Standard groupoids of von Neumann algebras

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Plan

Operator algebras

- $L^\infty(\mathcal{H})$ the set of all bounded linear operators
- lattice of projections $\mathcal{L} := \{p \in L^\infty(\mathcal{H}) : p^2 = p^* = p\}$
- partial isometries $\mathcal{U} := \{v \in L^\infty(\mathcal{H}) : v^* v \in \mathcal{L}\}$

- $\mathrm{Tr} : L^\infty(\mathcal{H})^+ \to [0,\infty]$, $\mathrm{Tr}\,(x) := \sum_{j\in J} \langle x\gamma_j \mid \gamma_j\rangle$ for any o.n.b. $(\gamma_j)_{j\in J}$
- $L^1(\mathcal{H}) := \{x \in L^\infty(\mathcal{H}) : \mathrm{Tr}\,(|x|) < \infty\}$
- duality pairing $\langle\cdot,\cdot\rangle \colon L^\infty(\mathcal{H}) \times L^1(\mathcal{H}) \to \mathbb{C}$, $\quad \langle x, y\rangle := \mathrm{Tr}\,(xy)$

$\rightsquigarrow$ $\boxed{L^1(\mathcal{H}) \text{ is predual to } L^\infty(\mathcal{H})}$ that is, $L^\infty(\mathcal{H}) \simeq (L^1(\mathcal{H}))^*$

- W^*-algebra $:=$ C^*-algebra $\mathfrak{M}$ with a predual $\mathfrak{M}_*$, hence $\mathfrak{M} = (\mathfrak{M}_*)^*$

$$\iff \mathfrak{M} \simeq \underbrace{\text{weakly closed } *\text{-subalgebra} \subseteq L^\infty(\mathcal{H})}_{\text{von Neumann algebra}}$$

Groupoids

Example: ~~direct product~~ disjoint union of two groups $G \sqcup H$

- canonical projection $G \sqcup H \to \{\mathbf{1}_G, \mathbf{1}_H\}$
- multiplication is partially defined, only for elements in the same fiber

Groupoid = associative partial multiplication on a set ("of arrows") $\mathcal{G}$:

- set of unit elements ("objects") $\mathcal{G}_0$
- object inclusion map $\boldsymbol{\epsilon}\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$
- source/target maps $\boldsymbol{s}, \boldsymbol{t}\colon \mathcal{G} \to \mathcal{G}_0$
- set of composable pairs $\mathcal{G} * \mathcal{G} := \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} \mid \boldsymbol{s}(\gamma_1) = \boldsymbol{t}(\gamma_2)\}$
- multiplication map $\mathcal{G} * \mathcal{G} \to \mathcal{G}$, $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$
- inversion map $\boldsymbol{\iota}\colon \mathcal{G} \to \mathcal{G}$

• should be denoted $\mathcal{G} \underset{\boldsymbol{s}}{\overset{\boldsymbol{t}}{\rightrightarrows}} \mathcal{G}_0$ but we keep it simple: $\mathcal{G} \rightrightarrows \mathcal{G}_0$

Groupoids associated to a W^*-algebra $\mathfrak{M}$

Example 1: unitary group $U(\mathfrak{M}) := \{u \in \mathfrak{M} : u^*u = uu^* = \mathbf{1}\} \rightrightarrows \{\mathbf{1}\}$
This is a Banach-Lie group.

Example 2: groupoid of partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

- objects: $\mathcal{L}(\mathfrak{M}) = \{p \in \mathfrak{M} \mid p = p^* = p^2\}$ orthogonal projections
- arrows: $\mathcal{U}(\mathfrak{M}) = \{v \in \mathfrak{M} \mid v^*v \in \mathcal{L}(\mathfrak{M})\}$ partial isometries
- object inclusion map $\boldsymbol{\epsilon}\colon \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{U}(\mathfrak{M})$
- source/target maps $\boldsymbol{s}, \boldsymbol{t}\colon \mathcal{U}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$, $\boldsymbol{s}(v) = v^*v$, $\boldsymbol{t}(v) = vv^*$
- multiplication: $v_1, v_2 \in \mathcal{U}(\mathfrak{M})$, $\boldsymbol{s}(v_1) = \boldsymbol{t}(v_2) \implies v_1v_2 \in \mathcal{U}(\mathfrak{M})$
- inversion map $\boldsymbol{\iota}\colon \mathcal{U}(\mathfrak{M}) \to \mathcal{U}(\mathfrak{M})$, $v \mapsto v^*$

This is a Banach-Lie groupoid.

Intermezzo

Why study group-like structures that arise from operator algebras?

Answer: for the sake of gaining a geometric perspective.
For, geometry = invariant theory of some transformation group(oid. . .), according to Felix Klein (1812) and others.

More recent history:

- R.V. Kadison: unitary groups (1952)

. . . .

- H. Porta, L. Recht: Grassmann manifolds (1987)

. . . .

- A. Odzijewicz, T. Ratiu: Poisson geometry on W^*-algebras ('03)
- E. Andruchow, G. Corach, M. Mbekhta: generalized inverses ('05)

. . . .

- A. Odzijewicz, A. Sliżewska: groupoids associated to W^*-algebras ('16)
- D. B., T.Goliński, G.Jakimowicz, F.Pelletier: Banach-Lie groupoids ('19)

Standard forms of W^*-algebras

• Every W^*-algebra can be realized as a standard von Neumann algebra.

That is, a von Neumann algebra $\mathfrak{M} \subseteq L^\infty(\mathcal{H})$, along with an antilinear unitary $J = J^{-1} \colon \mathcal{H} \to \mathcal{H}$ and a self-dual cone $\mathcal{P} \subset \mathcal{H}$ satisfying

(i) $J\mathfrak{M}J = \mathfrak{M}'$ $(:= \{y \in L^\infty(\mathcal{H}) \mid xy = yx$ for all $x \in \mathfrak{M}\})$;

(ii) $JxJ = x^*$ for $x \in \mathfrak{M} \cap \mathfrak{M}'$;

(iii) $J\gamma = \gamma$ for $\gamma \in \mathcal{P}$;

(iv) $xJxJ\mathcal{P} \subset \mathcal{P}$ for $x \in \mathfrak{M}$.

• The standard form of any W^*-algebra is unique up to unitary equivalence.

Standard groupoids of W^*-algebras

Theorem 1

- $(\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ standard form
- $\mathfrak{M}_*^+ := \{\rho \in \mathfrak{M}_* \mid \rho(x^* x) \geq 0 \text{ for all } x \in \mathfrak{M}\}$
- $E\colon \mathcal{H} \to \mathfrak{M}_*^+$, $E(\gamma) := \mid \gamma\rangle\langle\gamma \mid = \langle \gamma \mid \bullet \gamma\rangle$

$\rightsquigarrow$ groupoid $\mathcal{H} \rightrightarrows \mathfrak{M}_*^+$ having its source map E and inversion map J

Example: $\mathfrak{M} = L^\infty(T, \mu)$ commutative, $\mathfrak{M}_* = L^1(T, \mu)$

• $\mathcal{H} = L^2(T, \mu)$

• $\boldsymbol{s} = \boldsymbol{t}\colon L^2(T, \mu) \to L^1(T, \mu)^+, \quad \gamma \mapsto |\gamma|^2$

$\rightsquigarrow$ the standard groupoid $\mathcal{H} \rightrightarrows \mathfrak{M}_*^+$ is a group bundle

Standard groupoids as symplectic-like groupoids

• The base of the standard groupoid $\mathcal{H} \rightrightarrows \mathfrak{M}_*^+$ is a convex cone rather than a manifold.

• A. Odzijewicz & al.: $\mathfrak{M}_*$ carries a Poisson bracket $\{\cdot,\cdot\}$ for which

$$(\mathfrak{M}, [\cdot,\cdot]) \hookrightarrow (\mathcal{C}^\infty(\mathfrak{M}_*), \{\cdot,\cdot\})$$

Theorem 2

- The orbits of the standard groupoid of $\mathfrak{M}$ are weakly symplectic Banach manifolds.
  Its transitive sugbroupoids are Banach-Lie groupoids.
- Source/target maps are Poisson/anti-Poisson maps.
- The graph of the multiplication is an isotropic submanifold $\{(\gamma_1, \gamma_2, \gamma_1 \bullet \gamma_2) : (\gamma_1, \gamma_2) \in \mathcal{H} * \mathcal{H}\} \subseteq \mathcal{H} \times \mathcal{H} \times \overline{\mathcal{H}}$

Example: type I factors

- $\mathfrak{M}_0 = L^\infty(\mathcal{H}_0)$ with its predual $(\mathfrak{M}_0)_* = L^1(\mathcal{H}_0)$

- $\mathcal{H} = L^2(\mathcal{H}_0) := \{x \in \mathfrak{M}_0 \mid \mathrm{Tr}\,(x^*x) < \infty\}$
- $J\colon \mathcal{H} \to \mathcal{H}$, $J(x) := x^*$
- $\mathcal{P} := \{x \in \mathcal{H} \mid x \geq 0\}$
- for all $a \in \mathfrak{M}_0$ we define $\lambda(a)\colon \mathcal{H} \to \mathcal{H}$, $\lambda(a)\gamma := a\gamma$, and then $\mathfrak{M}_0 \simeq \lambda(\mathfrak{M}_0) =: \mathfrak{M} \subseteq L^\infty(\mathcal{H})$

$\rightsquigarrow (\mathfrak{M}, \mathcal{H}, J, \mathcal{P})$ standard form of $\mathfrak{M}_0$

$\rightsquigarrow L^2(\mathcal{H}_0) \rightrightarrows L^1(\mathcal{H}_0) \cap \mathcal{P}$ its corresponding standard groupoid with its source/target maps

$$\boldsymbol{s}, \boldsymbol{t}\colon L^2(\mathcal{H}_0) \to L^1(\mathcal{H}_0) \cap \mathcal{P}, \quad \boldsymbol{s}(\gamma) = \gamma^*\gamma,\ \boldsymbol{t}(\gamma) = \gamma\gamma^*$$

Standard groupoids of finite W^*-algebras

A W^*-algebra $\mathfrak{M}$ is *finite* if it every isometry in $\mathfrak{M}$ is unitary, i.e.,

$$u \in \mathfrak{M} \quad \& \quad u^*u = \mathbf{1} \implies uu^* = \mathbf{1}$$

Examples of finite W^*-algebras:

1. Type I: $M_n(\mathbb{C})$, $n = 1, 2, \ldots$
2. Type II: the von Neumann of any countable discrete group

For any W^*-algebra $\mathfrak{M}$ the following conditions are equivalent:

- $\mathfrak{M}$ is finite
- The orbits of the standard groupoid of $\mathfrak{M}$ are connected.

Standard groupoids of purely infinite W^*-algebras

$\mathfrak{M}$ is *purely infinite* (type III) if every nonzero projection is infinite:

$$p \in \mathcal{L}(\mathfrak{M}) \setminus \{0\} \implies (\exists u \in \mathfrak{M}) \quad p = u^* u > u u^*$$

If $\mathfrak{M}$ is a factor with separable predual, the following are equivalent:

- $\mathfrak{M}$ is purely infinite
- Each nonzero orbit of the standard groupoid of $\mathfrak{M}$ has exactly two connected components.

Standard groupoids of type III_1

Special (type III_1) purely infinite W^*-algebras occur in some models of relativistic QFT (Haag-Kastler-Araki-Borchers-. . .)

$\mathfrak{M}$ is *type III_1* if

1. $\mathfrak{M}$ is purely infinite
2. $\mathfrak{M}_*$ is separable
3. for every faithful state $\rho \in \mathfrak{M}_*^+$ its modular operator Δ_ρ has its spectrum equal to $[0, \infty)$.

If $\mathfrak{M}$ is a factor with separable predual, the following are equivalent:

- $\mathfrak{M}$ is type III_1
- The norm-closure of every orbit of the standard groupoid of $\mathfrak{M}$ is a sphere in $\mathfrak{M}_*^+$.