Schrödinger operator with continuous spectrum and turbulence in integrable systems

Vladimir Zakharov and Dmitry Zakharov
Department of Mathematics, University of Arizona, Tucson;
Courant Institute of Mathematical Sciences, NYU, New York

Introduction

We consider the Schrödinger equation on the real axis

$$-\psi'' + u(x)\psi = E\psi, \quad -\infty < x < \infty,$$

with a bounded potential $u(x)$.

A value of E belongs to the spectrum of $u(x)$ if there exists one or two independent bounded wave functions $\psi(x, E)$:

$$|\psi(x, E)| < 1, \quad -\infty < x < \infty.$$

The spectrum is a subset of the axis $-\infty < E < \infty$, and can have a quite complicated structure. We only consider the case when the spectrum is purely continuous.

A periodic one-gap potential is determined up to translation by the formula

$$u(x) = u_0(x) = 2\wp(x + i\omega' - x_0) + e_3.$$

Here $\wp(x)$ is the elliptic Weierstrass function with periods 2ω and $2i\omega'$. Its spectrum is $[-\kappa_2^2, -\kappa_1^2] \cup [0, \infty)$, where

$$\kappa_2^2 = e_1 - e_3, \quad \kappa_1^2 = e_2 - e_3, \quad e_1 > e_2 > e_3, \quad e_1 + e_2 + e_3 = 0,$$

$$[\wp'(z)]^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

The spectrum is doubly degenerate and reflectionless.

We will show that a general one-gap reflectionless potential is determined by two positive continuous functions $R_1(p)$ and $R_2(p)$, defined inside the allowed gap.

Bargmann potentials via the dressing method

We consider a $\overline{\partial}$-problem on the complex k-plane of the following kind:

$$\frac{\partial \chi}{\partial \overline{k}} = ie^{2ikx} T(k) \chi(x, -k).$$

Here $T(k)$ is a compactly supported distribution called the *dressing function* of the $\overline{\partial}$-problem. A solution of this equation is defined up to multiplication by a function of x, hence if a solution exists we can normalize it by the condition $\chi \to 1$ as $|k| \to \infty$.

Such a solution satisfies the integral equation

$$\chi(x,k) = 1 + \frac{i}{\pi}\iint \frac{e^{-2iqx}T(-q)\chi(x,q)}{k+q} dq d\overline{q},$$

where we normalize the integral in the following way:

$$\frac{1}{k} = \lim_{\varepsilon\to 0} \frac{\overline{k}}{|k|^2+\varepsilon^2}, \quad \frac{\partial}{\partial \overline{k}}\left(\frac{1}{k}\right) = \pi\delta(k).$$

Here $\delta(k)$ is the two-dimensional δ-function.

We now show that a solution of the $\overline{\partial}$-problem gives rise to a solution of the initial Schrödinger equation.

Theorem 1. *Suppose that the dressing function $T(k)$ has the property that the $\overline{\partial}$-problem has a unique solution χ normalized by the condition*

$$\chi(x,k) = 1 + o(1) \ \text{as} \ |k| \to \infty$$

on the set $\mathbb{C} \times \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}$ is an open subset. Denote

$$\chi(x,k) = 1 + \frac{i\chi_0(x)}{k} + O(k^{-2}), \quad u(x) = 2\frac{d}{dx}\chi_0(x).$$

Then the function $\chi(x,k)$ is a solution of the differential equation

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0,$$

and the function $\psi(x,k) = \chi(x,k)e^{-ikx}$ is a solution of the Schrödinger equation with $E = k^2$.

Theorem 2. *Let* $\kappa_1, \ldots, \kappa_N$ *and* $c_1, \ldots, c_n$ *be a collection of nonzero real numbers satisfying the following properties:*

1. $\kappa_m \neq \pm\kappa_n$ *for all* m *and* n.

2. $c_n/\kappa_n > 0$ *for all* n.

Consider the dressing function

$$T(k) = \pi \sum_{n=1}^{N} c_n \delta(k - i\kappa_n).$$

Then the $\overline{\partial}$*-problem has a unique solution* χ *satisfying the normalization condition* $\chi \to 1$ *as* $|k| \to \infty$.

This solution is a rational function of k *having simple poles at the points* $k = i\kappa_n$ *for* $n = 1, \ldots, N$*, and has the following form:*

$$\chi(x, k) = 1 + i \sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n},$$

where the $\chi_n(x)$ *are real-valued functions. The corresponding potential*

$$u(x) = 2\frac{d}{dx} \sum_{n=1}^{N} \chi_n(x)$$

is a reflectionless Bargmann potential having the finite discrete spectrum $-\kappa_1^2, \ldots, -\kappa_N^2$*, and* $\psi_n(x) = \chi_n(x)e^{\kappa_n x}$ *are the corresponding eigenfunctions. Furthermore, for each* n*, changing the signs of both* c_n *and* κ_n *does not change the potential* $u(x)$*.*

The $\overline{\partial}$-problem on the complex k-plane is reduced now to a system of linear equations on the residues $\chi_n(x)$:

$$\chi_n(x) = e^{-2\kappa_n x} c_n \chi(x, -i\kappa_n).$$

Writing this system out explicitly, and replacing $\chi_n(x) = \psi_n(x) e^{-\kappa_n x}$, we obtain the following system:

$$\psi_n(x) + c_n \sum_{m=1}^{N} \frac{e^{-(\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m} \psi_m(x) = c_n e^{-\kappa_n x}$$

The matrix of this system

$$A_{nm} = \delta_{nm} + \frac{c_n e^{-(\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m}$$

is the sum of an identity matrix and a Cauchy-like matrix, therefore its determinant is the sum of the principal minors of the Cauchy-like-matrix. This sum is indexed by subsets $I = \{i_1, \ldots, i_n\}$ of the index set $\{1, \ldots, N\}$ and can be explicitly evaluated as follows:

$$A = \det[A_{nm}] = \sum_{I \subset \{1,\ldots,N\}} \left[\prod_{\{i,j\} \subset I, i<j} \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2} \prod_{i \in I} \frac{c_i}{2\kappa_i} e^{-2\kappa_i x} \right].$$

By assumption, the quantities c_i/κ_i and $(\kappa_i - \kappa_j)^2$ are all positive, therefore each summand and hence all of A is positive, so the linear system has a unique solution. By Theorem 1, χ satisfies equation

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0,$$

and the corresponding potential $u(x)$ is

$$u(x) = 2\frac{d\chi_0}{dx} = 2\frac{d}{dx}\sum_{n=1}^{N} \chi_n(x).$$

To finish the proof, we consider what happens to potential $u(x)$ when we change the signs of one of the κ_n. A direct calculation shows that

$$A = \frac{c_n}{2\kappa_n} e^{-2\kappa_n x} \widetilde{A},$$

where $\widetilde{A}$ is the determinant of the matrix corresponding to the data $(\widetilde{\kappa}_i, \widetilde{c}_i)$, where

$$\widetilde{\kappa}_i = \begin{cases} \kappa_i, & i \neq n, \\ -\kappa_n, & i = n, \end{cases} \qquad \widetilde{c}_i = \begin{cases} \left(\frac{\kappa_i - \kappa_n}{\kappa_i + \kappa_n}\right)^2 c_i, & i \neq n, \\ -4\kappa_n^2 / c_n, & i = n. \end{cases}$$

The symmetric Riemann–Hilbert problem

We consider a Riemann–Hilbert problem that is a continuous analogue of the finite $\overline{\partial}$-problem of Theorem **??** that generates the Bargmann potentials.

Theorem 3. *Let* $0 < \kappa_1 < \kappa_2$ *be real numbers, and let* R_1 *and* R_2 *be two positive continuous functions on the interval* $[\kappa_1, \kappa_2]$*. Consider the dressing function*

$$T(k) = \pi \int_{\kappa_1}^{\kappa_2} R_1(p)\delta(k - ip)dp - \pi \int_{\kappa_1}^{\kappa_2} R_2(p)\delta(k + ip)dp.$$

Then the $\overline{\partial}$*-problem has a unique solution* χ *satisfying the normalization condition* $\chi \to 1$ *as* $|k| \to \infty$*. This function is analytic on the* k*-plane away from two cuts* $[i\kappa_1, i\kappa_2]$ *and* $[-i\kappa_2, -i\kappa_1]$ *on the imaginary axis. Denoting by* χ^+ *and* χ^- *the right and left boundary values of* χ *along the cuts*

$$\chi^{\pm}(x, k) = \lim_{\varepsilon \to 0} \chi(x, k \pm i\varepsilon), \quad k \in [-i\kappa_2, -i\kappa_1] \cup [i\kappa_1, i\kappa_2],$$

the function χ satisfies a symmetric Riemann–Hilbert problem on the cuts:

$$\chi^+(x, ip) - \chi^-(x, ip) = \pi i R_1(p) e^{-2px}[\chi^+(x, -ip) + \chi^-(x, -ip)],$$

$$\chi^+(x, -ip) - \chi^-(x, -ip) = -\pi i R_2(p) e^{2px}[\chi^+(x, ip) + \chi^-(x, ip)].$$

The function χ can be explicitly given as

$$\chi(x, k) = 1 + i \int_{\kappa_1}^{\kappa_2} \frac{f(x,p)}{k - ip} dp + i \int_{\kappa_1}^{\kappa_2} \frac{g(x,p)}{k + ip} dp,$$

where $f(x, p)$ and $g(x, p)$ are real-valued functions defined for real x and for $p \in [\kappa_1, \kappa_2]$. The corresponding potential of the Schrödinger operator is

$$u(x) = 2 \frac{d}{dx} \int_{\kappa_1}^{\kappa_2} [f(x,p) + g(x,p)] dp.$$

Proof. Given R_1 and R_2, we look for a solution of the $\overline{\partial}$-problem in the form

$$\chi(x,k) = 1 + i\int_{\kappa_1}^{\kappa_2} \frac{f(x,p)}{k-ip}dp + i\int_{\kappa_1}^{\kappa_2} \frac{g(x,p)}{k+ip}dp,$$

The jumps of χ along the cuts are

$$\chi^+(x,ip) - \chi^-(x,ip) = 2\pi i f(x,p), \quad \chi^+(x,-ip) - \chi^-(x,-ip) = 2\pi i g(x,p).$$

Plugging into the $\overline{\partial}$-problem, we see that χ satisfies the Riemann–Hilbert problem if f and g satisfy the following system of singular integral equations:

$$f(x,p) + R_1(p)e^{-2px}\left[\int_{\kappa_1}^{\kappa_2} \frac{f(x,q)}{p+q}dq + \fint_{\kappa_1}^{\kappa_2} \frac{g(x,q)}{p-q}dq\right] = R_1(p)e^{-2px}$$

$$g(x,p) + R_2(p)e^{2px}\left[\fint_{\kappa_1}^{\kappa_2} \frac{f(x,q)}{p-q}dq + \int_{\kappa_1}^{\kappa_2} \frac{g(x,q)}{p+q}dq\right] = -R_2(p)e^{2px}.$$

We solve this system by approximation. Fix N, and let $\Delta = (\kappa_2 - \kappa_1)/2N$. We subdivide the segment $[\kappa_1, \kappa_2]$ into $2N$ equal parts and denote

$$\lambda_1 = \kappa_1, \quad \mu_1 = \kappa_1 + \Delta, \quad \lambda_2 = \kappa_1 + 2\Delta, \quad \mu_2 = \kappa_1 + 3\Delta, \ldots$$

$$f_n(x) = f(x, \lambda_n), \quad g_n(x) = g(x, \mu_n), \quad \alpha_n = R_1(\lambda_n), \quad \beta_n = -R_2(\mu_n).$$

We approximate the integrals with their Riemann sums and obtain:

$$f_n(x) + \alpha_n e^{-2\lambda_n x} \left(\sum_{m=1}^{N+1} \frac{f_m(x)}{\lambda_n + \lambda_m} + \sum_{m=1}^{N} \frac{g_m(x)}{\lambda_n - \mu_m} \right) = \alpha_n e^{-2\lambda_n x},$$

$$g_n(x) + \beta_n e^{2\mu_n x} \left(\sum_{m=1}^{N+1} \frac{f_m(x)}{-\mu_n + \lambda_m} + \sum_{m=1}^{N} \frac{g_m(x)}{-\mu_n - \mu_m} \right) = \beta_n e^{2\mu_n x}.$$

We see that this system is equivalent to the system determining the eigenfunctions of a Bargmann potential having $2N+1$ solitons with poles $(\lambda_1, \ldots, \lambda_{N+1}, -\mu_1, \ldots, -\mu_N)$ and residues $(\alpha_1, \ldots, \alpha_{N+1}, \beta_1, \ldots, \beta_N)$.

According to the results of the last paragraph, this system has a unique solution for all x and gives a Bargmann potential with $2N+1$ solitons, which is bounded uniformly in N.

Hence, for sufficiently large N our system of singular integral equations can be approximated by an algebraic system, which has a unique solution. This approximation holds for $-L < x < L$, where

$$2\Delta R e^{\kappa_2 L} << 1, \quad R = \max(R_1(p), R_2(p)).$$

To increase L, we need to exponentially increase N:

$$N \simeq e^{\kappa_2 L}.$$

Nevertheless, for N sufficiently large, it is possible to include any point of the real axis in the interval $(-L, L)$. The resulting potentials are bounded in both directions.

□

Let $0 < \kappa_1 < \kappa_2$, and consider a symmetric pair of segments $[i\kappa_1, i\kappa_2]$ and $[-i\kappa_2, -i\kappa_1]$ on the imaginary axis. Let $R_1(\kappa)$ and $R_2(\kappa)$ be two functions defined on $[\kappa_1, \kappa_2]$, and consider the following dressing function that is the continuous analogue of the dressing function in Theorem 2:

$$T(k) = \pi \int_{\kappa_1}^{\kappa_2} R_1(\kappa)\delta(k - i\kappa)d\kappa + \pi \int_{\kappa_1}^{\kappa_2} R_2(\kappa)\delta(k + i\kappa)d\kappa,$$

where we integrate along the segments $[i\kappa_1, i\kappa_2]$ and $[-i\kappa_2, -i\kappa_1]$. This dressing function is the continuous analogue of the dressing function in Theorem 2.

To increase L we need to exponentially increase N: $N \simeq e^{bL}$. Nevertheless, for N sufficiently large, it is possible to include any point of the real axis in the interval $(-L, L)$. Proceeding as before, we determine that the eigenfunctions

$$\varphi(x, \kappa) = f(x, \kappa)e^{\kappa x}, \quad \psi(x, \kappa) = g(x, \kappa)e^{-\kappa x}$$

are bounded and orthonormal:

$$\int_{-\infty}^{\infty} \varphi(x, \kappa)\varphi(x, \kappa')dx = R_1(\kappa)\delta(\kappa - \kappa'),$$

$$\int_{-\infty}^{\infty} \psi(x, \kappa)\varphi(x, \kappa')dx = 0,$$

$$\int_{-\infty}^{\infty} \psi(x, \kappa)\psi(x, \kappa')dx = R_2(\kappa)\delta(\kappa - \kappa').$$

The functions $f(x,\kappa)$ and $g(x,\kappa)$ grow exponentially as $x \to -\infty$ and $x \to \infty$, respectively. The function $\chi_0(x)$ grows linearly in both diretions:

$$\chi_0 = -c_1 x + \chi_0(x), \text{ as } x \to -\infty, \quad \chi_0 = -c_2 x + \chi_0(x) \text{ as} x \to +\infty,$$

where $|\chi_0(x)| < const$ for $-\infty < x < \infty$.

All these statements are supported by numerical experiments, which will be published elsewhere.

We note that in the κ-region, in which both functions $R_1(\kappa)$ and $R_2(\kappa)$ are strictly positive, the spectrum is doubly degenerate. On the set of points where one of the functions vanishes, the spectrum is simple.

If $R_1(\kappa) = R_2(\kappa)$, then $g(x,\kappa) = -f(-x,\kappa)$, and the potential is symmetric $u(-x) = u(x)$. We note that for $R_1(\kappa) = R_2(\kappa)$ every finite approximation only gives an approximately symmetric potential, however, the accuracy of symmetry grows exponentially as $N \to \infty$.

Periodic one-gap potentials

The periodic one-gap potentials of the Schrödinger operator can be constructed from the symmetric Riemann–Hilbert problem. Let ω and ω' be positive real numbers, and consider the elliptic curve $E = \mathbb{C}/\Lambda$, where Λ is the period lattice generated by 2ω and $2i\omega'$. Denote by $\wp(z)$ the Weierstrass elliptic function associated to the lattice Λ. It satisfies the differential equation

$$[\wp'(z)]^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),$$

where the zeroes e_1, e_2, e_3 are real-valued, satisfy $e_1 + e_2 + e_3 = 0$, and we assume that $e_3 < e_2 < e_1$.

The function

$$u(x) = 2\wp(x - \omega - i\omega') + e_3$$

is a real-valued potential of the Schrödinger operator with period 2ω. Our goal is to construct a solution of Schrödinger equation that gives a solution of the symmetric Riemann–Hilbert problem.

We consider the following function $\varphi(x,z)$, where x is real and z is defined on the curve E:

$$\varphi(x,z) = \frac{\sigma(x-\omega-i\omega'+z)\sigma(\omega+i\omega')}{\sigma(x-\omega-i\omega')\sigma(\omega+i\omega'-z)}\exp[-\zeta(z)x].$$

A direct calculation shows that φ satisfies the Lamé equation

$$\varphi'' - [2\wp(x-\omega-i\omega') + \wp(z)]\varphi = 0.$$

Hence we see that φ is a solution of the Schrödinger equation if the parameter z satisfies the relation

$$k^2 = e_3 - \wp(z).$$

The Weierstrass function $\wp$ has degree two, hence for a generic complex value of k there are values of z on E that satisfy this relation. In order to make the function $\varphi(x,z)$ a well-defined function of k, we need to choose a branch of z.

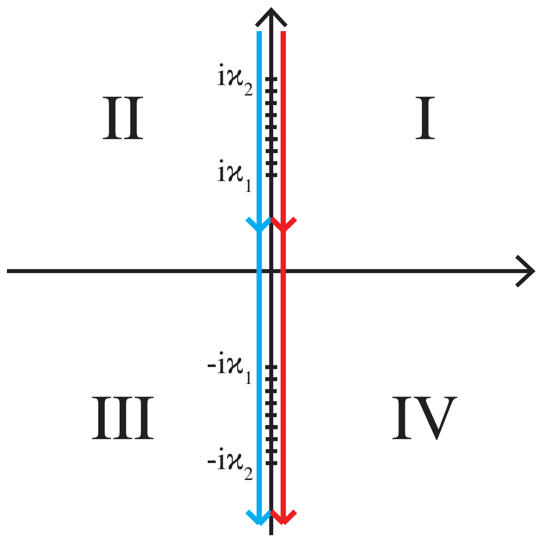
We choose the solution $z(k)$ of this relation that satisfies

$$z(k) = \frac{i}{k} + O\left(\frac{1}{k^2}\right) \text{ as } |k| \to \infty.$$

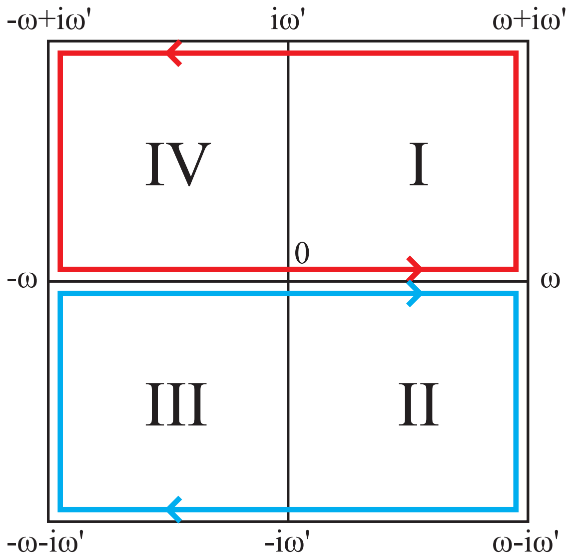
This branch defines a single-sheeted map from the complex k-plane with two cuts on the imaginary axis to a period rectangle of the lattice Λ centered at 0. The cuts on the imaginary axis are $[-i\kappa_2, -i\kappa_1]$ and $[i\kappa_1, i\kappa_2]$, where

$$\kappa_1 = \sqrt{e_2 - e_3}, \quad \kappa_2 = \sqrt{e_1 - e_3}.$$

The right and left sides of the top cut $[i\kappa_1, i\kappa_2]$ are mapped to the line segments joining ω to $\omega + i\omega'$ and $\omega - i\omega'$, respectively, and the right and left sides of the bottom cut $[-i\kappa_2, -i\kappa_1]$ are respectively mapped to the segments joining $-\omega'$ to $-\omega + i\omega'$ and $-\omega - i\omega'$.



The k-plane



The z-plane

The function φ satisfies the following properties:

$$\varphi(x, z + 2\omega) = \varphi(x, z), \quad \varphi(x, z + 2i\omega') = \varphi(x, z),$$

$$\overline{\varphi}(x, z) = \varphi(x, z) \text{ when } \overline{z} = z, \overline{x} = x.$$

Also (very important)

$$\overline{\varphi}(x,z) = \varphi(x,\overline{z})$$

for all z having real part ω.

Theorem 4. *Let $f(k)$ be the branch of the function*

$$f(k) = \sqrt{\frac{k+i\kappa_1}{k+i\kappa_2}}$$

satisfying $f(k) \to 1$ as $|k| \to \infty$. On the complex k-plane with two cuts $[i\kappa_1, i\kappa_2]$ and $[-i\kappa_2, -i\kappa_1]$ along the imaginary axis, define the function

$$\xi(x,k) = f(k)\varphi(x,z(k))e^{-ikx}.$$

Then the function $\xi(x,k)$ satisfies the equation

$$\xi'' + 2ik\xi' - u(x)\xi = 0, \quad \xi \to 1 \text{ as } |k| \to \infty$$

with potential $u(x)$ *given by the standard formulae. On the cuts, the function* ξ *satisfies the Riemann–Hilbert problem*

$$\xi^+(x,iq) - \xi^-(x,iq) = i\pi R_1(q)e^{2qx}\left[\xi^+(x,-iq) + \xi^-(x,-iq)\right],$$

$$\xi^+(x,-iq) - \xi^-(x,-iq) = -i\pi R_2(q)e^{-2qx}\left[\xi^+(x,iq) + \xi^-(x,iq)\right].$$

Here $q \in [\kappa_1, \kappa_2]$, *and* $\xi^+(x,\pm iq)$ *are the right hand values of the upper and lower cuts, and* $\xi^-(x,\pm iq)$ *are the left hand values on the upper and lower cuts. The functions* R_1 *and* R_2 *are*

$$R_1(q) = \frac{1}{\pi}h(q), \quad R_2(q) = \frac{1}{\pi h(q)}, \quad h(q) = \sqrt{\frac{(q-\kappa_1)(q+\kappa_2)}{(\kappa_2-q)(q+\kappa_1)}}$$

References

[1] V. Marchenko, Nonlinear equations and operator algebras, *Mathematics and its Applications (Soviet Series)*, 17, D. Reidel Publishing Co., Dordrecht, 1988

[2] D. Lundina, Compactness of sets of reflectionless potentials, *J. Soviet Math.* 48 (1990), no. 3, 290-297.

[3] V. Marchenko, The Cauchy problem for the KdV equation with non-decreasing initial data, in *What is integrability?*, Edited by V. E. Zakharov. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.

[4] V. Zakharov, S. Manakov, Construction of multidimensional nonlinear integrable systems and their solutions, *Funktsional. Anal. i Prilozhen.* (in Russian) 19 (1985), no. 2, 11-25, 96.

[5] S. Novikov, S. Manakov, L. Pitaevskii, V. Zakharov, Theory of solitons. The inverse scattering method. Translated from the Russian. Contemporary Soviet Mathematics. Consultants Bureau [Plenum], New York, 1984.

[6] V. Zakharov, Turbulence in integrable systems, *Stud. Appl. Math.* 122 (2009), no. 3, 219-234.

[7] D. Zakharov, V. Zakharov and S. Dyachenko, Bounded solutions of KdV and non-periodic one-gap potentials in quantum mechanics, Stud. Appl. Math., 2016

[8] S. Dyachenko, D. Zakharov, V. Zakharov, Primitive potentials and bounded solutions of the KdV equation, Physica D, 2016

[9] D. Zakharov, V. Zakharov and S. Dyachenko, Primitive potentials and turbulence in integrable systems, submitted to Phys. Letters A.