Circle and sphere bundles in noncommutative geometry

Based on joint work with J. Kaad and G. Landi

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For X a compact Hausdorff space, consider

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

The set $C(X)$ comes with

- vector space structure: for $f, g \in C(X)$ and $\lambda \in \mathbb{C}$

$$(\lambda f + g)(x) := \lambda f(x) + g(x), \qquad \forall x \in X;$$

- commutative product: for $f, g \in C(X)$:

$$(fg)(x) := f(x)g(x), \quad \forall x \in X;$$

- unit: the function identically equal to 1; and
- an involution $* : C(X) \to C(X)$ given by

$$f^*(x) := \overline{f(x)}.$$

There is a natural norm on the space $C(X)$, given by

$$\|f\| = \sup_{x \in X} |f(x)|. \tag{1}$$

with respect to which $C(X)$ is *a Banach *-algebra*.

The norm satisfies

$$\boxed{\|f^* f\| = \|f\|^2.}$$

$C(X)$ is a *commutative C^*-algebra*.

Example

Let X consist of n-points. $C(X) \simeq \mathbb{C}^n$ with the usual vector space structure, coordinate-wise multiplication and complex conjugation, and norm

$$\|(z_1, \dots z_n)\|^2 = \max\{\overline{z_i} z_i \ | i = 1, \dots, n\}$$

Any point $P \in X$ can be thought of as a functional

$$\sigma_P : C(X) \to \mathbb{C}, \quad \sigma_P(f) := f(P),$$

and it satisfies

$$\sigma_P(fg) = \sigma_P(f)\sigma_P(g), \quad \sigma_P(1) = 1,$$

i.e. σ_P is a *character* (also, a *pure state*).

All characters on $C(X)$ are of this form and the set of characters $\Sigma(C(X))$ is *homeomorphic* to X.

Theorem (Gelfand Duality)

*Let A be a commutative unital C^*-algebra. Then there is a *-isomorphism*

$$\boxed{A \simeq C(\Sigma(A))}$$

of commutative C^*-algebras.

Definition

A C^*-algebra is a Banach *-algebra A with the property that

$$\boxed{\|a^* a\| = \|a\|^2},$$

for all $a \in A$.

Some examples

- The algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices with conjugate transpose and the operator norm
$$\|A\| = \sup_{x \in \mathbb{C}^n, \|x\|=1} \|Ax\|;$$
- The algebra $B(H)$ of bounded operators on a Hilbert space, with operator adjoint, and operator norm
$$\|A\| = \sup_{x \in H, \|x\|=1} \|Ax\|;$$

$B(H)$ is the prototypical example of C*-algebra.

Theorem (Gelfand–Naimark–Seagal)

Let A be a C^*-algebra. Then there exist a Hilbert space H and an injective $*$-homomorphism $\pi : A \to B(H)$.

Every C*-algebra can be embedded into the bounded operators on a Hilbert space.

Main Idea

Motivated from Gelfand duality, look at noncommutative C*-algebras of operators as algebras of functions on some *noncommutative space*.

The circle:

$$S^1 := \{z \in \mathbb{C} \mid \bar{z}z = 1\}.$$

The C*-algebra $C(S^1)$ is the closure of the *Laurent polynomials*

$$\frac{\mathbb{C}[\zeta, \bar{\zeta}]}{\langle \bar{\zeta}\zeta = 1 \rangle}.$$

We represent $C(S^1)$ via multiplication operators on the Hilbert space

$$H = L^2(S^1) \simeq \ell^2(\mathbb{Z}).$$

Under this isomorphism, multiplication by $e^{2\pi i\theta}$ is mapped to the bilateral shift

$$U(e_n) = (e_{n+1}), \quad U^*(e_n) = e_{n-1}.$$

$C(S^1)$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{Z}))$ that contains the *unitary* U.

Now instead consider the Hilbert space $\ell^2(\mathbb{N})$ and the shift operator

$$T(e_n) = (e_{n+1})$$

Its adjoint is not invertible

$$T^*(e_n) = \begin{cases} e_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}.$$

The *Toeplitz algebra* $\mathcal{T}$ is the smallest C*-subalgebra of $B(\ell^2(\mathbb{N}))$ that contains T.

It is not commutative since $T^*T = \mathrm{Id}$ and $TT^* = 1 - P_{\ker(T^*)}$.

Elements of $\mathcal{T}$ commute up to compact operators:

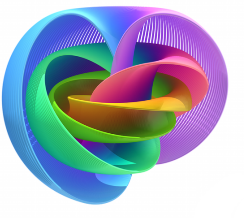
$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

The spectrum $\Sigma(\mathcal{T})$ (defined as the set of pure states) is the disk $\mathbb{D} \subseteq \mathbb{C}$.

The algebra $C(S^1)$ is the "boundary" of a noncommutative disk.

Topology	**Operator algebra**
topological space	C^*-algebra
point	pure state
vector bundle	finitely generated projective module
Hermitian vector bundle	finitely generated projective *Hilbert* module
circle bundle	Cuntz–Pimsner algebra
sphere bundle	WIP

The Hopf bundle I



Principal circle bundle

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$$

Look at S^3 inside $\mathbb{C}^2$:

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \overline{z_1} z_1 + \overline{z_2} z_2 = 1\}.$$

Circle action defined component-wise: for every $\lambda \in S^1$,

$$\alpha_\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2).$$

The orbit space is the two sphere S^2.

In physics: connections on the Hopf bundle describe magnetic monopole potentials.

The Hopf projection $\pi : S^3 \to S^2$ dualises to an inclusion of C*-algebras

$$C(S^2) \overset{\iota}{\hookrightarrow} C(S^3).$$

Circle action on $C(S^3)$, such that $C(S^2)$ is the fixed point algebra. The coordinate algebra

$$C(S^3) \supseteq \mathcal{O}(S^3) := \frac{\mathbb{C}[z_1, z_2, \overline{z_1}, \overline{z_2}]}{\langle \overline{z_1} z_1 + \overline{z_2} z_2 = 1 \rangle}$$

admits a vector space decomposition

$$\mathcal{O}(S^3) \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

where each $\mathcal{L}_n$ is the space of elements of $\mathcal{O}(S^3)$ that transform under the circle action as

$$\phi \mapsto \lambda^{-n} \phi, \qquad \forall \lambda \in S^1$$

Each $\mathcal{L}_n$ is a bimodule over $\mathcal{L}_0 \simeq \mathcal{O}(S^2)$ and it is finitely generated projective.

The condition that the bundle is *principal* translates into the *algebraic* condition that the grading

$$\mathcal{O}(S^3) \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

is *strong*, i.e.

$$\mathcal{L}_n \otimes_{\mathcal{L}_0} \mathcal{L}_m \simeq \mathcal{L}_{n+m}.$$

This is in turn equivalent to

$$\mathcal{L}_1 \otimes_{\mathcal{L}_0} \mathcal{L}_{-1} \simeq \mathcal{L}_0 \simeq \mathcal{L}_{-1} \otimes_{\mathcal{L}_0} \mathcal{L}_1,$$

i.e. the module $\mathcal{L}_1$ is *invertible*, hence the module of sections of a line bundle.

Moreover, for $k \geq 1$ we have

$$\bigoplus_{n \geq 0} \mathcal{L}_{kn} \simeq \mathcal{O}(L^3(k)),$$

the algebra of coordinate functions on the lens space

$$L^3(k) \simeq S^3/\mathbb{Z}_k.$$

The Peter–Weyl decomposition allows to decompose the coordinate algebra of a circle bundle into *sums of powers of line bundles* and to characterise principal circle bundles.

The PW view-point is useful because:

- many C*-algebras have this structure (e.g. higher dimensional spheres and lens spaces)
- it is suitable for K-theory computations.

For every $k \geq 1$, we have an exact sequence in K-theory:

$$\begin{array}{ccccc} K^0(S^2) & \xrightarrow{\alpha} & K^0(S^2) & \xrightarrow{\pi^*} & K^0(L^3(k)) \\ \delta_{1,0}\uparrow & & & & \downarrow\delta_{0,1} \\ K^1(L^3(k)) & \xleftarrow[\pi^*]{} & K^1(S^2) & \xleftarrow[\alpha]{} & K^1(S^2) \end{array} , \qquad (2)$$

where

$$\alpha := 1 - [\xi^{\otimes k}],$$

and ξ is the canonical bundle, i.e. its module of section is the completion of $\mathcal{L}_1$.
Using the fact that

$$K^0(S^2) = \mathbb{Z}^2, \quad K^1(S^2) = 0,$$

we can write $\alpha = 1 - \mathrm{A}$, with $\mathrm{A} \in \mathrm{Mat}_2(\mathbb{Z})$, and

$$K^0\big(L^3(k)\big) \simeq \mathrm{Coker}(1 - \mathrm{A}), \qquad K^1\big(L^3(k)\big) \simeq \mathrm{Ker}(1 - \mathrm{A}).$$

Hilbert modules generalize the notion of Hilbert space with the field $\mathbb{C}$ replaced by a C*-algebra B.

A Hilbert module is a pair $(E, \langle\ ,\ \rangle_B)$, where

- E is a right B-module with an Hermitian B-valued inner product; and
- E is complete in the norm
$$\|\xi\|^2 := \|\langle \xi, \xi \rangle_B\|.$$

Operations on Hilbert modules: direct sums, tensor products.

The adjointable operators

$$\mathrm{End}^*(E) := \{T : E \to E \mid \exists T^* : E \to E : \langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle\},$$

form a C*-algebra.

Define the C*-algebraic dual

$$E^* := \{\lambda_\xi, \xi \in E \mid \lambda_\xi(\eta) = \langle \xi, \eta \rangle\} \subseteq \mathrm{Hom}^*(E, B).$$

Let E be a finitely generated projective Hilbert bimodule over a unital C*-algebra B. We say that E is a *self-Morita equivalence* over B if

$$E \otimes_B E^* \simeq B \simeq E^* \otimes_B E.$$

Example

Let $B = C(X)$. Then $E = \Gamma(L)$, the module of sections of a Hermitian line bundle $L \to X$ is a self-Morita equivalence over B.

Self Morita equivalences over B form a group: $\mathrm{Pic}(B)$. For $B = C(X)$ we have

$$\mathrm{Pic}(B) = H^2(X, \mathbb{Z}) \rtimes \mathrm{Aut}(B).$$

Out of internal tensor products, construct

$$\mathcal{F}_E := B \oplus \bigoplus_{n \geq 1} E^{\otimes n}$$

On $\mathcal{F}_E$ define the *shift operators* by

$$T_\eta(\xi_1 \otimes \cdots \xi_n) = \eta \otimes \xi_1 \otimes \cdots \xi_n, \quad T_\eta b = \eta \cdot b.$$

These are adjointable with adjoints

$$T_\eta^*(\xi_1 \otimes \cdots \xi_n) = \phi\left(\langle \eta, \xi_1 \rangle\right) \xi_2 \otimes \cdots \otimes \xi_n, \quad T_\eta^*(b) = 0$$

The *Toeplitz algebra* $\mathcal{T}_E$ as the C*-subalgebra of $\mathcal{L}(\mathcal{F}_E)$ generated by the shifts.

If E is a self-Morita equivalence bimodule, we can define the two-sided Fock module

$$\mathcal{F}_{\mathbb{Z}}(E) := \bigoplus_{n \in \mathbb{Z}} E^{(n)}$$

where $E^{(n)} := E^{\otimes n}$ for $n > 0$, $E^{(0)} = B$ and $E^{(n)} := (E^*)^{\otimes n}$ for $n < 0$.
On $\mathcal{F}_{\mathbb{Z}}(E)$ we consider bilateral shift operators S_ξ, $\xi \in E$.

Definition

The *Cuntz–Pimsner algebra* of E, denoted $\mathcal{O}_E$, is the smallest C^*-subalgebra of $\mathrm{End}^*(\mathcal{F}_{\mathbb{Z}}(E))$ which contains all the bilateral shift operators.

We have an exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{F}(E)) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} \mathcal{O}_E \longrightarrow 0.$$

Both $\mathcal{T}_E$ and $\mathcal{O}_E$ come endowed with a circle action.
We denote by $\mathcal{O}_E^\gamma$ the fixed point algebra for this action.

Proposition (A.–Rennie)

E is a self-Morita equivalence bimodule if and only if $\mathcal{O}_E^\gamma \simeq B$.

Theorem (A.–Kaad–Landi)

Pimsner algebras of self-Morita equivalences are quantum principal circle bundles.

Examples: q-deformations

This construction naturally lends itself to the computation of invariants (K-theory).

Pimsner 1997: The defining extension is semi-split. Hence it induces six term exact sequences in KK-theory.

These simplify by using:

- The class of the correspondence $E \in KK(B, B)$;
- The class of the Morita equivalence $[\mathcal{F}_E] \in KK(\mathcal{K}_B(\mathcal{F}_E), B)$;
- The class of the KK-equivalence $[\alpha]^{-1} \in KK(\mathcal{T}_E, B)$, which is the inverse to the class of the inclusion $\alpha : B \hookrightarrow \mathcal{T}_E$.

These satisfy:

$$[\mathcal{F}_E] \otimes_B (1 - [E]) = [j] \otimes_{\mathcal{T}_E} [\alpha]^{-1}$$

Let [ext] be the class of the defining extension and
$[\partial] := [\mathrm{ext}] \otimes_{\mathcal{K}(\mathcal{F}_E)} [\mathcal{F}_E] \in KK_1(\mathcal{O}_E, B)$ the class of the product.
For $C = \mathbb{C}$ we get exact sequences in K-theory:

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-[E]} & K_0(B) & \xrightarrow{j_*} & K_0(\mathcal{O}_E) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_E) & \xleftarrow[j_*]{} & K_1(B) & \xleftarrow[1-[E]]{} & K_1(B) \end{array}, \tag{3}$$

We start from a collection $\{X_n\}_{n\in\mathbb{N}}$ of f.g.p. modules over B such that

1. $X_0 = B$
2. X_{n+m} is a complemented submodule of $X_n \otimes_B X_m$ for all $n, m \in \mathbb{N}$.

On

$$\mathcal{F}_X := \bigoplus_{n\geq 0} X_n,$$

we consider operators

$$T_\xi(\eta) = p_{n+m}(\xi \otimes \eta), \qquad \xi \in X_n, \eta \in X_m.$$

We define the Toeplitz algebra $T(X)$ as the subalgebra of $\mathcal{L}(\mathcal{F})$ generated by the shift operators.

The Cuntz–Pimsner algebra of the subproduct system X is realised as quotient *by a suitable invariant ideal*:

$$0 \longrightarrow \mathcal{I}_X \longrightarrow T(X) \xrightarrow{\pi} O(X) \longrightarrow 0. \tag{4}$$

Example

Let $B = \mathbb{C}$, $E = \mathbb{C}^d$ and consider the symmetric tensor product

$$X_n = \mathrm{Sym}^n(\mathbb{C}^d).$$

Then $\mathcal{F}_X$ is the symmetric Fock space.
The resulting extension is the Toeplitz extension for the odd spheres:

$$0 \longrightarrow \mathcal{K}(H^2(S^{2d-1})) \longrightarrow \mathcal{T}_d \xrightarrow{\pi} C(S^{2d-1}) \longrightarrow 0.$$

Let $\tau : SU(2) \to U(H)$ be a strongly continuous unitary representation of a Hilbert space H, $\dim(H) < \infty$. Define

$$\det(\tau, H) = \{\xi \in H \otimes H \mid (\tau(g) \otimes \tau(g))\xi = \xi \quad \forall g \in SU(2)\}.$$

Inductively construct a subproduct system X where

- $X_0 = \mathbb{C}$;
- $X_1 = E$;
- $X_2 = \det(\tau, H)^{\perp} \subseteq H \otimes H$;
- $X_n := \cap_{k+l=n} X_k \otimes X_l$.

Example

If $\tau = \rho_1 : SU(2) \to U(\mathbb{C}^2)$ is the fundamental representation, then

$$X_n = \mathrm{Sym}^n(\mathbb{C}^2)$$

Let τ be an irreducible $SU(2)$ representation. Then

- $\det(\tau, H)$ is a one dimensional vector space;
- for all $n \geq 1$ we have

$$\dim(X_{n+1}) = \dim(X_1) \cdot \dim(X_n) - \dim(X_{n-1}).$$

- the sequence of quotients

$$\left\{ \frac{\dim(X_n)}{\dim(X_{n-1})} \right\}_{n=0}^{\infty}$$

  converges.

Example

Let $\tau = \rho_2 : SU(2) \to U(\mathbb{C}^3)$ be the triplet (a.k.a. coadjoint) representation. Then the dimensions of the fibers are the even Fibonacci numbers:

$$\{\dim X_n\}_{n=0}^{\infty} = \{1, 3, 8, 21, 55, 144, \dots\}, \quad \text{and} \quad \frac{\dim X_n}{\dim X_{n+1}} \to \frac{3-\sqrt{5}}{2} \sim 0.382.$$

Theorem

Let $\rho_n : SU(2) \to U(\mathbb{C}^{n+1})$ an irreducible representation. Then the Toeplitz algebra $T(\tau, H)$ is KK-equivalent to $\mathbb{C}$.

The Cuntz–Pimsner algebra $O(\tau, H)$ fits into an exact sequence

$$0 \longrightarrow K_1(O) \longrightarrow K_0(\mathbb{C}) \xrightarrow{\mathbf{1}_{\mathbb{C}} - [H] + [\det(\tau, H)]} K_0(\mathbb{C}) \xrightarrow{i_*} K_0(O) \longrightarrow 0$$

Corollary

For every $n \geq 1$, let $\tau = \rho_n : SU(2) \to U(\mathbb{C}^{n+1})$ we have

$$K_0(O(\rho_n, \mathbb{C}^{n+1})) \cong \mathbb{Z}/(n-1)\mathbb{Z} \qquad K_1(O(\rho_n, \mathbb{C}^{n+1})) \cong \begin{cases} \mathbb{Z} & n = 1, \\ \{0\} & \textit{otherwise.} \end{cases}$$

- Within the NCG dictionary, Cuntz–Pimsner algebras of self-Morita equivalencs are a model for circle bundles.
- There is evidence that Cuntz–Pimsner algebas of subproduct systems are suitable to encode spherical symmetries.
- K-theoretic Gysin sequences.
- Open questions:
  - Universal properties / description in terms of genators and relations.
  - Extend our results from the case of Hilbert spaces to that of Hilbert modules.