$.

Notation:
$D := \mathrm{Der}(A, A \otimes A)$, bimodule of double derivations (noncommutative vector fields)

$T_A D$ - the tensor algebra of the bimodule D (noncommutative polyvector fields)

$T_A D$ is a graded algebra, with A placed in degree 0 and double derivations D in degree 1.

Double brackets (after Van den Bergh)

A **double bracket** on A is a linear map $\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$ which has the following properties:

$$\{\!\{a, bc\}\!\} = b\,\{\!\{a, c\}\!\} + \{\!\{a, b\}\!\}\,c \quad \text{and} \quad \{\!\{a, b\}\!\} = -\,\{\!\{b, a\}\!\}^{\circ}\ ,$$

where $(u \otimes v)^{\circ} = v \otimes u$.

We will use Sweedler notation, writing elements $x \in A \otimes A$ as $x = x' \otimes x''$, skipping the summation symbol.

Thus, we write $\{\!\{a, b\}\!\} = \{\!\{a, b\}\!\}' \otimes \{\!\{a, b\}\!\}''$.

With each double bracket $\{\!\{-,-\}\!\}$ one associates a (single) bracket $\{-,-\} : A \otimes A \to A$ given by

$$\{a, b\} := \{\!\{a, b\}\!\}'\,\{\!\{a, b\}\!\}''\ .$$

$\{-,-\}$ is not antisymmetric in general, and it satisfies Leibniz identity only in the second argument.

Schouten bracket on $T_A D$

Recall: Starting from A, we use the bimodule $D = \mathrm{Der}(A, A \otimes A)$ to construct the tensor algebra $T_A D$.
Nontrivial fact (Van den Bergh): The algebra $T_A D$ admits a canonical double bracket with the following properties:

$$\{\!\{a, b\}\!\} = 0\,, \quad \{\!\{\alpha, a\}\!\} = \alpha(a) \quad \forall a, b \in A \quad \forall \alpha \in D$$

$$\{\!\{\alpha, \beta\}\!\} \in D \otimes A + A \otimes D \qquad \forall \alpha, \beta \in D$$

Moreover, this bracket is a **double Poisson bracket** in an appropriate sense. Note that $T_A D$ is a graded algebra, and the above double bracket on $T_A D$ admits a graded version, called the **double Schouten–Nijenhuis bracket**,
$\{\!\{-,-\}\!\}_{SN} : T_A D \otimes T_A D \to T_A D \otimes T_A D$ and the associated **Schouten–Nijenhuis bracket** $\{-,-\}_{SN} : T_A D \otimes T_A D \to T_A D$, with both brackets being of degree -1.

(Quasi-)Poisson brackets and moment maps

Using the Schouten–Nienhuis bracket on $T_A D$, we can formulate compactly the notions of a double (quasi-)Poisson bracket on A and the corresponidng moment maps. Double brackets on A can be produced from double bivectors, i.e. elements $\mathrm{P} \in (T_A D)_2$. Explicitly, for $\delta_1, \delta_2 \in D$, the double bracket associated with $\delta_1\delta_2$ is given by

$$\{\!\{a, b\}\!\} = \delta_2(b)'\delta_1(a)'' \otimes \delta_1(a)'\delta_2(b)'' - \delta_1(b)'\delta_2(a)'' \otimes \delta_2(a)'\delta_1(b)''$$

A double bracket associated to a bivector P is called **Poisson** if $\{\mathrm{P}, \mathrm{P}\}_{SN} = 0$ modulo commutators, and **quasi-Poisson** if $\{\mathrm{P}, \mathrm{P}\}_{SN} = \frac{1}{6}E^3$ modulo commutators.

For a double Poisson bracket, a **moment map** is an element $\mu \in A$ such that $\{\mathrm{P}, \mu\}_{SN} = -E$.
For a double quasi-Poisson bracket, a **multiplicative moment map** is an invertible element $\Phi \in A$ such that $\{\mathrm{P}, \Phi\}_{SN} = -\frac{1}{2}(E\Phi + \Phi E)$.

Representation spaces

A path from noncommutative to commutative geometry goes through representation spaces.

For any $n \in \mathbb{N}$, the representation space $\mathrm{Rep}(A, n)$ is the space of all algebra maps $\varrho : A \to \mathrm{Mat}_n(\mathbb{C})$.
On $\mathrm{Rep}(A, n)$ we have a natural action of GL_n (by changing basis).

Each element $a \in A$ can be viewed as a matrix-valued function on $\mathrm{Rep}(A, N)$, whose value at a point ϱ is given by $\varrho(a)$. The ring of functions $\mathcal{O}(\mathrm{Rep}(A, n))$ is generated by the functions a_{ij} for $a \in A$, $i, j = 1, \ldots, n$ satisfying the relations $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Guiding principle: Notions of noncommutative geometry should induce the usual geometric notions on the representation spaces.

Induced notions on representation spaces

1. A double bracket $\{\!\{-,-\}\!\}$ on A induces a bracket on the space $\mathrm{Rep}(A,n)$ by the formula:

$$\{a_{ij}, b_{uv}\} = \{\!\{a,b\}\!\}'_{uj}\ \{\!\{a,b\}\!\}''_{iv}\ .$$

In general, this will be just an antisymmetric bracket. If $\{\!\{-,-\}\!\}$ is a double (quasi-)Poisson bracket, then the induced bracket is (quasi-)Poisson in the standard sense.

Useful formula: $\{\mathrm{tr}\, a, \mathrm{tr}\, b\} = \mathrm{tr}\{a,b\}$ for $a, b \in A$.

2. A noncommutative moment map $\mu \in A$ induces a matrix-valued function $(\mu_{ij})_{i,j=1,\dots n}$ on $\mathrm{Rep}(A,n)$. It has the properties of the usual moment map w.r.t. GL_n-action on $\mathrm{Rep}(A,n)$.

The same for a multiplicative moment map Φ, in agreement with [Alekseev–Malkin–Meinreken].

Representation spaces as GL_n- spaces

As a result, if an algebra A is equipped with a double (quasi-)Poisson bracket and a (multiplicative) moment map, then the space $\mathrm{Rep}(A, n)$ is a (quasi-)Hamiltonian GL_n-space.

In such situations we can perform a (quasi-)Hamiltonian reduction $\Rightarrow \quad \mu^{-1}(\lambda)//\,\mathrm{GL}_n$ or $\Phi^{-1}(q)//\,\mathrm{GL}_n$

The above general theory can be made very concrete in the case of quivers $\quad \Rightarrow \quad$ a good source of interesting Poisson varieties.

Quiver varieties (Nakajima) and multiplicative quiver varieties (Crawley-Boevey – Shaw) are particular subclasses.

Quivers: double derivations

Take $Q = (Q, I)$, a quiver with vertex set I and arrow set Q.
Let $\overline{Q}$ be the double of Q, obtained by adjoining to every arrow $a : i \to j$ its opposite, $a^* : j \to i$, using the convention $(a^*)^* = a$.
We write $\mathbb{C}\overline{Q}$ for the **path algebra** of $\overline{Q}$, where the multiplication is defined by concatenation of paths.
As an algebra, $\mathbb{C}\overline{Q}$ is generated by arrows $a \in \overline{Q}$ and vertices (viewed as zero paths $e_i, i \in I$).

For all $a \in \overline{Q}$, define a double derivation $\frac{\partial}{\partial a}$ on $\mathbb{C}\overline{Q}$ which acts as

$$\frac{\partial b}{\partial a} = \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Here $t(a)$, $h(a)$ denote tail and head of a.

Quivers: double Poisson structure

The following two theorems are due to Van den Bergh.

Theorem 1. For any quiver Q, the path algebra of $\overline{Q}$ admits a double Poisson bivector

$$\mathsf{P} = \sum_{a \in Q} \frac{\partial}{\partial a} \frac{\partial}{\partial a^*}$$

with a moment map

$$\mu = \sum_{a \in Q} (aa^* - a^* a) \, .$$

Quivers: double quasi-Poisson structure

Fix some ordering $<$ on $\overline{Q}$. Define $\epsilon(a) = \pm 1$ depending on whether a is in Q or not.

Theorem 2. For any quiver Q, the (localised) path algebra of $\overline{Q}$ admits a double quasi-Poisson bivector given by

$$\mathrm{P} = \frac{1}{2}\sum_{a\in\overline{Q}} \epsilon(a)(1 + a^* a)\frac{\partial}{\partial a}\frac{\partial}{\partial a^*} - \frac{1}{2}\sum_{a<b\in\overline{Q}} \left(\frac{\partial}{\partial a^*}a^* - a\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial b^*}b^* - b\frac{\partial}{\partial b}\right),$$

with a multiplicative moment map

$$\Phi = \prod_{a\in\overline{Q}} (1 + aa^*)^{\epsilon(a)}.$$

Here the product is taken with respect to the chosen ordering $<$.

A one-loop quiver: Poisson case

A simple example: a quiver Q with one vertex and one loop, x.
The double quiver has two loops, x and $y = x^*$.

Path algebra: $\mathbb{C}\overline{Q} = \mathbb{C}\langle x, y\rangle$.

Double Poisson bracket:
$\{\!\{x, x\}\!\} = \{\!\{y, y\}\!\} = 0$, $\{\!\{x, y\}\!\} = -\{\!\{y, x\}\!\} = 1 \otimes 1$.

Moment map: $\mu = xy - yx$.
Noncommutative Hamiltonian quotient:

$$\Pi^\lambda := \mathbb{C}\langle x, y\rangle / \{xy - yx = \lambda\}$$

A one-loop quiver: quasi-Poisson case

Same quiver, so $\mathbb{C}\overline{Q} = \mathbb{C}\langle x, y\rangle$.
Double quasi-Poisson bracket:

$$\{\!\{x, x\}\!\} = \frac{1}{2}\left(x^2 \otimes 1 - 1 \otimes x^2\right) , \quad \{\!\{y, y\}\!\} = \frac{1}{2}\left(1 \otimes y^2 - y^2 \otimes 1\right) ,$$

$$\{\!\{x, y\}\!\} = 1 \otimes 1 + \frac{1}{2}(yx \otimes 1 + 1 \otimes xy + y \otimes x - x \otimes y)$$

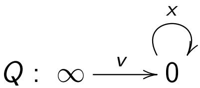
$$\{\!\{y, x\}\!\} = -1 \otimes 1 - \frac{1}{2}(1 \otimes yx + xy \otimes 1 + x \otimes y - y \otimes x) .$$

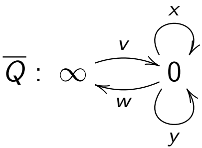
Multiplicative moment map: $\Phi = (1 + xy)(1 + yx)^{-1} = xzx^{-1}z^{-1}$, where $z := y + x^{-1}$. Noncommutative Hamiltonian quotient:

$$\Lambda^q := \mathbb{C}\langle x^{\pm 1}, z^{\pm 1}\rangle / \{xz = qzx\}$$

Adding framing

The algebras $\mathbb{C}\langle x, y\rangle/\{xy - yx = \lambda\}$ and $\mathbb{C}\langle x^{\pm 1}, z^{\pm 1}\rangle/\{xz = qzx\}$ do not have n-dimensional representations unless $\lambda = 0$ or $q^n = 1$.
To allow finite-dimensional representations, we extend the quiver:





Here we associate idempotents e_0, e_∞ to each of the vertices, so the path algebra is generated by x, y, v, w together with e_0, e_∞.
Note that the brackets are of a "local" nature, i.e. the brackets between x, y are essentially the same as without framing. By definition, all the brackets are linear over $B = \mathbb{C}e_0 \oplus \mathbb{C}e_\infty$.

Adding framing (continued)

In the additive case, the only nonzero brackets are $\{\!\!\{x, y\}\!\!\} = -\{\!\!\{y, x\}\!\!\} = e_0 \otimes e_0$, $\{\!\!\{v, w\}\!\!\} = -\{\!\!\{w, v\}\!\!\} = e_\infty \otimes e_\infty$.

In the multiplicative case the brackets are as follows:

$$\{\!\!\{x, x\}\!\!\} = \frac{1}{2}\left(x^2 \otimes e_0 - e_0 \otimes x^2\right), \quad \{\!\!\{z, z\}\!\!\} = \frac{1}{2}\left(e_0 \otimes z^2 - z^2 \otimes e_0\right)$$

$$\{\!\!\{x, z\}\!\!\} = \frac{1}{2} zx \otimes e_0 + \frac{1}{2} e_0 \otimes xz + \frac{1}{2}(z \otimes x - x \otimes z)$$

$$\{\!\!\{v, v\}\!\!\} = \{\!\!\{w, w\}\!\!\} = 0, \quad \{\!\!\{v, w\}\!\!\} = e_\infty \otimes e_0 + \frac{1}{2} e_\infty \otimes vw + \frac{1}{2} wv \otimes e_0$$

$$\{\!\!\{x, v\}\!\!\} = \frac{1}{2} e_0 \otimes xv - \frac{1}{2} x \otimes v, \quad \{\!\!\{x, w\}\!\!\} = \frac{1}{2} wx \otimes e_0 - \frac{1}{2} w \otimes x$$

$$\{\!\!\{z, v\}\!\!\} = \frac{1}{2} e_0 \otimes zv - \frac{1}{2} z \otimes v, \quad \{\!\!\{z, w\}\!\!\} = \frac{1}{2} wz \otimes e_0 - \frac{1}{2} w \otimes z$$

Additive case: Calogero–Moser system

In the additive case, the moment map after framing changes to

$$\mu = xy - yx + vw - wv = (xy - yx + vw)e_0 - wve_\infty \,.$$

Consider representations $V = V_0 \oplus V_\infty$ of $\overline{Q}$ of dimension $(n, 1)$, i.e. $V = \mathbb{C}^n \oplus \mathbb{C}$. The value of the moment map also becomes two-component quantity $\lambda_0 \,\mathrm{Id}_n + \lambda_\infty \,\mathrm{Id}_1$. Choose $\lambda_0 = 1$, $\lambda_\infty = -n$.

The arrows of the quiver are represented by linear maps between the spaces at the vertices, i.e. we need to prescribe $X, Y \in \mathrm{Mat}_{n\times n}$, $v \in \mathrm{Mat}_{n\times 1}$, $w \in \mathrm{Mat}_{1\times n}$. On such matrix data the group $G = \mathrm{GL}_n$ acts by $g.(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, wg^{-1})$.

(There is also the action of GL_1 at V_∞ but it can be neglected.)

Hence, the moment map equations are

$$XY - YX + vw = \mathrm{Id}_n \,, \qquad wv = -n \,.$$

(Note that the second relation is automatic corollary of the first.)

Additive case: Calogero–Moser system (continued)

As a result, the Hamiltonian quotient gives the nth Calogero–Moser space (Kazdan–Kostant–Sternberg (1978), WIlson (1998)):

$$\mathcal{C}_n := \{X, Y \in \mathrm{Mat}_{n\times n} \mid \mathrm{rank}(XY - YX - \mathrm{Id}_n) = 1\} // \mathrm{GL}_n \,.$$

$\mathcal{C}_n$ is a smooth affine variety of $\dim = 2n$. By construction, it is a Poisson variety.

Local coordinates:

An open susbset of $\mathcal{C}_n$ consists of points with diagonalisable X. Then, modulo GL_n action, we have:

$$X = \mathrm{diag}(x_1, \ldots, x_n)\,, \qquad Y_{ij} = \delta_{ij} p_i + (1 - \delta_{ij}) \frac{1}{x_i - x_j}$$

Here $p_1, \ldots, p_n$ are arbitrary, so x_i, p_j give $2n$ local coordinates on $\mathcal{C}_n$.

Claim: We have $\{x_i, x_j\} = \{p_i, p_j\} = 0$, $\{x_i, p_j\} = \delta_{ij}$.

Additive case: Calogero–Moser system (continued)

Now:

$$\{\!\{y, y\}\!\} = 0 \quad \Rightarrow \quad \{y^i, y^j\} = 0 \quad \Rightarrow \quad \{\operatorname{tr} Y^i, \operatorname{tr} Y^j\} = 0 \quad \text{for all } i, j.$$

Thus, we obtain n functions in involution $h_i := \frac{1}{i} \operatorname{tr} Y^i$ on $\mathcal{C}_n$, i.e. an integrable system.

Recall: $Y_{ij} = \delta_{ij} p_i + (1 - \delta_{ij}) \frac{1}{x_i - x_j}$.

E.g.,

$$h_1 = \operatorname{tr} Y = p_1 + \cdots + p_n$$

$$h_2 = \frac{1}{2} \operatorname{tr} Y^2 = \frac{1}{2} \sum_{i=1}^{n} p_i^2 - \sum_{i<j}^{n} \frac{1}{(x_i - x_j)^2}$$

Thus, we recover the Calogero–Moser system of n particles.

Moreover, we easily find the dynamics induced by h_i on (X, Y) as

$$\frac{d}{dt} X = Y^{i-1}, \quad \frac{d}{dt} Y = 0.$$

Thus, $X(t) = X(0) + tY^{i-1}$, and $x_i(t)$ are found as eigenvalues of $X(t)$.

Multiplicative case: Ruijsenaars–Schneider system

The multiplicative moment map for tadpole quiver becomes

$$\Phi = xzx^{-1}z^{-1}(1+vw)(1+wv)^{-1}\,.$$

Choosing the dimension of representations suitably, the result of quasi-Hamiltonian reduction can be described as

$$\mathcal{C}_{n,q} = \{X, Z \in \mathrm{GL}_n \mid \operatorname{rank}\left(XZX^{-1}Z^{-1} - q\,\mathrm{Id}\right) = 1\} /\!/ \,\mathrm{GL}_n\ .$$

By construction, this is a Poisson variety. It is smooth when $q^n \neq 1$.

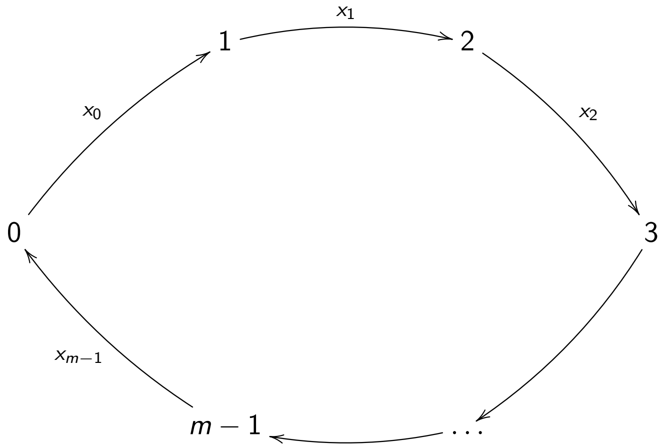
Note that $\{\!\{z, z\}\!\} = \frac{1}{2}\left(e_0 \otimes z^2 - z^2 \otimes e_0\right)$, so $\{\!\{z, z\}\!\} \neq 0$.

Nevertheless, we still have $\{z^i, z^j\} = 0$ and so $\{\operatorname{tr} Z^i\,, \operatorname{tr} Z^j\} = 0$ for all i, j. Thus, we obtain an integrable system on $\mathcal{C}_{n,q}$.

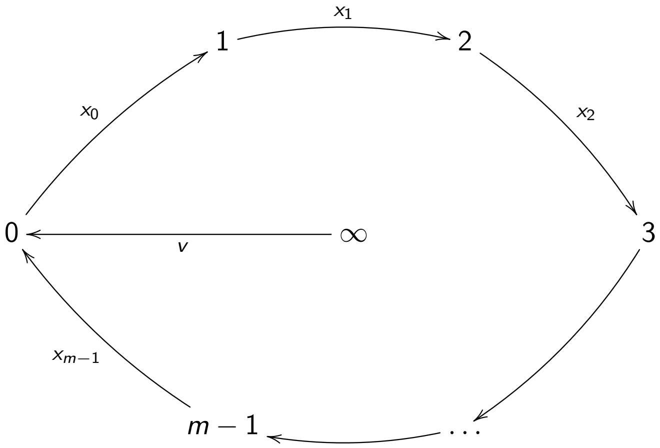
In suitable local coordinates (and after some work!), one can identify it with the Ruijsenaars–Schneider system (cf. Fock–Rosly (1999)):

$$\operatorname{tr} Z = \sum_{i=1}^{n} \prod_{j:j\neq i}^{n} \frac{1 - q x_i x_j^{-1}}{1 - x_i x_j^{-1}} e^{p_i}\,, \quad \text{with } \{x_i, p_j\} = \delta_{i,j} x_i\,.$$

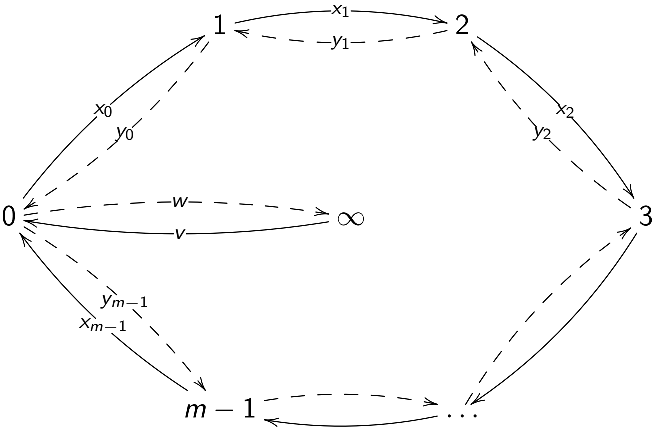
Cyclic quiver Q



Framed quiver Q_∞



Doubled framed quiver $\overline{Q}_\infty$



Cyclic quiver: integrable systems

Using similar methods, in case of a cyclic quiver with m verices we obtain:

1. Additive case: Calogero–Moser system for $W = S_n \rtimes \mathbb{Z}_m$ (O.C.–Silanyev (2017)).
2. Multiplicative case: New families of integrable systems generalising the Ruijsenaars–Schenider system (O.C.–Fairon (2017))

In general, these are complicated, but the simplest Hamiltonians can be written explicitly for any m.
Related quantum integrable systems also appeared in:
(a) O.C.–Etingof (2013), in the context of Macdonald theory
(b) Braverman–Etingof–Finkelberg (2016), in the context of cyclotomic DAHAs
(c) Braverman–Finkelberg–Nakajima (2016), in the context of qunatised K-theoretic Coulomb branches of certain supersymmetric gauge theories

Example: $m = 2$

When $m = 2$ (cyclic quiver with two vertices), we obtain an integrable quantum Hamiltonian as follows:

$$H^{(2)}_{q,t} = \sum_{i=1}^{n} a_i T_i^2 + \sum_{i<j}^{n} b_{ij} T_i T_j + \alpha \sum_{i=1}^{n} \prod_{k \neq i}^{n} \frac{1 - t x_i x_k^{-1}}{1 - x_i x_k^{-1}} x_i^{-1} T_i + \beta \sum_{i=1}^{n} x_i^{-1} \,,$$

where the coefficients a_i, b_{ij} are given by

$$\begin{aligned} a_i =& (q x_i)^{-1} \prod_{j \neq i}^{n} \frac{(1 - t x_i x_j^{-1})(1 - q t x_i x_j^{-1})}{(1 - x_i x_j^{-1})(1 - q x_i x_j^{-1})} \,, \\ b_{ij} =& \frac{(t-1)(t-q)(x_i^{-1} + x_j^{-1})}{(1 - q x_i x_j^{-1})(1 - q x_j x_i^{-1})} \prod_{l \neq i,j}^{n} \frac{(1 - t x_i x_l^{-1})(1 - t x_j x_l^{-1})}{(1 - x_i x_l^{-1})(1 - x_j x_l^{-1})} \,. \end{aligned}$$

Here $T_i = e^{\hat{p}_i}$, and α, β arbitrary parameters. The classical case is obtained by setting $q = 1$, $T_i = e^{p_i}$.

Summary (so far)

The above framework provides three main classes of examples of noncommutative (quasi-)Poisson spaces:

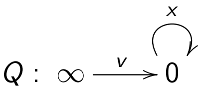
1. Noncommutative cotangent bundles $T_A D$ with canonical double Poisson bracket.
2. Path algebras of doubled quivers with double Poisson bracket.
3. Path algebras of doubled quivers with double quasi-Poisson bracket of Van den Bergh.

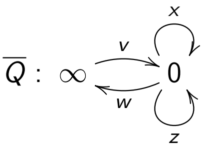
Each of these classes provides a large family of interesting Poisson varieties obtained by Hamiltonian or quasi-Hamiltonian reduction.

Open question: Which of these varieties host interesting integrable systems?

Other interesting examples

Above we discussed





Spin generalisation: Similar quiver, but with multiple framing, i.e. d arrows $v_1, \dots, v_d$ and d arrows $w_1, \dots, w_d$.
Additive case: Gibbons–Hermsen system (spin Calogero–Moser system). Multiplicative case: spin Ruijsenaars–Schneider system. In the latter case, this provides a Hamiltonian formulation of the spin RS system, answering an old open question by Arutyunov–Frolov (O.C.–Fairon (2018)). Also, one can use multiple framing fro the cycilc quiver. This leads to new spin versions of the Calogero–Moser system for $W = S_n \wr \mathbb{Z}_m$ (O.C.–Silantyev (2017)), as well as their relativistic versions (Fairon (2019)).

THANK YOU!