QUANTUM DIRICHLET FORMS AND THEIR RECENT APPLICATIONS

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ABSTRACT. We will discuss the notion of classical Dirichlet forms, quadratic forms giving rise to Markov semigroups on the spaces of the form $L^2(X, \mu)$, and its quantum generalizations, defined in terms of von Neumann algebras. Some very recent applications of such quantum Dirichlet forms will be presented and further directions of research outlined.

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PLAN OF THE LECTURES

Lecture 1 **C_0-semigroups of operators and classical Dirichlet forms**: C_0-semigroups of operators and their generators; quadratic forms; Beurling-Deny conditions; some examples.

Lecture 2 **Quantum Dirichlet forms**: noncommutative L^p-spaces (tracial and non-tracial case); quantum Markov semigroups; noncommutative Beurling-Deny conditions.

Lecture 3 **Recent applications and perspectives**: Haagerup property for von Neumann algebras; quantum convolution semigroups; open problems.

The lectures should be accessible to the audience having a general functional analytic background and some knowledge of operator algebras.

1. Lecture 1

The originating idea of the classical theory of operator semigroups comes from the desire to describe physical evolutions which are in some sense 'time-invariant', in the sense that what happens to the system between time t and $t+s$ depends only on the time distance s (and the state of the system at time t). In probability such behaviour is usually called the Markov property.

Definition 1.1. Let X be a Banach space. A C_0-semigroup of operators is a family $(P_t)_{t\geq 0}$ of bounded linear operators on X such that

(i) $P_0 = \mathrm{id}_X$;
(ii) $P_{t+s} = P_t \circ P_s, \quad s,t \geq 0$;
(iii) $\lim_{t\to 0^+} P_t x = x, \quad x \in X$.

The last property is usually called the *strong* continuity or *point-norm* continuity. Sometimes we need to talk about C_0^*-*semigroups*: if Y is a Banach space then $(P_t)_{t\geq 0}$ is called a C_0^*-semigroup on $X = Y^*$ if it is a family of bounded linear weak*-continuous operators on Y^* such that

$$\lim_{t\to 0^+} (P_t x)(y) = x(y), \qquad x \in X, y \in Y.$$

Definition 1.2. Given a C_0-semigroup of operators $(P_t)_{t\geq 0}$ on X define

$$\mathrm{Dom}(L) := \left\{ x \in X : \lim_{t\to 0^+} \frac{P_t x - x}{t} \text{ exists} \right\}$$

and further $L : \mathrm{Dom}(L) \to X$ by the formula

$$Lx = \lim_{t\to 0^+} \frac{P_t x - x}{t}, \qquad x \in \mathrm{Dom}(L).$$

We have the following fundamental result.

Theorem 1.3. *Let $(P_t)_{t\geq 0}$ be a C_0-semigroup of operators on a Banach space X. The map $L : \mathrm{Dom}(L) \to X$ defined above, called the* generator of the semigroup $(P_t)_{t\geq 0}$ *is a densely defined, closed, linear operator, determining the semigroup uniquely. Further the following conditions are equivalent:*

(i) $\mathrm{Dom}(L) = X$*;*
(ii) L *is bounded;*
(iii) $(P_t)_{t\geq 0}$ *is* norm continuous *(or* uniformly continuous*), i.e.* $\lim_{t\to 0^+} \|P_t - P_0\| = 0$.

In the latter case we have for each $x \in X$ the formula

$$P_t x = \exp(tL)(x) = \sum_{n=0}^{\infty} \frac{(tL)^n x}{n!}.$$

In general the following question is difficult: when is a closed densely defined operator L a generator of a C_0-semigroup?

Theorem 1.4 (Hille-Yoshida)**.** *Let $L : \mathrm{Dom}(L) \to X$ be a linear operator (*$\mathrm{Dom}(L) \subset X$*). The following are equivalent:*

(i) *L is a generator of a C_0-semigroup of contractions (i.e. $\|P_t\| \leq 1, t \geq 0$);*

(ii) *L is closed, densely defined, and for all $\lambda > 0$ we have that the operator $\lambda \mathrm{id}_X - L$ is invertible and*

$$\|\lambda(\lambda \mathrm{id}_X - L)^{-1}\| \le 1$$

(i.e. L satisfies a certain spectral condition).

How much easier things are if X is say a Hilbert space (to be denoted H)? Let $\xi, \eta \in \mathsf{H}$. Then we can ask when do the limits of the form

$$\lim_{t\to 0_+} \left\langle \xi, \frac{\eta - P_t\eta}{t} \right\rangle$$

exist (obviously this is the case for $\eta \in \mathrm{Dom}(L)$). If further all the operators P_t are self-adjoint, then the usual polarisation identity implies that all the information is contained in the densely defined *quadratic* form

$$Q(\xi) := \lim_{t\to 0^+} \left\langle \xi, \frac{\xi - P_t\xi}{t} \right\rangle.$$

Note that then $Q : \mathrm{Dom}(Q) \to \mathbb{R}$.

Theorem 1.5. *Let H be a Hilbert space. There is a 1-1 correspondence between the following three classes of objects:*

(i) *C_0-semigroups $(P_t)_{t\ge 0}$ of self-adjoint contractions on H;*
(ii) *(unbounded) positive self-adjoint operators A on H;*
(iii) *closed, densely defined quadratic forms Q on H.*

Very roughly speaking the correspondences are as follows: $-A$ is the generator of $(P_t)_{t\ge 0}$; we have $P_t = \exp(-tA)$ (in the sense of the functional calculus for self-adjoint operators), and $Q(\cdot) = \|A^{\frac{1}{2}} \cdot \|^2$.

Definition 1.6. Let (Ω, μ) be a space with a (non-negative) measure. A Markov semigroup on $\mathsf{L}^\infty(\Omega, \mu)$ is a C_0^*-semigroup $(P_t)_{t\ge 0}$ on $\mathsf{L}^\infty(\Omega, \mu) = \mathsf{L}^1(\Omega, \mu)^*$ such that

(i) $P_t 1 \le 1$, $P_t f \ge 0$, $f \in \mathsf{L}^\infty(\Omega, \mu)_+$, $t \ge 0$;
(ii) $\int_\Omega f d\mu = \int_\Omega P_t f d\mu$, $f \in \mathsf{L}^\infty(\Omega, \mu)_+$, $t \ge 0$.

Such a semigroup is called symmetric if for all bounded $f, g \in \mathsf{L}^2(\Omega, \mu)$

$$\int_\Omega \bar{f} P_t g d\mu = \int_\Omega \overline{P_t f} g d\mu.$$

It is called conservative if $P_t 1 = 1$, $t \ge 0$.

All such semigroups restrict/extend to C_0-semigroups of (positivity preserving) contractions on each of the $\mathsf{L}^p(\Omega, \mu)$-spaces for $p \in [1, \infty)$.

Example 1.7. Consider the Euclidean space with the Lebesgue measure: $(\mathbb{R}^n, \lambda)$ and define for each $t \ge 0, f \in \mathsf{L}^\infty(\mathbb{R}^n, \lambda)$

$$(P_t f)(s) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-\frac{\|s - r\|^2}{4t}) f(r) dr, \quad s \in \mathbb{R}^n.$$

This defines a Markov semigroup – a so-called *heat semigroup* on $\mathbb{R}^n$. In fact it is a *translation invariant* conservative Markov semigroup, i.e. one of the form

$$P_t f = \mu_t \star f, \quad t \ge 0, f \in \mathsf{L}^\infty(\mathbb{R}^n, \lambda),$$

where μ_t is a probability measure on $\mathbb{R}^n$.

The generator of the corresponding L^2-semigroup is the *Laplace operator*: the closure of the map $-\Delta$, where

$$(\Delta f)(s) = \sum_{i=1}^{n} \frac{\partial^2}{\partial s_i^2} f(s_1, \ldots, s_n)$$

for f in the *Schwarz space* $\mathcal{S}(\mathbb{R}^n) \subset \mathsf{L}^2(\mathbb{R}^n, \lambda)$. The corresponding quadratic form is

$$Qf = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial s_i} \right|^2 ds$$

for $f \in H^1(\mathbb{R}^n) = \{f \in \mathsf{L}^2(\mathbb{R}^n) : \frac{\partial f}{\partial s_i} \in \mathsf{L}^2(\mathbb{R}^n), i = 1, \ldots, n\}$.

Definition 1.8. Let (Ω, μ) be a space with a (non-negative) measure. Denote by $P_\wedge$ the orthogonal projection onto the closed convex set $\{f \in \mathsf{L}^2(\Omega, \mu) : 0 \leq f \leq 1\}$. A densely defined closed quadratic form Q on $\mathsf{L}^2(\Omega, \mu)$ is called Dirichlet if for every $f \in \mathsf{L}^2(\Omega, \mu)_\mathbb{R}$ we have

$$f \in \mathrm{Dom}(Q) \Longrightarrow P_\wedge f \in \mathrm{Dom}(Q) \text{ and } Q(P_\wedge f) \leq Q(f).$$

Theorem 1.9 (Beurling-Deny)**.** *Let (Ω, μ) be a space with a (non-negative) measure. There is a 1-1 correspondence between:*

(i) *symmetric Markov semigroups on $\mathsf{L}^\infty(\Omega, \mu)$;*
(ii) *Dirichlet forms on $\mathsf{L}^2(\Omega, \mu)$,*

If the measure μ is finite, then the Markov semigroup in question is conservative if and only if $Q(1_\Omega) = 0$.

We can chose whether we prefer to work with real or complex $\mathsf{L}^2(\Omega, \mu)$. The closedness condition can be replaced by lower semicontinuity, and with forms defined everywhere, but sometimes taking value $+\infty$.

Let G be a locally compact group. A family of probability measures $(\mu_t)_{t\geq 0}$ on G is called a *convolution semigroup* if we have $\mu_0 = \delta_e$, $\mu_{t+s} = \mu_t \star \mu_s, s, t \geq 0$ and $\int_G f d\mu_t \stackrel{t\to 0^+}{\longrightarrow} f(e)$ for all $f \in C_b(G)$.

Theorem 1.10. *Let G be a locally compact group (with the Haar measure denoted dg). Then there is a 1-1 correspondence between the following classes of objects:*

(i) *translation invariant symmetric conservative Markov semigroups on (G, dg);*
(ii) *translation invariant Dirichlet forms on $\mathsf{L}^2(G, dg)$ (modulo multiplication by a positive number);*
(iii) *convolution semigroups of probability measures on G;*
(iv) *Lévy processes on G, that is G-valued stochastic processes indexed by $\mathbb{R}^+$ with independent, identically distributed increments.*

Note that the maps P_t as above, given by the prescription

$$(P_t f)(s) = \int_G f(r^{-1}s) d\mu_t(r),$$

map continuous bounded functions into continuous bounded functions: this is usually called the *Feller property* and is of big importance in classical probability.

2. Lecture 2

The aim of this lecture is to present some of the earlier ideas in the quantum setting. We will first replace the space (Ω, μ) by the algebra $\mathsf{L}^\infty(\Omega, \mu)$, and then consider general, not necessarily commutative algebras which 'look like' $\mathsf{L}^\infty(\Omega, \mu)$ – von Neumann algebras.

Definition 2.1. A von Neumann algebra M is a weak*-closed unital *-subalgebra of the algebra $B(\mathsf{H})$ for some Hilbert space H (equivalently: a *-subalgebra $\mathsf{M} \subset B(\mathsf{H})$ such that $\mathsf{M} = \mathsf{M}''$ – the algebra is equal to its bicommutant). We say that $\varphi : \mathsf{M}_+ \to [0, \infty]$ is a normal semifinite faithful weight on M, when it is a homogeneous, additive map such that

(i) $\mathfrak{n}_\varphi = \{x \in \mathsf{M} : \varphi(x^*x) < \infty\}$ is weak*-dense in M (semifiniteness);
(ii) when $x_i \nearrow x$, then $\varphi(x) \leq \limsup_{i \in \mathcal{I}} \varphi(x_i)$ (lower semicontinuity/normality)
(iii) $\varphi(x^*x) = 0$ implies $x = 0$ (faithfulness).

We call such a weight a state if $\varphi(1) = 1$. Weights extend to linear functionals on $\mathfrak{m}_\varphi = \operatorname{span}\{x \in \mathsf{M}_+ : \varphi(x) < \infty\}$; so normal faithful states can be viewed as special subclass of usual bounded functionals on M. Finally φ as above is called tracial if for all $x, y \in \mathfrak{m}_\varphi$ we have $\varphi(xy) = \varphi(yx)$.

Example 2.2. Consider the following examples:

(i) $\mathsf{M} = \mathsf{L}^\infty(\Omega, \mu) \subset B(\mathsf{L}^2(\Omega, \mu))$, $\varphi(f) = \int f d\mu$;
(ii) $\mathsf{M} = M_n = B(\mathbb{C}^n)$ (the algebra of n by n complex matrices), $\varphi = \frac{1}{n}\mathrm{Tr}$ (tracial state), or $\varphi(\cdot) = \mathrm{Tr}(D\cdot)$, where D is a *density matrix*: positive-definite matrix of trace 1;
(iii) $\mathsf{M} = B(\ell^2)$, $\varphi(\cdot) = \mathrm{Tr}(D\cdot)$, where D is a density matrix (positive trace class operator of trace 1), which yields a non-tracial state; or $\varphi = \mathrm{Tr}$ – which yields a tracial weight;
(iv) G-discrete group, $\mathsf{H} = \ell^2(G)$. For $g \in G$ let $\lambda_g \in B(\ell^2(G))$ be a (right) shift operator: $\lambda_g(\delta_h) = \delta_{gh}$, $h \in G$. Then define $\mathsf{M} = \mathrm{VN}(G) = \{\lambda_g : g \in G\}'' \subset B(\ell^2(G))$. Then the canonical tracial state on $\mathrm{VN}(G)$ is $\varphi = \omega_{\delta_e}$, i.e. $\varphi(x) = \langle \delta_e, x\delta_e \rangle, x \in \mathrm{VN}(G)$. The construction of $\mathrm{VN}(G)$ generalises to the situation where G is an arbitrary locally compact group, with φ becoming the so-called *Plancherel weight*. If G is abelian, we have $\mathrm{VN}(G) = \mathsf{L}^\infty(\hat{G})$ and the Plancherel weight of G is simply the Haar measure of $\hat{G}$.

Given a map $\Phi : \mathsf{M} \to \mathsf{M}$ and $n \in \mathbb{N}$ we can always define 'entrywise' a map $\Phi^{(n)} : \mathsf{M} \otimes M_n \to \mathsf{M} \otimes M_n$, where $\mathsf{M} \otimes M_n$ is the von Neumann algebra identified as the algebra of n by n matrices with entries in M. A map Φ as above is called *positive* if $\Phi(\mathsf{M}_+) \subset \mathsf{M}_+$, and *completely positive* if each $\Phi^{(n)}$ is positive.

Definition 2.3. Let (M, φ) be as above. A quantum Markov semigroup is a C_0^*-semigroup of normal maps $(P_t)_{t \geq 0}$ on $\mathsf{M} = (\mathsf{M}_*)^*$ such that

(i) $P_t 1 \leq 1$, and each P_t is completely positive ($t \geq 0$);
(ii) $\varphi(f) = \varphi(P_t f)$, $f \in \mathsf{M}_+$, $t \geq 0$.

The symmetry condition becomes in general more complicated! We can associate to a pair (M, φ) non-commutative L^p-spaces, but the way of doing this is non-trivial.

If φ is tracial, the procedure is simpler. We can just consider

$$\mathfrak{m}^{(p)} := \{x \in \mathsf{M} : \varphi(|x|^p) < \infty\}, \quad p \in [1, \infty)$$

and complete it with respect to the norm

$$\|x\|_p = \varphi(|x|^p)^{\frac{1}{p}}.$$

However, when φ is not tracial, this is not a norm!

There are several constructions in the non-tracial case, we will use the one due to Haagerup, based on the Tomita-Takesaki theory, concerning the behaviour of the non-tracial states or weights. We will just list some properties of the resulting Banach spaces:

- $\mathsf{L}^p(\mathsf{M},\varphi)$ are certain spaces of (unbounded) operators on a larger Hilbert space $\mathcal{H}$, closed under taking adjoints and positive parts;
- we have natural isomorphisms $\mathsf{L}^\infty(\mathsf{M},\varphi) \approx \mathsf{M}$, $\mathsf{L}^1(\mathsf{M},\varphi) \approx \mathsf{M}_*$;
- but there are trivial intersections between different spaces, for example $\mathsf{L}^\infty(\mathsf{M},\varphi) \cap \mathsf{L}^2(\mathsf{M},\varphi) = \{0\}$;
- there are different ways of getting from M into $\mathsf{L}^2(\mathsf{M},\varphi)$. Advanced Tomota-Takesaki theory allows us in a sense to write always $\varphi(\cdot) = \mathrm{Tr}(D\cdot)$, where D is a certain 'density-like' operator. Symbolically we may describe the *GNS-embedding* $x \mapsto xD^{\frac{1}{2}}$ and the *KMS-embedding* as $x \mapsto D^{\frac{1}{4}}xD^{\frac{1}{4}}$. We will denote the latter by $\iota^{(2)} : \mathfrak{n}_\varphi \to \mathsf{L}^2(\mathsf{M},\varphi)$.

All that originates from the automorphism group σ_t acting on M, the so-called *modular automorphism group*, ruling the non-traciality of φ:

$$\varphi(xy) = \varphi(y\sigma_i(x)),$$

for 'good' $x, y \in \mathsf{M}$. We have in fact

$$\sigma_t = D^{it}xD^{-it}, \quad x \in \mathsf{M}, t \in \mathbb{R}.$$

Consider the following informal computation:

$$\begin{aligned}\varphi(xy) &= \mathrm{Tr}(Dxy) = \mathrm{Tr}(yDx) = \mathrm{Tr}(y(DxD^{-1})D) = \mathrm{Tr}(Dy(D^{-1}xD))\\ &= \varphi(y(D^{-1}xD)) = \varphi(y\sigma_i(x))\end{aligned}$$

Definition 2.4. A quantum Markov semigroup $(P_t)_{t\geq 0}$ on (M,φ) is said to be KMS-symmetric if for each $t \geq 0$ the prescription

$$P_t^{(2)}(\iota^{(2)}(x)) = \iota^{(2)}(P_t x), \quad x \in \mathfrak{n}_\varphi$$

is well-defined and yields a bounded self-adjoint operator on $\mathsf{L}^2(\mathsf{M},\varphi)$.

Example 2.5. If $(\mathsf{M},\varphi) = (\mathsf{L}^\infty(\Omega,\mu), \int \cdot d\mu)$, then quantum Markov semigroups on (M,φ) are precisely the Markov semigroups on (Ω,μ) discussed in Lecture 1.

Example 2.6. Let G be again a discrete group, $\mathsf{M} = \mathrm{VN}(G)$, φ–canonical trace. Suppose that $\psi : G \to \mathbb{R}$ is a conditionally negative definite symmetric function, i.e. a function such that

(i) $\forall_{g\in G} \quad \psi(g) = \psi(g^{-1})$;
(ii) $\forall_{n\in\mathbb{N}}\forall_{\lambda_1,\ldots,\lambda_n\in\mathbb{C}}\forall_{g_1,\ldots,g_n\in G} \quad \sum_{i=1}^n \lambda_i = 0 \Longrightarrow \sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\psi(g_i^{-1}g_j) \geq 0.$

Then the family of maps $(P_t)_{t\geq 0}$ on $\mathrm{VN}(G)$ given by the formulas

$$P_t(\lambda_g) = \exp(-t\psi)\lambda_g, \quad g \in G, t \geq 0,$$

forms a quantum Markov semigroup of *Herz-Schur multipliers.*

Example 2.7. If $(\mathsf{M},\varphi) = (M_n, \mathrm{tr})$, then every quantum Markov semigroup on (M,φ) ais norm continuous and we can in fact characterise the generators:

$$P_t x = \exp(tL)x, \quad x \in M_n, t \geq 0,$$

with L of the *Lindblad* or *Gorini-Kossakowski-Sudarshan form*:

$$Lx = -i[H,x] + \frac{1}{2}\sum_{\alpha}\left([V_\alpha x, V_\alpha^*] + [V_\alpha, xV_\alpha^*]\right), \quad x \in M_n.$$

Here $H = H^* \in M_n$, $V_\alpha \in M_n$, $\sum_\alpha [V_\alpha, V_\alpha^*] = 0$, and $[A,B] := AB - BA$ denote the *commutators*. There are other variations of this form, for example:

$$Lx = -i[H,x] + E(x) - \frac{1}{2}\{E(1), x\}, \quad x \in M_n,$$

where H is as before, $E : M_n \to M_n$ is completely positive and $\{A,B\} := AB + BA$ denotes the *anticommutator*.

We are ready to discuss the Dirichlet forms in the quantum context.

Definition 2.8. Let (M, φ) be as above. Denote by $P_\wedge$ the orthogonal projection onto the closed convex set $\{f \in \mathsf{L}^2(\mathsf{M}, \varphi) : 0 \leq f \leq D^{\frac{1}{2}}\}$. A densely defined closed quadratic form Q on $\mathsf{L}^2(\mathsf{M}, \varphi)$ is called Dirichlet if for every $f \in \mathsf{L}^2(\mathsf{M}, \varphi)_{\mathbb{R}}$ we have

$$f \in \mathrm{Dom}(Q) \Longrightarrow P_\wedge f \in \mathrm{Dom}(Q) \text{ and } Q(P_\wedge f) \leq Q(f).$$

The form Q as above is called completely Dirichlet if for every n the natural associated quadratic form on $\mathsf{L}^2(\mathsf{M} \otimes M_n, \varphi \otimes \mathrm{tr}_n)$ is Dirichlet.

Theorem 2.9 (Goldstein+Lindsay, Cipriani, AS+ Viselter)**.** *Let* (M, φ) *be as above. There is a 1-1 correspondence between:*

(i) *quantum KMS-symmetric Markov semigroups on* (M, φ)*;*
(ii) *Dirichlet forms on* $\mathsf{L}^2(\mathsf{M}, \varphi)$*.*

If φ *is a state, then the quantum Markov semigroup in question is conservative if and only if* $Q(D^{\frac{1}{2}}) = 0$*.*

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