ON THE CURRENT ALGEBRA REPRESENTATIONS AND QUANTUM MANY-PARTICLE INTEGRABLE HAMILTONIAN MODELS

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1. Abstract

There is developed the G. Goldin's current algebra representation scheme for reconstructing quantum Hamiltonian and symmetry operators in case of quantum integrable spatially many- and one-dimensional Schrödinger type dynamical systems.

In the report we are interested mainly in studying local current algebra representations in suitably renormalized Fock spaces and their applications to constructing the related finite-particle factorized representations for corresponding secondly-quantized many-particle Hamiltonian operators. As examples we have studied in detail the factorized structure of Hamiltonian operators, describing such quantum integrable spatially many- and one-dimensional models as generalized oscillatory, Calogero-Sutherland, Coulomb type and Nonlinear Schrödinger dynamical systems of spin-less bose-particles.

Main topics to be discussed are as follows:

1.1. **An integrable many-particle oscillatory quantum model.** As a first application of the local current algebra representation construction devised above, we will consider a simple nonrelativistic oscillatory quantum model of interacting bose-particles in the m-dimensional Euclidean space $(\mathbb{R}^m; <\cdot|\cdot>), m \in \mathbb{Z}_+$, described by the secondly quantized Hamiltonian operator

$$\mathrm{H}^{(\omega)} = \frac{1}{2}\int_{\mathbb{R}^m} dx < \nabla\psi^+(x)|\nabla\psi(x) > + \frac{1}{2}\int_{\mathbb{R}^m} dx < \omega x|\omega x > \psi^+(x)\psi(x), \tag{1.1}$$

acting on the corresponding Fock space Φ, and parametrized by the positive definite frequency matrix $\omega \in End\ \mathbb{R}^m$. The following proposition holds.

Proposition 1.1. *The quantum oscillatory Hamiltonian operator (1.1) allows on the suitable Fock space Φ the factorized representation*

$$\mathrm{H}^{(\omega)} = \frac{1}{2}\int_{\mathbb{R}^m} dx < \mathrm{K}^+(x) + \omega x\rho(x))|\rho(x)^{-1}(\mathrm{K}(x) + \omega x\rho(x)) > + \frac{1}{2}\mathrm{tr}\omega\ N. \tag{1.2}$$

Its ground state $|\Omega^{(\omega)}) \in \Phi$ satisfies the conditions

$$\mathrm{H}^{(\omega)}|\Omega^{(\omega)}) = \frac{1}{2}\mathrm{tr}\omega\ N|\Omega^{(\omega)}),\ (\mathrm{K}(x) + \omega x\rho(x))|\Omega^{(\omega)}) = 0 \tag{1.3}$$

for all $x \in \mathbb{R}^m$.

Moreover, as for any $x, y \in \mathbb{R}^m$ there holds the equality

$$[< \mathrm{D}^{(\omega),+}(x)|\rho(x)^{-1}\mathrm{D}^{(\omega)}(x) >, < \mathrm{D}^{(\omega),+}(y)|\rho(y)^{-1}\mathrm{D}^{(\omega)}(y) >] = 0, \tag{1.4}$$

where, by definition, the local operator

$$\mathrm{D}^{(\omega)}(x) := \mathrm{K}(x) + \omega x\rho(x), \tag{1.5}$$

the next operators

$$\mathrm{H}^{(\omega,p)} = \frac{1}{2}\int_{\mathbb{R}^m} dx \left(< \mathrm{D}^{(\omega),+}(x)|\rho(x)^{-1}\mathrm{D}^{(\omega)}(x) >\right)^p \tag{1.6}$$

on the Fock space Φ are *a priori* commuting to each other, that is

$$[\mathrm{H}^{(\omega,p)}, \mathrm{H}^{(\omega,q)}] = 0 \tag{1.7}$$

for any integers $p, q \in \mathbb{Z}_+$. Thus, the following quantum integrability proposition holds.

Proposition 1.2. *The nonrelativistic oscillatory quantum model (1.1) of interacting bose-particles in the m-dimensional space $\mathbb{R}^m$ possesses a countable hierarchy of the commuting to each other symmetric operators (1.6) on the suitable Fock space Φ represents a quantum integrable model.*

The obtained this way differential operators $H_N^{(\omega,p)} : L_2^{(s)}(\mathbb{R}^{mN};\mathbb{C}) \to L_2^{(s)}(\mathbb{R}^{mN};\mathbb{C}), p \in \mathbb{Z}_+$, are respectively, also commuting to each other, as this follows from (1.7), giving rise to the quantum integrability of the N-particle oscillatory Hamiltonian model $\mathrm{H}_N^{(\omega)} = \mathrm{H}_N^{(\omega,1)} + \frac{1}{2}\mathrm{tr}\omega\ N$ for arbitrary finite $N \in \mathbb{Z}_+$.

1.2. **A generalized integrable many-particle oscillatory quantum model.** A generalized quantum oscillatory model of bose-particles in $\mathbb{R}^m$ is described by the N-particle Hamiltonian operator

$$H_N := \frac{1}{2}\sum_{j=\overline{1,N}} < \nabla_{x_j}|\nabla_{x_j} > + \frac{1}{2}\sum_{j,k=\overline{1,N}} < \omega_N(x_j - x_k)|\omega_N(x_j - x_k) > \tag{1.8}$$

on $L_2^{(s)}(\mathbb{R}^{mN};\mathbb{C})$, parametrized by a positive definite interaction matrix $\omega_N \in End\ \mathbb{R}^m, N \in \mathbb{Z}_+$. In the case when this interaction matrix depends on the particle number $N \in \mathbb{Z}_+$ as $\omega_N = \bar{\omega}\sqrt{N/2}$ for some constant positive definite matrix $\tilde{\omega} \in End\ \mathbb{R}^m$, the corresponding to (1.8) secondly quantized Hamiltonian operator is representable as

$$\mathrm{H} = \frac{1}{2}\int_{\mathbb{R}^m} dx < \nabla\psi^+(x)|\nabla\psi(x) > + \frac{\mathrm{N}}{4}\int_{\mathbb{R}^m\times\mathbb{R}^m} dxdy\psi^+(x)\psi^+(y)\psi(y)\psi(x) < \bar{\omega}(x-y)|\bar{\omega}(x-y) >, \tag{1.9}$$

acting on a suitably chosen Fock type representation space Φ.

Consider now a quasi-local operator mapping $\mathrm{D}(x) : \Phi \to \Phi^m, x \in \mathbb{R}^m$, equal to

$$\mathrm{D}(x) := \psi^+(x)\nabla\psi(x) + \int_{\mathbb{R}^m} dy < \bar{\omega}(x-y) : \rho(x)\rho(y) :, \tag{1.10}$$

and construct the next operator expression:

$$\tilde{\mathrm{H}} = \frac{1}{2}\int_{\mathbb{R}^m} < \mathrm{D}^+(x)|\rho(x)^{-1}\mathrm{D}(x) > . \tag{1.11}$$

Then the following proposition holds.

Proposition 1.3. *The operator expression (1.11) is equivalent on the Fock space Φ to the secondly quantized Hamiltonian operator (1.9):*

$$\tilde{\mathrm{H}} = \mathrm{H} - \frac{\mathrm{tr}\bar{\omega}}{2}\ \mathrm{N}(\mathrm{N}-1). \tag{1.12}$$

Remark 1.4. It is worthy to remark here that owing to its construction, the operator mappings $< \mathrm{D}^+(x)|\rho(x)^{-1}\mathrm{D}(x) >: \Phi \to \Phi, x \in \mathbb{R}^m$, are commuting to each other, that is

$$[< \mathrm{D}^+(x)|\rho(x)^{-1}\mathrm{D}(x) >, < \mathrm{D}^+(y)|\rho(y)^{-1}\mathrm{D}(y) >] = 0 \tag{1.13}$$

for any $x, y \in \mathbb{R}^m$. This naturally makes it possible to construct a countable hierarchy of commuting to each other operators $\mathrm{H}^{(p)} : \Phi \to \Phi, p \in \mathbb{Z}_+$, where

$$\mathrm{H}^{(p)} := \int_{\mathbb{R}^m} dx \left(< \mathrm{D}^+(x)|\rho(x)^{-1}\mathrm{D}(x) >\right)^p, \tag{1.14}$$

that is

$$[\mathrm{H}^{(p)}, \mathrm{H}^{(p)}] = 0 \tag{1.15}$$

for all $p, q \in \mathbb{Z}_+$. The latter, in particular, means that our generalized quantum oscillatory model (1.8) is also integrable.

1.3. **The Calogero-Sutherland quantum model: the current algebra representation, the Hamiltonian reconstruction and integrability.** The periodic Calogero-Sutherland quantum bosonic model on the finite interval $[0,l] \simeq \mathbb{R}/[0,l]\mathbb{Z}$ is governed by the N-particle Hamiltonian

$$H_N := -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \sum_{j\neq k=\overline{1,N}} \frac{\pi^2\beta(\beta-1)}{l^2\sin^2[\frac{\pi}{l}(x_j - x_k)]} \tag{1.16}$$

in the symmetric Hilbert space $L_2^{(s)}([0,l]^N;\mathbb{C})$, where $N \in \mathbb{Z}_+$ and $\beta \in \mathbb{R}$ is an interaction parameter. As it was stated in a very interesting and highly speculative works [11], there exists linear differential operators

$$\mathcal{D}_j := \frac{\partial}{\partial x_j} - \frac{\pi\beta}{l} \sum_{k=\overline{1,N}, k\neq j} ctg[\frac{\pi}{l}(x_j - x_k)] \tag{1.17}$$

for $j = \overline{1,N}$, such that the Hamiltonian (1.16) is factorized as the bounded from below symmetric operator

$$H_N = \sum_{j=\overline{1,N}} \mathcal{D}_j^+ \mathcal{D}_j + E_N, \tag{1.18}$$

where

$$E_N = \frac{1}{3}\left(\frac{\pi\beta}{l}\right)^2 N(N^2-1) \tag{1.19}$$

is the groundstate energy of of the Hamiltonian operator (1.16), that is there exists such a vector $|\Omega_N) \in L_2^{(s)}([0,l]^N;\mathbb{C})$, satisfying for any $N \in \mathbb{Z}_+$ the eigenfunction condition

$$H_N|\Omega_N) = E_N|\Omega_N) \tag{1.20}$$

and equals

$$|\Omega_N) =_{j<k=\overline{1,N}} \left(\sin[\frac{\pi}{l}(x_j - x_k)]\right)^\beta. \tag{1.21}$$

Being interested additionally in proving the quantum integrability of the Calogero-Sutherland model (1.16), we will proceed to its second quantized representation [2, 5] and studying it by means of the current algebra representation approach, devised and developed before in [7, 8, 9, 10, 12, 13].

The secondly quantized form of the Calogero-Sutherland Hamiltonian operator (1.16) looks as

$$\mathrm{H} = \int_0^l dx \psi_x^+(x)\psi_x(x) + \left(\frac{\pi}{l}\right)^2 \beta(\beta-1) \int_0^l dx \int_0^l dy \frac{\psi^+(x)\psi^+(y)\psi(y)\psi(x)}{\sin^2[\frac{\pi}{l}(x-y)]}, \tag{1.22}$$

acting in the corresponding Fock space $\Phi := \oplus_{n\in\mathbb{Z}_+}\Phi_n,\ \Phi_n \simeq L_2^{(s)}([0,l]^n;\mathbb{C}), n \in \mathbb{Z}_+$. Having defined the operator

$$\begin{aligned} \mathrm{D}(x) = \psi^+(x)\psi_x(x) - \\ -\tfrac{\pi\beta}{2l}\textstyle\int_0^l dy\ ctg[\tfrac{\pi}{l}(x-y)] : \rho(x)\rho(y) : \end{aligned} \tag{1.23}$$

acting in the Fock space Φ, one can formulate the following proposition, first stated in [12], using completely different approach.

Proposition 1.5. *The secondly quantized Hamiltonian operator (1.22) in the Fock space Φ is representable in dual to (1.18) factorized form as*

$$\mathrm{H} = \int_0^l dx \mathrm{D}^+(x)\rho(x)^{-1}\mathrm{D}(x) + \mathrm{E}, \tag{1.24}$$

where the ground state energy operator $\mathrm{E}:\Phi\rightarrow\Phi$ *equals*

$$\mathrm{E} = \frac{1}{3}\left(\frac{\pi\beta}{l}\right)^2 : \mathrm{N}^3 : + \left(\frac{\pi\beta}{l}\right)^2 : \mathrm{N}^2 :, \tag{1.25}$$

where

$$\mathrm{N} := \int_0^l \rho(x)dx \tag{1.26}$$

is the particle number operator, and satisfies the determining conditions

$$(\mathrm{H}-\mathrm{E})|\Omega) = 0, \quad \mathrm{D}(x)|\Omega) = 0 \tag{1.27}$$

on the suitably renormalized vacuum ground state $|\Omega) \in \Phi$ *for all* $x \in \mathbb{R}/[0,l]\mathbb{Z}$. *Moreover, for any integer* $N \in \mathbb{Z}_+$ *the corresponding projected vector* $|\Omega_N) := |\Omega)|_{\Phi_N}$ *there satisfies the following eigenfunction relationships:*

$$
\begin{aligned}
\mathrm{N}|\Omega_N) &= N|\Omega_N), \quad \mathrm{E}|\Omega_N) = \left(\frac{1}{3}\left(\frac{\pi\beta}{l}\right)^2 : \mathrm{N}^3 : + \left(\frac{\pi\beta}{l}\right)^2 : \mathrm{N}^2 : \right)|\Omega_N) = \\
&= \left[\frac{1}{3}\left(\frac{\pi\beta}{l}\right)^2 (N^3 - 3N^2 + 2N) + \left(\frac{\pi\beta}{l}\right)^2 N(N-1)\right]|\Omega_N) = \\
&= \left[\frac{1}{3}\left(\frac{\pi\beta}{l}\right)^2 (N^3 - 3N^2 + 2N + 3N^2 - 3N)\right]|\Omega_N) = \\
&= \left[\frac{1}{3}\left(\frac{\pi\beta}{l}\right)^2 N(N^2-1)\right]|\Omega_N) := E_N|\Omega_N),
\end{aligned} \tag{1.28}
$$

coinciding exactly with the result (1.19).

Remark 1.6. When deriving the expression (1.28), we have used the identity

$$
\begin{aligned}
\rho(x)\rho(y) &= \ :\rho(x)\rho(y): + \rho(y)\delta(x-y), \\
\rho(x)\rho(y)\rho(z) &= \ :\rho(x)\rho(y)\rho(z): + :\rho(x)\rho(y):\delta(y-z) + \\
&+ :\rho(y)\rho(z):\delta(z-x) + :\rho(z)\rho(x):\delta(x-y) + :\rho(x)\delta(y-z)\delta(z-x),
\end{aligned} \tag{1.29}
$$

which holds [8, 13] for the density operator $\rho : \Phi \to \Phi$ at any points $x, y, z \in \mathbb{R}/[0,l]\mathbb{Z}$.

1.4. **An integrable many-particle Coulomb type quantum model on axis.** A many particle Coulomb type quantum bose model on axis is governed by the N-particle Hamiltonian

$$
\begin{aligned}
H_N \ :&= -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \sum_{j\neq k=\overline{1,N}} \frac{\alpha}{|x_j - x_k|} + \\
&+ \frac{\alpha^2}{3} \sum_{j\neq k\neq s=\overline{1,N}} \left[\ln|x_j - x_k| \frac{(x_j-x_k)(x_j-x_s)}{|x_j-x_k||x_j-x_s|} \ln|x_j - x_s| + \right. \\
&+ \ln|x_k - x_j| \frac{(x_k-x_j)(x_k-x_s)}{|x_k-x_j||x_k-x_s|} \ln|x_k - x_s| + \\
&+ \left. \ln|x_s - x_j| \frac{(x_s-x_j)(x_s-x_k)}{|x_s-x_j||x_s-x_k|} \ln|x_s - x_k| \right]
\end{aligned} \tag{1.30}
$$

acting in the Hilbert space $L_2^{(s)}(\mathbb{R}^N;\mathbb{C}), N \in \mathbb{Z}_+$, is parametrized by a real-valued interaction parameter $\alpha \in \mathbb{R}\backslash\{0\}$, which modulates both the *binary* and *ternary* particle interactions. Its secondly quantized representation in a suitably chosen Fock space Φ, looks as

$$
\begin{aligned}
\mathrm{H} \ &= \int_{\mathbb{R}} dx \psi_x^+(x) \ \psi_x(x) > + \int_{\mathbb{R}^2} dxdy \frac{\alpha}{|x-y|} : \rho(x)\rho(y) : + \\
+\frac{\alpha^2}{3}\int_{\mathbb{R}^3} dxdydz \ &: \ \rho(x)\rho(y)\rho(z) : \left[\ln|x-y| \frac{(x-y)(x-z)}{|x-y||x-y|} \ln|x-z| + \right. \\
&+ \left. \ln|y-x| \frac{(y-z)(y-x)}{|y-z||y-x|} \ln|y-x| + \ln|z-x| \frac{(z-x)(z-y)}{|z-x||z-y|} \ln|z-y| \right],
\end{aligned} \tag{1.31}
$$

modulo the infinite renormalization constant operator, responsible for the coinciding points $x = y \in \mathbb{R}$ of the Coulomb and logarithmic type interaction potentials. Introduce now at any point $x \in \mathbb{R}$ the quasi-local operator expression

$$
\mathrm{D}^{(\varepsilon)}(x) := \psi^+(x)\psi_x(x) - \alpha \int_{\mathbb{R}^2} dy : \rho(x)\rho(y) : s(x-y;\varepsilon), \tag{1.32}
$$

acting in the Fock space Φ, and construct the following operator:

$$
\tilde{\mathrm{H}}^{(\varepsilon)} = \int_{\mathbb{R}} dx < \mathrm{D}^{(\varepsilon),+}(x)|\rho(x)^{-1}\mathrm{D}^{(\varepsilon)}(x) > . \tag{1.33}
$$

Then one can formulate the next proposition.

Proposition 1.7. *The many-particle Coulomb type Hamiltonian operator (1.31) in a suitably chosen Fock space Φ is weakly equivalent, as $\varepsilon \to 0$, to the operator expression (1.33), and satisfies the following regularized limiting relationship:*

$$reg \lim_{\varepsilon\to 0} \tilde{\mathrm{H}}^{(\varepsilon)} := \lim_{\varepsilon\to 0} \left(\tilde{\mathrm{H}}^{(\varepsilon)} - \frac{\alpha^2}{3\varepsilon^2}\mathrm{N}(\mathrm{N}^2-1) \right) = \mathrm{H}. \tag{1.34}$$

1.5. **Quantum many-particle Hamiltonian dynamical system on axis with $\beta\delta$-interaction, its quantum symmetries and integrability.** In this Section we will consider a quantum non-relativistic many-particle bose-system on the axis $\mathbb{R}$, governed by the Hamiltonian operator:

$$H_N := -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \beta \sum_{j\neq k=\overline{1,N}} \delta(x_j - x_k), \tag{1.35}$$

where $\alpha, \beta \in \mathbb{R}$ are interaction constants, and acting in the symmetric Hilbert space $L_2^{(s)}(\mathbb{R}^N;\mathbb{C})$, $N \in \mathbb{Z}_+$. The corresponding secondly quantized expression for the Hamiltonian operator (1.35) in the related Fock space $\Phi \simeq \sum_{n\in\mathbb{Z}_+}^{\oplus} L_2^{(s)}(\mathbb{R}^n;\mathbb{C})$ equals

$$\mathrm{H} = \int_{\mathbb{R}} dx(\psi_x^+\psi_x + \beta\psi^+\psi^+\psi\psi), \tag{1.36}$$

where the creation ψ^+- and annihilation $\psi-$operators satisfy the canonical commutator relationships

$$\begin{aligned} [\psi(x), \psi^+(y)] &= \delta(x-y), \\ [\psi^+(x), \psi^+(y)] &= 0 = [\psi(x), \psi(y)] \end{aligned} \tag{1.37}$$

for any $x, y \in \mathbb{R}$. The Hamiltonian operator (1.36) via the Heisenberg recipe [5, 14] naturally generates on the creation $\psi^+ : \Phi \to \Phi$ and annihilation $\psi : \Phi \to \Phi$ operators the following quantum Schrödinger type evolution flow :

$$\begin{aligned} d\psi/dt \; &: = \frac{1}{i}[\mathrm{H}, \psi] = -i\psi_{xx} + 2i\beta\psi^+\psi^2, \\ d\psi^+/dt \; &: = \frac{1}{i}[\mathrm{H}, \psi^+] = i\psi_{xx}^+ - 2i\beta(\psi^+)^2\psi \end{aligned} \tag{1.38}$$

with respect to the temporal parameter $t \in \mathbb{R}$.

Let us define in the Fock space Φ the following structural operator:

$$\mathrm{D}^{(\varepsilon)}(x) := \psi^+(x)\psi_x(x) - \beta \int_{\mathbb{R}} dy \vartheta_\varepsilon(x-y) : \rho(x)\rho(y) :, \tag{1.39}$$

where for any $\varepsilon > 0$ the expression $\vartheta_\varepsilon(x-y) := \vartheta(x-y-\varepsilon) = \{1, \text{if } \; x > y-\varepsilon\} \wedge \{0, \text{if } \; x \leq y+\varepsilon\}$ for $x, y \in \mathbb{R}$ denotes the shifted classical Heaviside ϑ-function, and construct the following quantum operator:

$$\mathrm{H}^{(\varepsilon)} := \int_{\mathbb{R}} dx \mathrm{D}^{(\varepsilon),+}(x)\rho(x)^{-1}\mathrm{D}^{(\varepsilon)}(x). \tag{1.40}$$

The next proposition (1.40) states an equivalence of the quantum Hamiltonian operator (1.36) and the weak operator limit $\lim_{\varepsilon\to 0} \mathrm{H}^{(\varepsilon)}$.

Proposition 1.8. *The many-particle quantum operator (1.36) in a suitably chosen Fock space Φ is weakly equivalent, as $\varepsilon \to 0$, to the operator expression (1.40), and satisfies the following regularized limiting relationship:*

$$reg \lim_{\varepsilon\to 0} \tilde{\mathrm{H}}^{(\varepsilon)} := \lim_{\varepsilon\to 0} \left(\tilde{\mathrm{H}}^{(\varepsilon)} - \beta^2\mathrm{N}^3/3 \right) = \mathrm{H}. \tag{1.41}$$

Remark 1.9. It is worthy to mention that the following generalized quantum many-particle Hamiltonian dynamical bose system

$$\begin{aligned} H_N \; &: = -\sum_{j=\overline{1,N}} \frac{\partial^2}{\partial x_j^2} + \beta \sum_{j\neq k=\overline{1,N}} \delta(x_j - x_k) + \\ &+ i\alpha \sum_{j\neq k=\overline{1,N}} \left(\frac{\partial}{\partial x_j} \circ \delta(x_j - x_k) \; + \delta(x_j - x_k) \circ \frac{\partial}{\partial x_k} \right), \end{aligned} \tag{1.42}$$

where $\alpha, \beta \in \mathbb{R}$ are interaction constants, and acting in the symmetric Hilbert space $L_2^{(s)}(\mathbb{R}^N;\mathbb{C})$, $N \in \mathbb{Z}_+$ with $(\alpha\delta + \beta\delta')$-interaction potential is also integrable, as it was before proved in [14, 15, 4] by means of the quantum inverse scattering transform in a suitably constructed Fock space Φ. This fact, eventually, can suggest that there exists a local current algebra representation for the Hamiltonian (1.42), allowing a suitable finite-particle operator construction for the related structural operator $\mathrm{D}(x) : \Phi \to \Phi, x \in \mathbb{R}$, factorizing the secondly quantized Hamiltonian operator $\mathrm{H} \simeq \int_R dx \mathrm{D}^+(x)\rho(x)^{-1}\mathrm{D}(x)$ up to some renormalizing constant operators in the Fock space.

2. Conclusion

In the work we succeeded in developing an effective algebraic scheme of constructing density operator and functional representations for the canonical local quantum current algebra and its application to quantum Hamiltonian and symmetry operators reconstruction. We analyzed the corresponding factorization structure for quantum Hamiltonian operators, governing spatially many- and one-dimensional integrable dynamical systems. The quantum generalized oscillatory, Calogero-Sutherland, Coulomb type and Nonlinear Schrödinger models of spin-less bose-particles were analyzed in detail.

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