Banach Poisson-Lie groups and the restricted Grassmannian

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Example of bounded operator with unbounded triangular truncation [Davidson, Nest Algebras]

$$
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \frac{1}{n} & \ddots \\
\ddots & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \ddots \\
\ddots & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \ddots \\
\ddots & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ddots \\
\ddots & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \ddots \\
\ddots & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \ddots \\
\ddots & -\frac{1}{n} & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
$$

- Does there exists a trace class operator whose triangular truncation is not trace class? Yes, ask D. Beltiță!
- Exercise: Find a concrete example!

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

$\Gamma_+ = \{g = e^f, f \text{ holomorphic in } \mathbb{D}, f(0) = 0\}$

$\mathrm{Gr}^{(n)} = \{w \in \mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}) \ : z^n W \subset W\}.$

$\Gamma^+_W = \{g \in \Gamma_+ \ : g^{-1} W \cap \mathcal{H}_- = \{0\}\}.$

Proposition 5.1 in [SW85] :
$\forall W \in \mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H})$, $\exists! \Phi_W(g,z)$ called the Baker function of W, defined for $g \in \Gamma^+_W$ and $z \in \mathbb{S}^1$, such that

(i) $\Phi_W(g,\cdot) \in W$ for each fixed $g \in \Gamma^+_W$

(ii) $\Phi_W(g,z) = g(z)(1 + \sum_1^\infty a_i(g) z^{-i})$, a_i are analytic functions on Γ^+_W and extend to meromorphic functions on the whole of Γ^+.

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

Proposition 5.5 in [SW85] :
Set $D = \frac{\partial}{\partial x}$. For each integer $r \geq 2$, there is a unique differential operator P_r of the form $P_r = D^r + p_{r2}D^{r-2} + \cdots + p_{r,r-1}D + p_{rr}$ such that $\frac{\partial \Phi_W}{\partial t_r} = P_r \Phi_W$.

Denote by $\mathcal{C}^{(n)}$ the space of all operators P_n associated to subspaces W in $\mathrm{Gr}^{(n)}$ evaluated at $t_2 = t_3 = \cdots = 0$.

Proposition 5.13 in [SW85] The action of Γ_+ on $\mathrm{Gr}^{(n)}$ induces an action on the space $\mathcal{C}^{(n)}$. For $r \geq 1$, the flow $W \mapsto \exp(t_r z^r)W$ on $\mathrm{Gr}^{(n)}$ induces the r-th KdV flow on $\mathcal{C}^{(n)}$.

Key Observation : $\Gamma_+ \subset \mathrm{B}^+_{\mathbf{res}}(\mathcal{H})$.

Bruhat-Poisson structure of finite-dimensional Grassmannians [Lu-Weinstein, 1990]

Proposition :

- $\mathrm{SU}(n)$ and $\mathrm{SB}(n, \mathbb{C})$ are dual Poisson-Lie groups
- the Grassmannians $\mathrm{Gr}(p, n) = \mathrm{SU}(n)/\,\mathrm{S}(U(p) \times U(n))$ are Poisson homogeneous spaces
- $\mathrm{SB}(n, \mathbb{C})$ acts on $\mathrm{Gr}(p, n)$ by Dressing transformations
- the symplectic leaves of $\mathrm{Gr}(p, n)$ are the Schubert-Bruhat cells and coincides with the orbits under the action of $\mathrm{SB}(n, \mathbb{C})$

cf I. Marshall's talk!

Hilbert-Schmidt result

$$\begin{array}{ll} \mathsf{GL}_2(\mathcal{H}) = & \mathsf{GL}(\mathcal{H}) \cap \{\mathrm{Id} + A, A \text{ Hilbert-Schmidt}\} \\ \mathsf{U}_2(\mathcal{H}) = & \mathsf{U}(\mathcal{H}) \cap \{\mathrm{Id} + A, A \text{ Hilbert-Schmidt}\} \\ \mathsf{B}_2^+(\mathcal{H}) = & \{\alpha, \alpha^{-1} \text{ upper triangular of the form Id+ Hilbert-Schmidt} \\ & \text{with stricktly positive coeff. on diagonal}\}. \end{array}$$

$$\begin{array}{rcccl} \langle\cdot,\cdot\rangle & : & \mathfrak{u}_2(\mathcal{H}) \times \mathfrak{b}_2^+(\mathcal{H}) & \to & \mathbb{R} \\ & & (u, b) & \mapsto & \mathrm{Im}\ \mathrm{Tr}(ub) \end{array}$$ is a strong duality pairing

Theorem :

(1) $\mathsf{U}_2(\mathcal{H})$ and $\mathsf{B}_2^+(\mathcal{H})$ are dual Hilbert Poisson-Lie groups with $\pi_g^u := R_g^* \pi_r^u(g)$, $\pi_g^b := R_g^* \pi_r^b(g)$ and
$\pi_r^u : \mathsf{U}_2(\mathcal{H}) \to \Lambda^2 \mathfrak{b}_2^+(\mathcal{H})^*(\mathfrak{b}_2^+(\mathcal{H}))$ defined by
$$\pi_r^u(u)(b_1, b_2) = \mathrm{Im}\ \mathrm{Tr}\ p_{\mathfrak{u}_2}(u^{-1} b_1 u) \left[p_{\mathfrak{b}_2^+}(u^{-1} b_2 u) \right],$$
$\pi_r^b : \mathsf{B}_2^\pm(\mathcal{H}) \to \Lambda^2 \mathfrak{u}_2(\mathcal{H})^*(\mathfrak{u}_2(\mathcal{H}))$ defined by
$$\pi_r^b(g)(x_1, x_2) = \mathrm{Im}\ \mathrm{Tr}\ p_{\mathfrak{b}_2^+}(g^{-1} x_1 g) \left[p_{\mathfrak{u}_2}(g^{-1} x_2 g) \right].$$

(2) $\mathsf{Gr}^0_{\mathrm{res}}(\mathcal{H})$ is a $\mathsf{U}_2(\mathcal{H})$-Poisson homogeneous space

Consequence : Using Iwasawa decomposition $GL_2(\mathcal{H}) = \mathsf{U}_2(\mathcal{H})\, \mathsf{B}_2^+(\mathcal{H})$, [D. Beltiță, 2006], one can solve associated Lax equations.

Poisson manifold modelled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathcal{C}^1$ away from the origin)

(2) Leibniz rule does not imply existence of Poisson tensor (there exists derivation of order greater then 1)

(3) existence of Hamiltonian vector field is not automatic

Definition of a Banach Poisson manifold

Definition of a Poisson tensor :

M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha,\beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition of a Poisson Manifold :

A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Definition of Banach Poisson-Lie groups

Definition : A Banach Poisson-Lie group B is a Banach Lie group equipped with a Banach Poisson manifold structure such that the group multiplication $m : B \times B \to B$ is a Poisson map, where $B \times B$ is endowed with the product Poisson structure.

Proposition : Let B be a Banach Lie group and $(B, \mathbb{U}, \pi)$ a Banach Poisson structure on B. Then B is a Banach Poisson-Lie group if and only if

1. $\mathbb{U}$ is invariant under left and right multiplications by elements in B,
2. the subspace $\mathfrak{u} := \mathbb{U}_e \subset \mathfrak{b}^*$, where e is the unit element of B, is invariant under the coadjoint action of B on $\mathfrak{b}^*$ and the map

$$\begin{array}{rccl} \pi_r & : & B & \to & \Lambda^2\mathfrak{u}^*(\mathfrak{u}) \\ & & g & \mapsto & R^*_{g^{-1}}\pi_g, \end{array}$$

is a 1-cocycle on B with respect to the coadjoint representation of B in $\Lambda^2\mathfrak{u}^*(\mathfrak{u})$.

Banach Lie bialgebras

Definition : Let $\mathfrak{b}$ be a Banach Lie algebra, and a duality pairing $\langle\cdot,\cdot\rangle_{\mathfrak{b},\mathfrak{u}}$ between $\mathfrak{b}$ and a normed vector space $\mathfrak{u}$. One says that $\mathfrak{b}$ is a Banach Lie bialgebra with respect to $\mathfrak{u}$ if

(1) $\mathfrak{b}$ acts continuously by coadjoint action on $\mathfrak{u}$.

(2) there is a 1-cocycle $\theta\ :\mathfrak{b}\to\Lambda^2\mathfrak{u}^*(\mathfrak{u})$ with respect to the adjoint representation of $\mathfrak{b}$ on $\Lambda^2\mathfrak{u}^*(\mathfrak{u})$, i.e. satisfying

$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = {} & \theta(y)(\mathrm{ad}^*_x\alpha,\beta)+\theta(y)(\alpha,\mathrm{ad}^*_x\beta)\\ & -\theta(x)(\mathrm{ad}^*_y\alpha,\beta)-\theta(x)(\alpha,\mathrm{ad}^*_y\beta)\end{aligned}$$

where $x,y\in\mathfrak{b}$ and $\alpha,\beta\in\mathfrak{u}$.

Banach Lie bialgebras versus Manin triple

Definition : [A. A. Odzijewicz, T. Ratiu, 2003]
We will say that $\mathfrak{b}$ is a Banach Lie-Poisson space with respect to $\mathfrak{u}$ if $\mathfrak{u}$ is in duality with $\mathfrak{b}$ and is a Banach Lie algebra $(\mathfrak{u}, [\cdot,\cdot]_{\mathfrak{u}})$ which acts continuously on $\mathfrak{b}$ by coadjoint action.

Theorem :
Consider two Banach Lie algebras $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ and $(\mathfrak{u}, [\cdot,\cdot]_{\mathfrak{u}})$ in duality. Denote by $\mathfrak{g}$ the Banach space $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$ with norm $\|\cdot\|_{\mathfrak{g}} = \|\cdot\|_{\mathfrak{b}} + \|\cdot\|_{\mathfrak{u}}$. The following assertions are equivalent.

(1) $\mathfrak{b}$ is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to $\mathfrak{u}$;

(2) $(\mathfrak{g}, \mathfrak{b}, \mathfrak{u})$ is a Manin triple for the natural non-degenerate symmetric bilinear map

$$\begin{array}{rccl} \langle\cdot,\cdot\rangle_{\mathfrak{g}} : & \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{K} \\ & (x,\alpha)\times(y,\beta) & \mapsto & \langle x,\beta\rangle_{\mathfrak{b},\mathfrak{u}} + \langle y,\alpha\rangle_{\mathfrak{b},\mathfrak{u}}. \end{array}$$

Duality pairing between $\mathfrak{b}_{\mathrm{res}}(\mathcal{H})$ and $\mathfrak{u}_{1,2}(\mathcal{H})$

$L_{\mathrm{res}}(\mathcal{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), B \text{ and } C \text{ Hilbert-Schmidt}\}$
$L_{1,2}(\mathcal{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A \text{ and } C \text{ Trace class}, B \text{ and } C \text{ Hilbert-Schmidt}\}$

Definition : [T. Goliński, A. Odzijewicz, 2010]
For $A = \begin{pmatrix} A_{++} & A_{-+} \\ A_{-+} & A_{--} \end{pmatrix} \in L_{1,2}(\mathcal{H})$ define the restricted trace of A by
$\mathrm{Tr}_{\mathrm{res}} A = \mathrm{Tr} A_{++} + \mathrm{Tr} A_{--}$.

Proposition 2.1 in [GO10] : $\forall A \in L_{1,2}(\mathcal{H})$, $\forall B \in L_{\mathrm{res}}(\mathcal{H})$, $AB \in L_{1,2}(\mathcal{H})$, $BA \in L_{1,2}(\mathcal{H})$ and $\mathrm{Tr}_{\mathrm{res}} AB = \mathrm{Tr}_{\mathrm{res}} BA$.

Proposition : The following continuous bilinear map is a duality pairing between $\mathfrak{b}^+_{\mathrm{res}}(\mathcal{H})$ and $\mathfrak{u}_{1,2}(\mathcal{H})$

$$\begin{array}{rccl} \langle \cdot, \cdot \rangle_{L_{\mathrm{res}}, L_{1,2}} : & L_{\mathrm{res}}(\mathcal{H}) \times L_{1,2}(\mathcal{H}) & \longrightarrow & \mathbb{R} \\ & (A, B) & \longmapsto & \mathrm{Im}\ \mathrm{Tr}_{\mathrm{res}}\,(AB)\,. \end{array}$$

Consequence of slide 1 : $\mathfrak{u}_{1,2}(\mathcal{H})$ is not preserved by the coadjoint action of $\mathfrak{b}^+_{\mathrm{res}}$. No Manin triple structure on $\mathfrak{b}^+_{\mathrm{res}}(\mathcal{H}) \oplus \mathfrak{u}_{1,2}(\mathcal{H})$!
However there exists a Banach Poisson-Lie group structure on $\mathrm{B}_{\mathrm{res}}(\mathcal{H})$...