Integrable Probability and the Role of Painlevé Functions

XXXV Workshop on Geometric Methods in Physics

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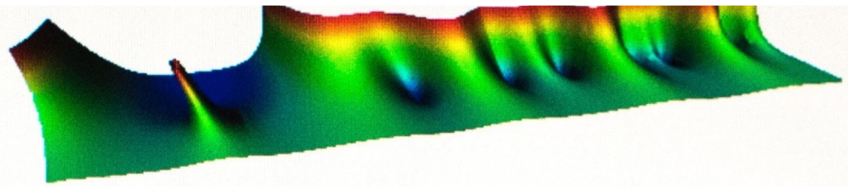
Paul Painlevé (1863-1933)

What are Painlevé Functions?



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 _DLMF Update; Version 1.0.9_
2014-04-25 _DLMF Update; Version 1.0.8; errata & improved MathML_
2014-03-21 _DLMF Update; Version 1.0.7; **New Features improve Math & 3D Graphics**_
2013-08-16 _Bille C. Carlson, DLMF Author, dies at age 89_
More news

The story I want to tell
is how Painlevé functions intersect with probability theory (in the form of limit theorems) and how these theoretical predictions have been experimentally confirmed in the laboratory.

The experiments involve stochastically growing interfaces. Physicists call all this KPZ Universality.

Let's see the experimental results first.

Work of K.Takeuchi & M.Sano
in 2010

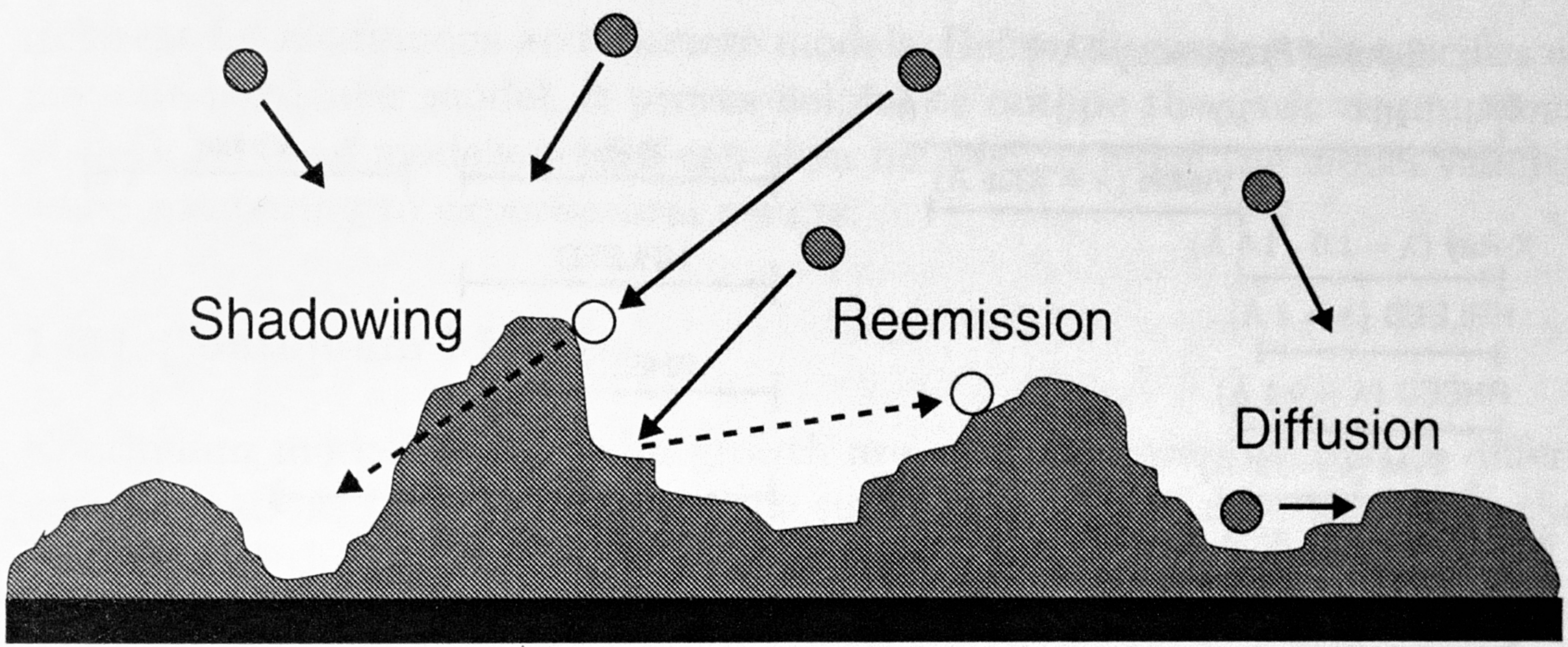


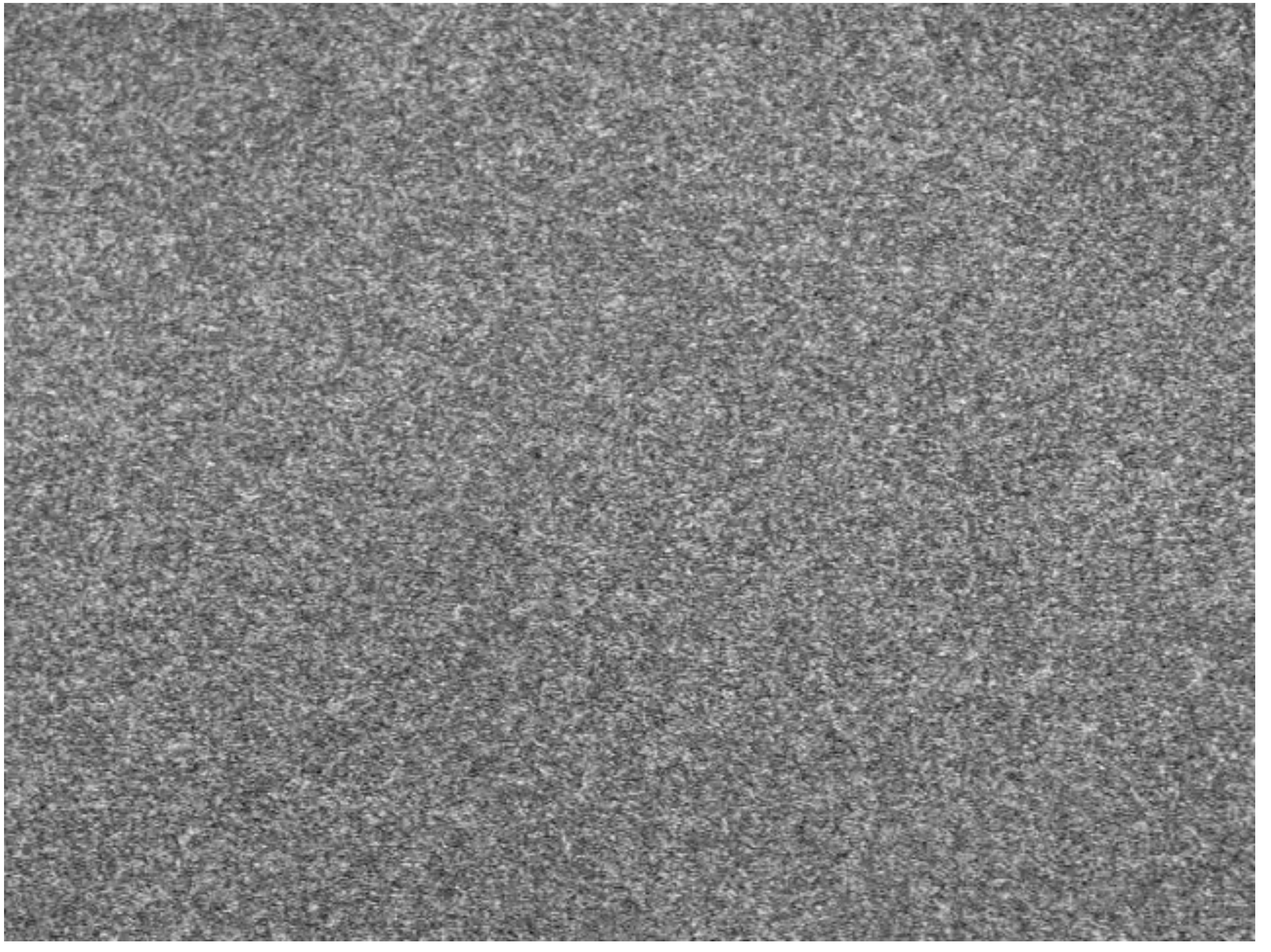
Fig. 1.3. Diagram of growth effects including diffusion, shadowing, and reemission that may affect surface morphology during thin film growth. The incident particle flux may arrive at the surface with a wide angular distribution depending on the deposition methods and parameters.

KPZ Phenomenology

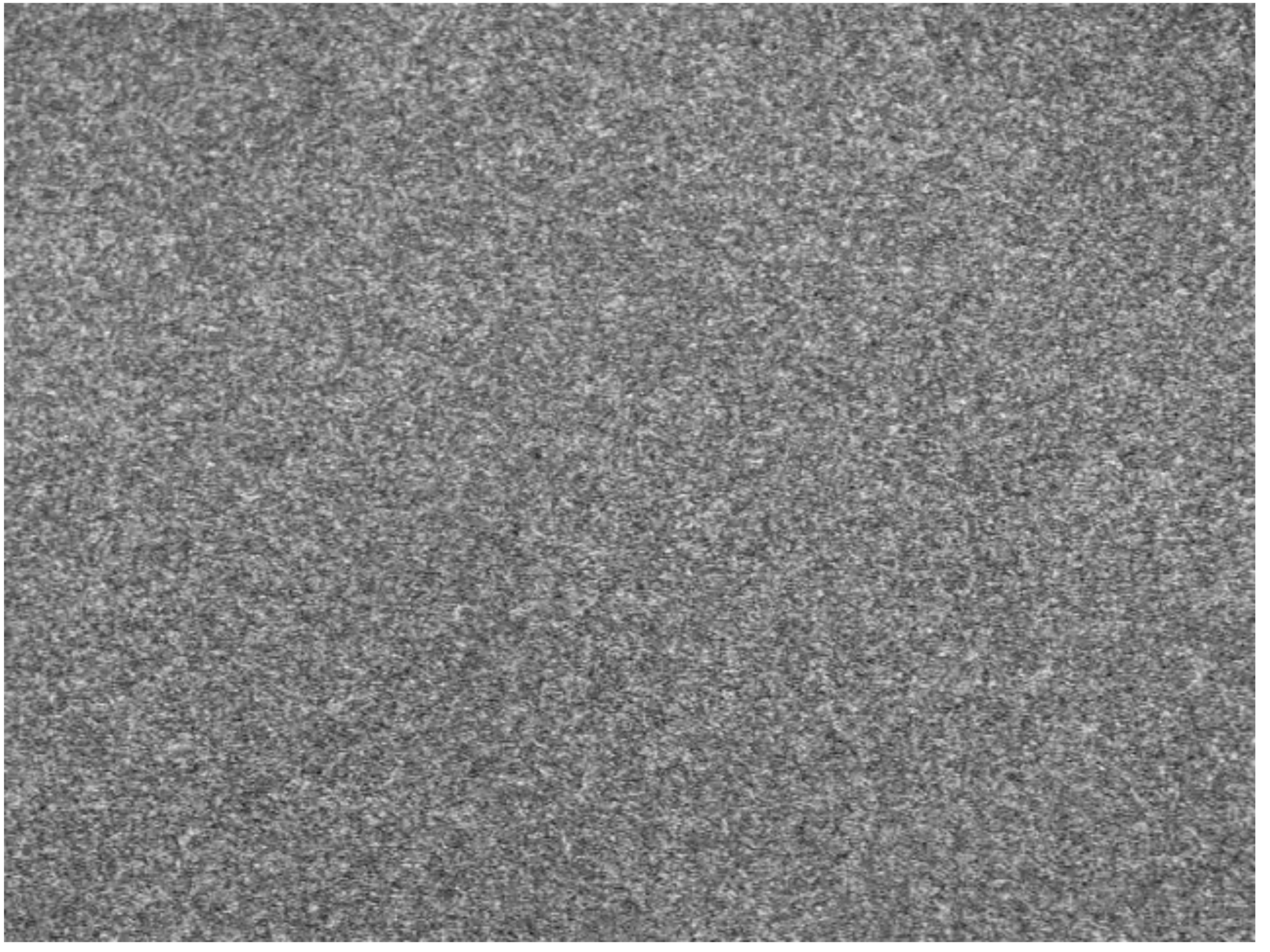
- Stochastic growth normal to the surface
- Kardar-Parisi-Zhang (1986)
- Basic object: (random) height function h(x,t)
- Satisfies the KPZ equation (nonlinear stochastic PDE):

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \lambda \left(\frac{\partial h}{\partial x} \right)^2 + \sqrt{D}\, \eta(x,t)$$

$$h \sim v_\infty t + (\Gamma t)^{1/3} \chi, t \to \infty$$



Stochastic growth in liquid crystals: Droplet initial condition



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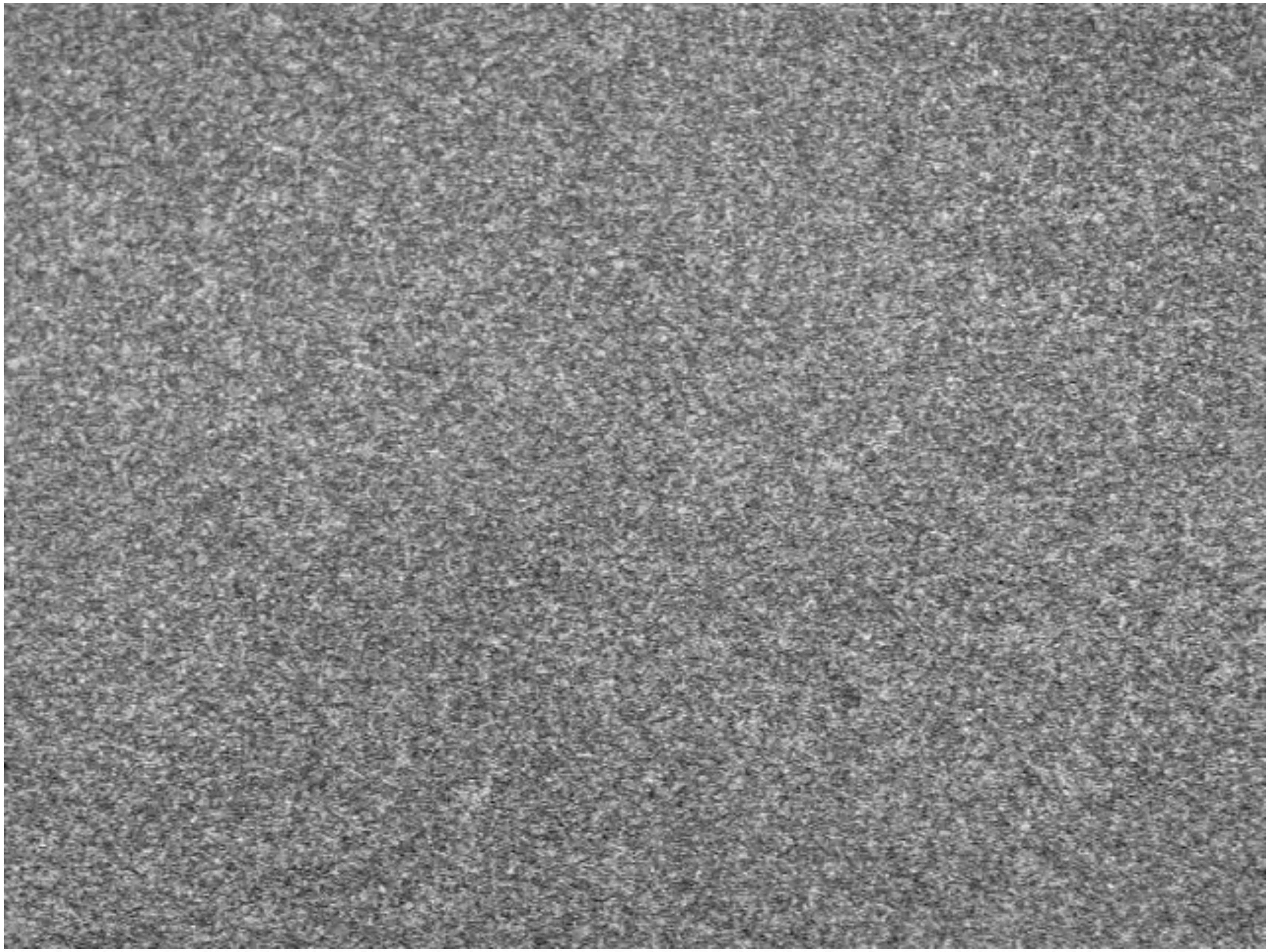
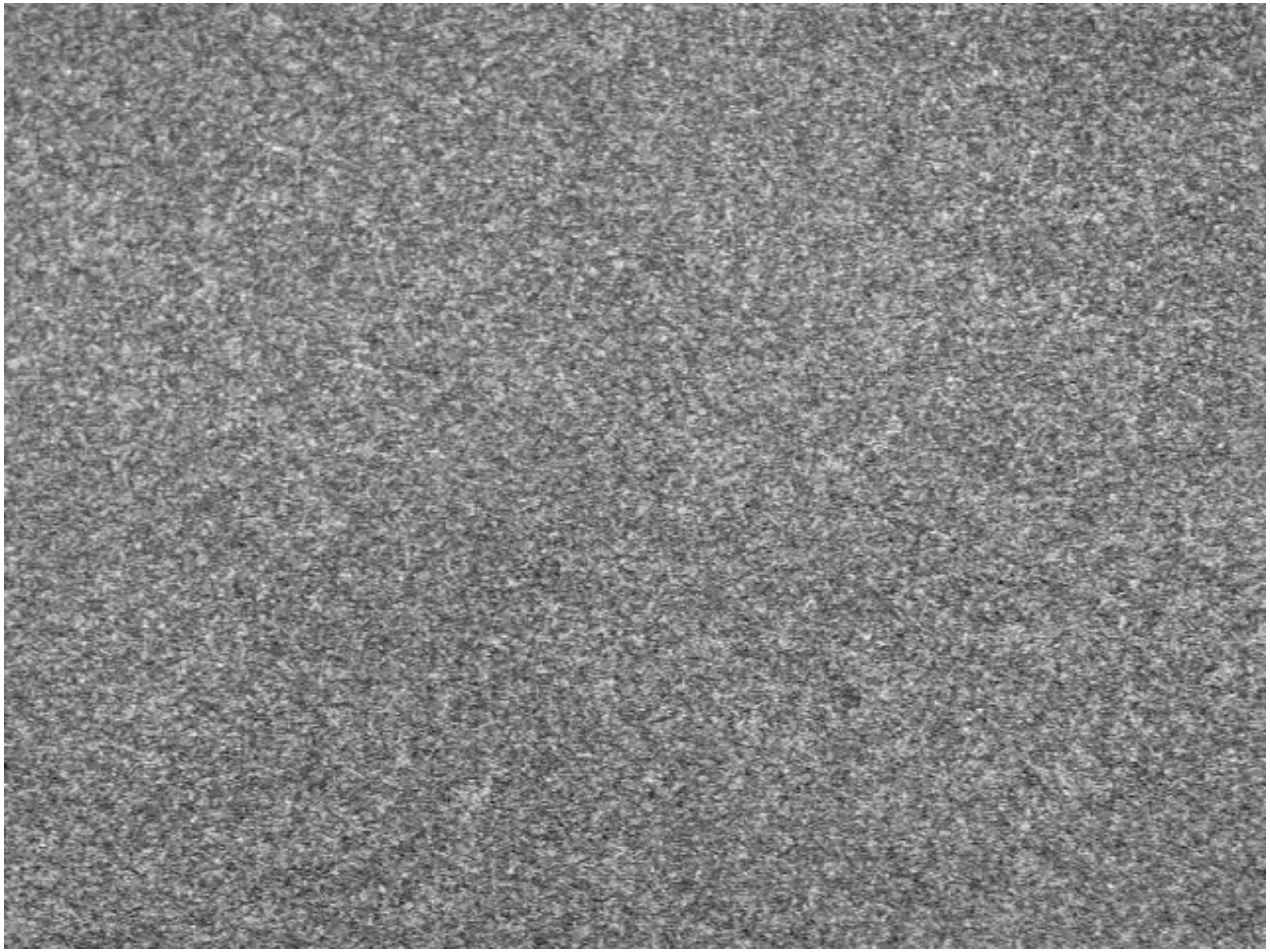


Stochastic growth in liquid crystals: Flat initial condition



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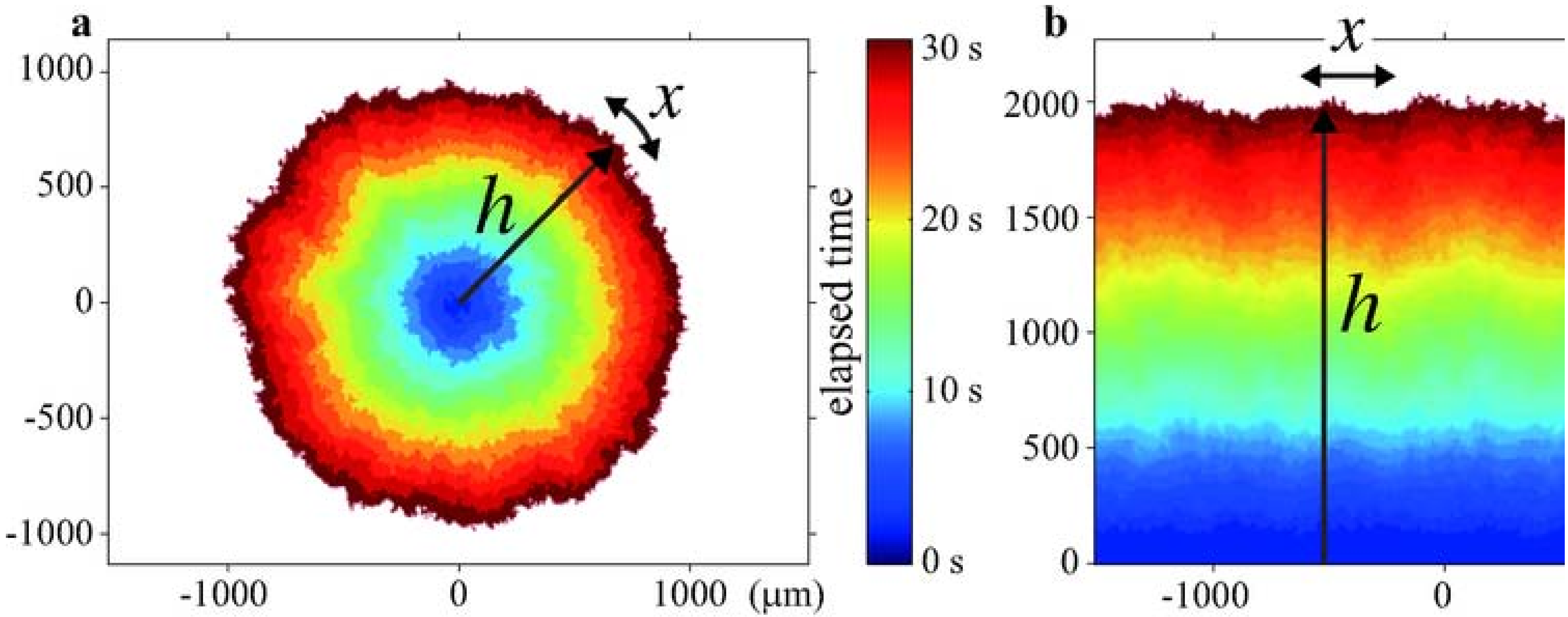


Binarised snapshots at successive times are shown with different colours. Indicated in the colour bar is the elapsed time after the laser emission. The local height $h(x, t)$ is defined in each case as a function of the lateral coordinate x along the mean profile of the interface (a circle for a and a horizontal line for b). See also **Supplementary Movies 1 and 2**.

Height function h(x,t)

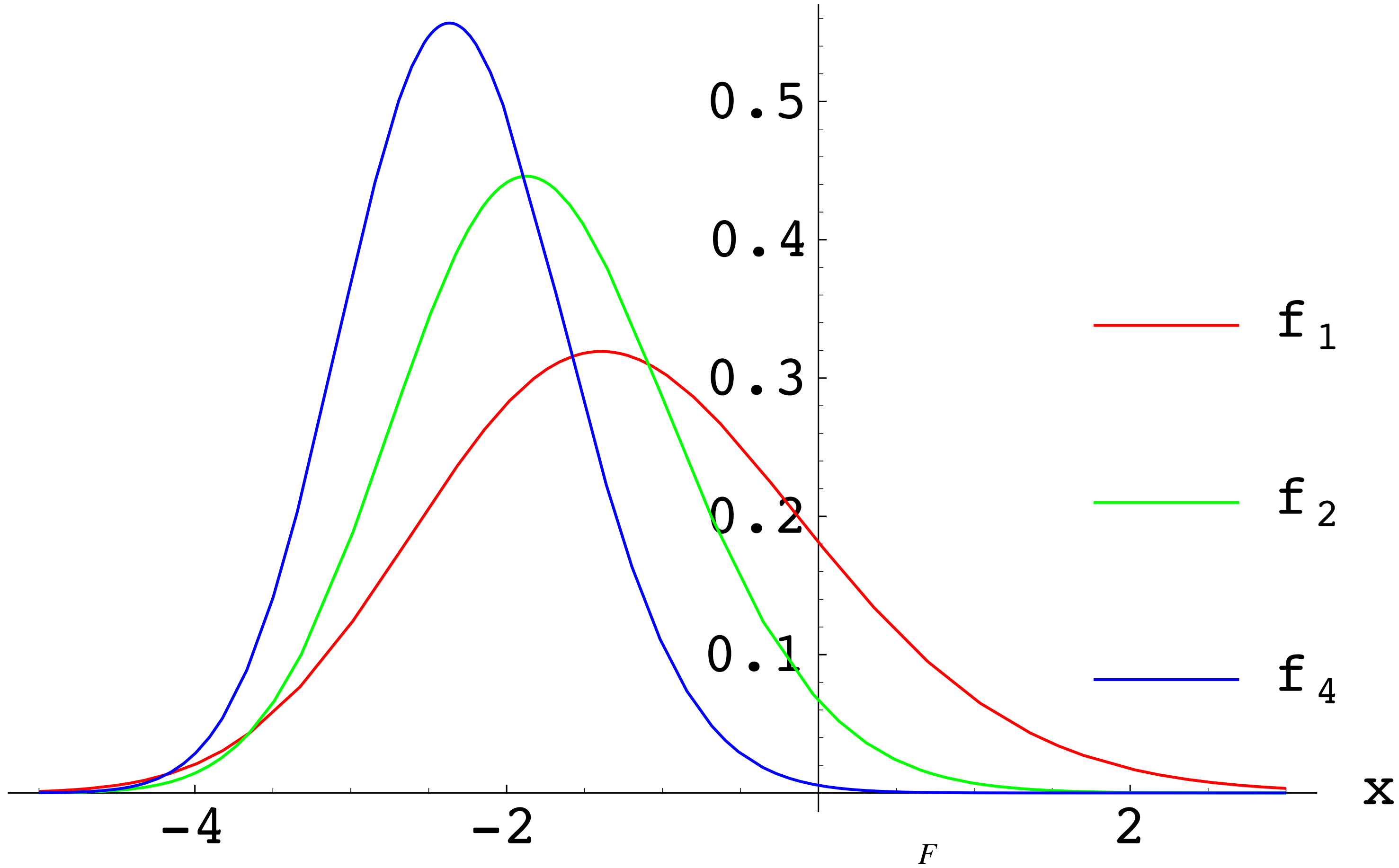
Distribution Functions F_1 and F_2

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right)$$

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty q(x)\,dx\right) F_2(s)^{1/2}$$

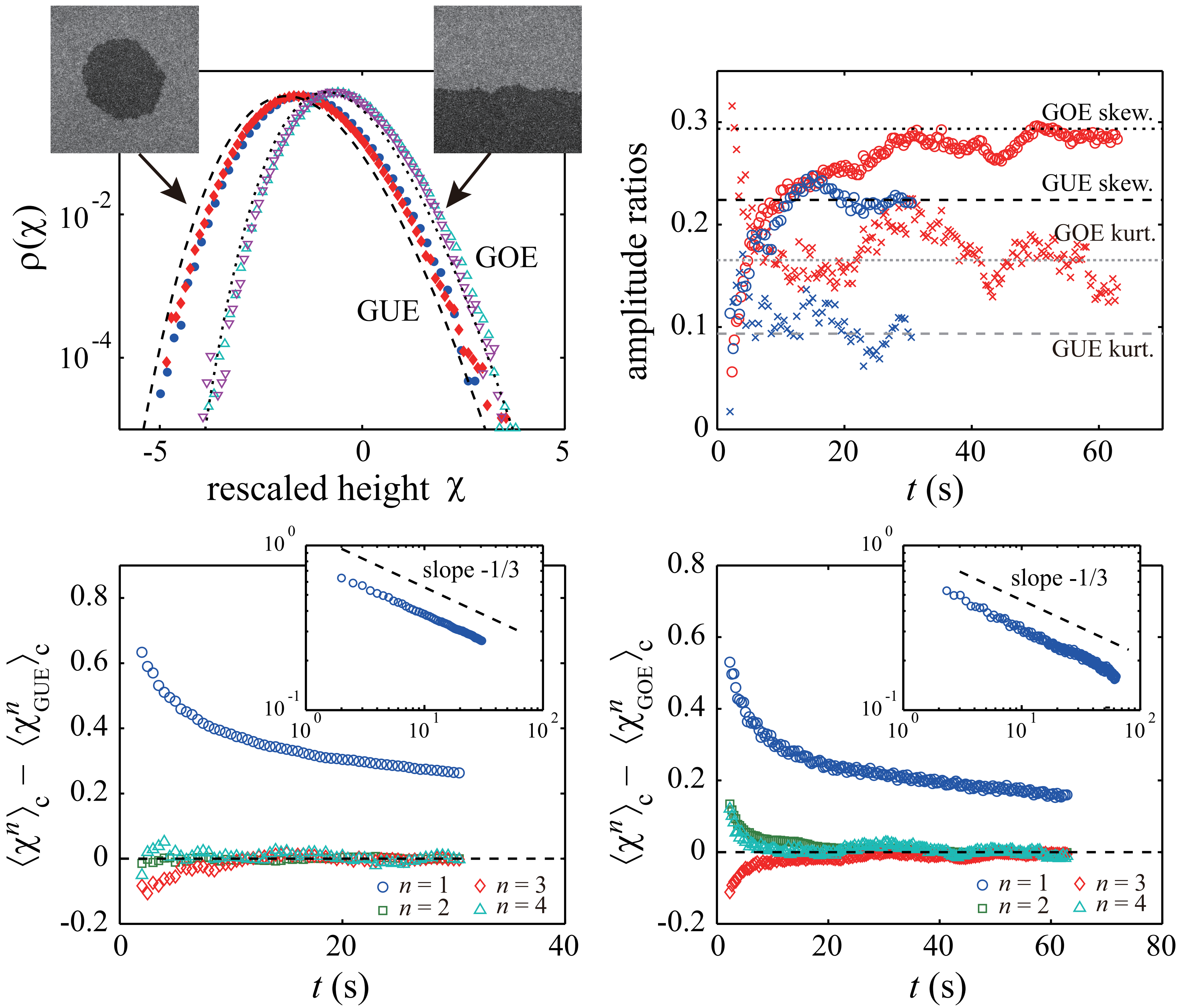
$$\frac{d^2q}{dx^2} = xq + 2q^3 \quad q(x) \sim \mathrm{Ai}(x),\ \ x \to \infty$$

Painlevé II, Hastings-McCleod



$$f_\beta(x) = \frac{dF_\beta(x)}{dx}, \;\; \beta = 1, 2, 4$$

Distribution	Skewness	Kurtosis
F_1	0.293...	0.165...
F_2	0.224...	0.093...
F_4	0.165...	0.049...



K. Takeuchi & M. Sano, "Evidence for geometry-dependent universal fluctuations of the Kardar-Parisi-Zhang interfaces in liquid-crystal turbulence", Journal of Statistical Physics 147 (2012), 853-890. arXiv:1203.2530. (Earlier Phys. Rev. Lett.)

The distributions F_1 and F_2 first arose as the limiting distribution (size of the matrices->infinity) of the largest eigenvalue in the the Gaussian Orthogonal Ensemble (GOE, F_1) and the Gaussian Unitary Ensemble (GUE, F_2). Harold Widom & CT (1992-96).

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Since then it has been shown that these are the limiting distributions for the largest eigenvalue for a broad class of random matrices (Soshnikov, Its & Bleher, Deift et al., Tao & Vu, H.-T. Yau et al., ...)

- Question 1: Why Painlevé functions?
- Question 2: What does all this have to do with growth processes?

Partial Answer to #1

- For random matrix models with invariant measures, many distribution functions can be expressed as Fredholm determinants (Gaudin, Mehta 1960s): Det(I-K)

- For unitary ensembles, the kernel of the operator K has an "integrable structure"

$$K(x,y) = \frac{\varphi(x)\psi(y) - \varphi(y)\psi(x)}{x-y}$$

$$\frac{d}{dx}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}(x) = \Omega(x)\begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix}$$

$$\Omega : \text{rational entries, trace zero}$$

$\bullet F_2 = \det(I - K), \;\; \varphi(x) = \mathrm{Ai}(x), \psi(x) = \mathrm{Ai}'(x)$

K acts on $L^2(s, \infty)$

$\bullet$In general, K acts on $L^2(J)$, $J = (a_1, a_2) \cup \cdots \cup (a_{2n-1}, a_{2n})$

$\tau(a) := \det(I - K)$ satisfies a total system of PDEs

Simplest cases PDE reduce to ODEs of Painleve type

M. Adler & P. van Moerbeke have a Virasoro algebra explanation for the appearance of Painlevé functions

- Universality of F_1 and F_2 extends to non-invariant measures, e.g. Wigner matrices. In some sense these are the "nonintegrable cases" since there is no Fredholm determinant representation of the distribution functions

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- Similar to a CLT for Bernoulli random variables and a general CLT.

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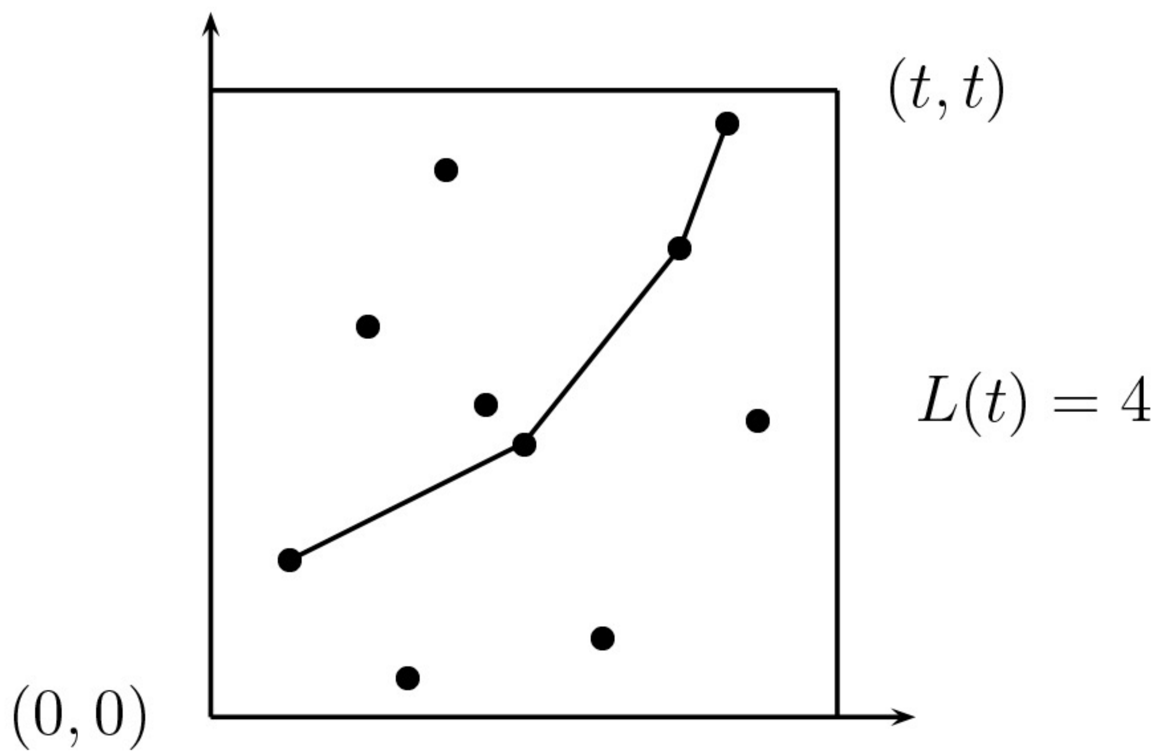
- Physicists formulated many discrete models that they argued should have the same behavior as the KPZ equation—KPZ Universality
- We look at "Last passage percolation"

Poisson process in square $(0,t) \times (0,t)$. Pick $\mathcal{N}$ point in the square where

$$\mathbb{P}(\mathcal{N} = N) = \frac{e^{-t^2}(t^2)^N}{N!}$$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \underline{3} & 7 & 1 & 9 & \underline{6} & 4 & 2 & \underline{8} & \underline{10} & 5 \end{pmatrix}$$

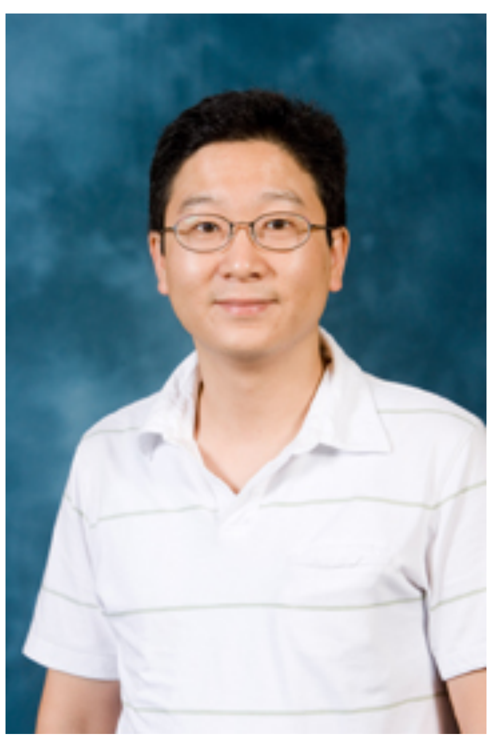
$$L_{10} = 4$$

$$\mathbb{P}(L(t) = \ell) = \sum_{N=0}^{\infty} \frac{e^{-t^2}(t^2)^N}{N!} \mathbb{P}(L_N = \ell) \quad \text{(see Aldous–Diaconis)}$$

where L_N is the length of the longest increasing subsequence of S_N.

Baik–Deift–Johansson Theorem
1999

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L(t)-2t}{t^{1/3}} \leq x\right) = F_2(x)$$



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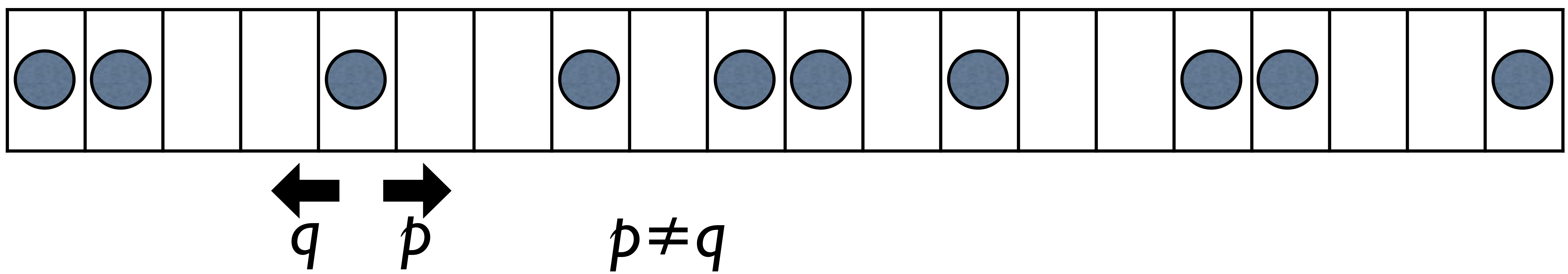
- However, all these models were of the DETERMINANTAL CLASS. KPZ equation not a determinantal process!

ASEP on Integer Lattice

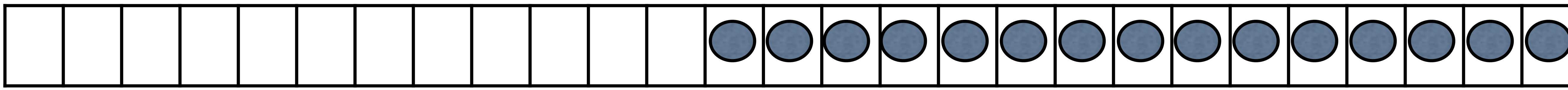


- Each particle has an alarm clock -- exponential distribution with parameter one

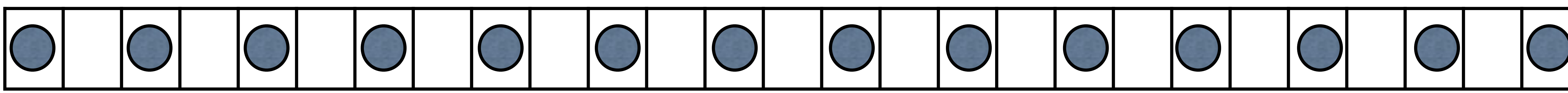
- When alarm rings particle jumps to right with probability p and to the left with probability q

- Jumps are suppressed if neighbor is occupied

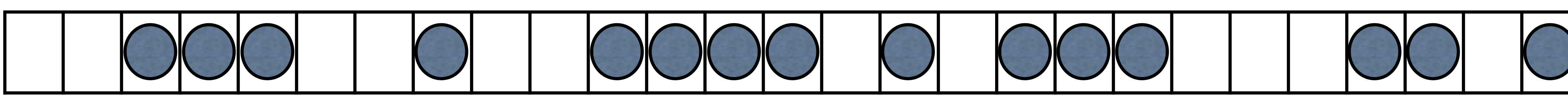
Initial Conditions



Step Initial Condition, $q>p$



Flat Initial Condition



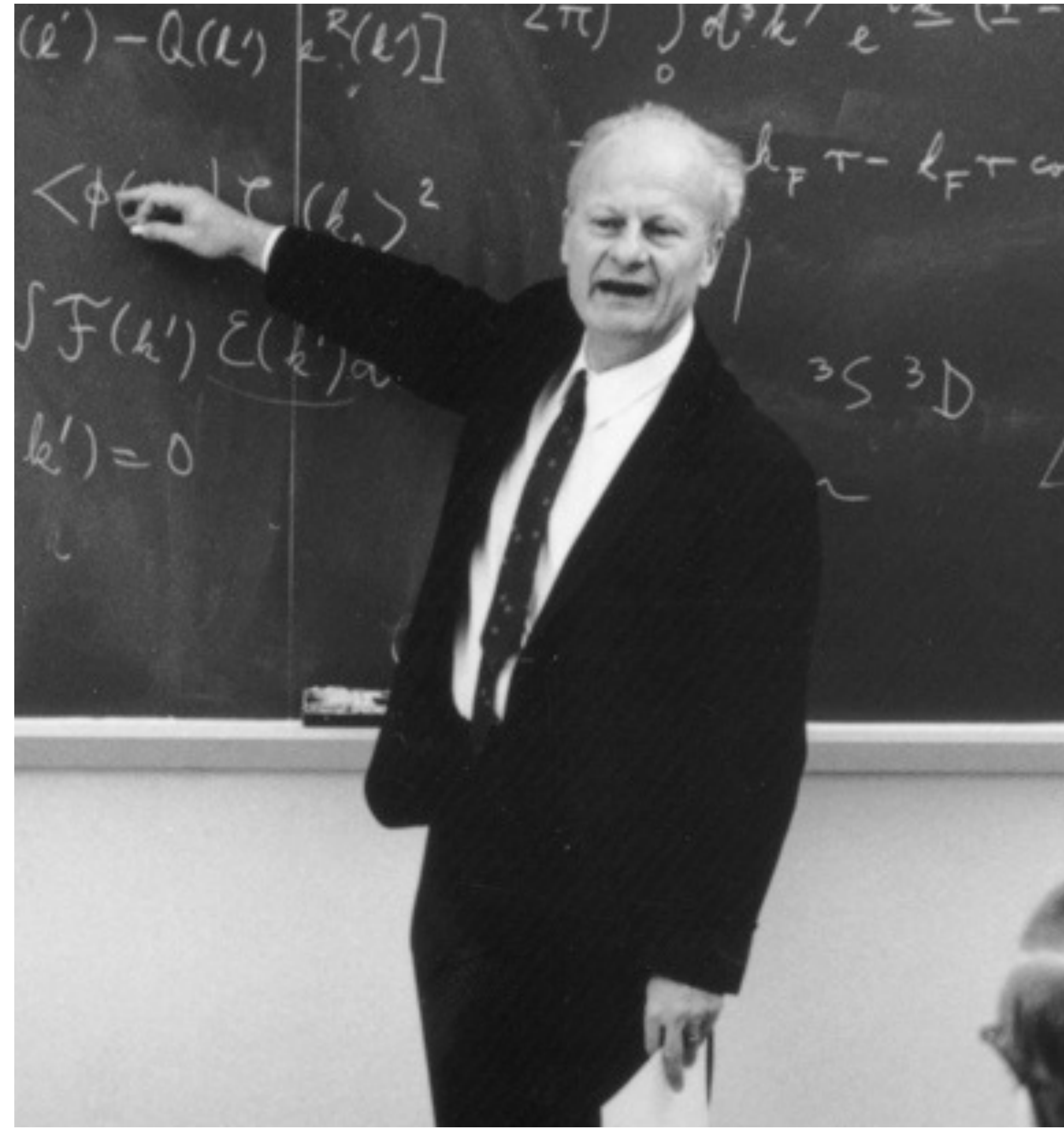
Random: Product Bernoulli measure

Integrable Structure of ASEP

We solve the Kolmogorov forward equation ("master equation") for the transition probability Y→X:

$$P_Y(X;t)$$

Main idea comes from the **Bethe Ansatz** (1931)



Hans Bethe
1906-2005

The Differential Equation

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The Differential Equation

- First consider case of two particles, N=2. State is specified by giving the positions of the two particles $x_1 < x_2$
- Write master equation for two cases $x_2 > x_1+1$ and $x_2 = x_1+1$
- First case particles do not interact with each other (no exclusion effect) and second case exclusion must be taken into account.

The differential equations are

- $x_2 > x_1 + 1$:

$$\frac{d}{dt}u(x_1, x_2) = p\,u(x_1 - 1, x_2) + q\,u(x_1 + 1, x_2) +$$

$$p\,u(x_1, x_2 - 1) + q\,u(x_1, x_2 + 1) - 2u(x_1, x_2)$$

- $x_2 = x_1 + 1$:

$$\frac{d}{dt}u(x_1, x_2) = p\,u(x_1 - 1, x_2) + q\,u(x_1, x_2 + 1) - u(x_1, x_2)$$

We could have simply one equation but then the RHS would have nonconstant coefficients.

Formally subtract the second equation from the first equation when $x_2 = x_1 + 1$:

$$p\,u(x_1, x_1) + q\,u(x_1 + 1, x_1 + 1) - u(x_1, x_1 + 1) = 0$$

If the first equation holds for *all* x_1 and x_2 and this last *boundary condition* holds for all x_1, then the second equation holds when $x_2 = x_1 + 1$. So an equation with nonconstant coefficients has been replaced with an equation with constant coefficients plus a boundary condition.

Solving the DE, $N = 2$

- Since DE is constant coefficient and holds for all $(x_1, x_2) \in \mathbb{Z}^2$ easy to see that a solution is

$$\xi_1^{x_1} \xi_2^{x_2} \mathrm{e}^{t(\varepsilon(\xi_1)+\varepsilon(\xi_2))}, \;\; \xi_1, \xi_2 \in \mathbb{C}$$

  where

$$\varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1$$

- Permuting ξ_j also gives a solution. Since equation is linear—take linear combination

$$u(x_1, x_2; t) = \int_{\mathcal{C}} \int_{\mathcal{C}} \left[A_{12}(\xi) \xi_1^{x_1} \xi_2^{x_2} + A_{21}(\xi) \xi_2^{x_1} \xi_1^{x_2}\right] \mathrm{e}^{t(\varepsilon(\xi_1)+\varepsilon(\xi_2))} \, d\xi_1 d\xi_2$$

- Apply boundary condition to the integrand (!):

$$A_{21}(\xi_1, \xi_2) = -\frac{p + q\xi_1\xi_2 - \xi_2}{p + q\xi_1\xi_2 - \xi_1} A_{12}(\xi_1, \xi_2)$$

- Impose initial condition $u(x_1, x_2; 0) = \delta_{x_1, y_1} \delta_{x_2, y_2}$

-

$$A_{12} = \xi_1^{-y_1 - 1} \xi_2^{-y_2 - 1}$$

- Choose contour $\mathcal{C}$ so that nonzero poles of A_{21} lie outside of $\mathcal{C}$, then initial condition satisfied.

Solving the DE, General N

Remarkably, this generalizes to arbitrary (finite) number of particles N (H. Widom & CT, 2008)

- $$P_Y(X;t) = \sum_\sigma \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i-1} e^{t\varepsilon(\xi_i)} \right) \, d^N\xi$$

- $$A_\sigma = \mathrm{sgn}(\sigma) \left[\prod_{i<j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}) / \prod_{i<j} f(\xi_i, \xi_j) \right]$$
  $$f(\xi, \xi') = p + q\xi\xi' - \xi$$

- Poles of A_σ lie outside contour $\mathcal{C}$.

Simplication

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- By some remarkable combinatoric identities plus analysis can (1) take limit N->infinity and then (2) simplify the result for marginal distr.

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- $P_Y(X;t)$: Sum of N! terms
- For step initial condition, compute marginal distribution of the m^{th} particle from the left: $\mathrm{Prob}(x_m(t)<x)$
- By some remarkable combinatoric identities plus analysis can (1) take limit N->infinity and then (2) simplify the result for marginal distr.
- Here is the final result before any asymptotics

$$\tau := \frac{p}{q} < 1, \;\; \gamma := q - p, \;\; f(\mu, z) := \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} z^k$$

$$\mathbb{P}\left(x_m(t/\gamma) \le x\right) = \int \prod_{k=0}^{\infty} (1 - \mu\tau^k) \det\left(I + \mu J\right) \; \frac{d\mu}{\mu}$$

$$J(\eta, \eta') = \int_{\mathcal{C}_\rho} \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')} \frac{\zeta^m}{(\eta')^{m+1}} \; \frac{f(\mu, \zeta/\eta')}{\zeta - \eta} \, d\zeta$$

$$\varphi_\infty(\eta) = (1 - \eta)^{-x} \mathrm{e}^{\eta t/(1-\eta)}, \;\; 1 < \rho < 1/\tau$$

Universality Theorem

$$\tau = \frac{p}{q},\ \ \gamma = q - p,\ \ \sigma = \frac{m}{t},\ \ c_1 = -1 + 2\sqrt{\sigma},\ \ c_2 = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}$$

Theorem (TW, 2009):

For ASEP with step initial condition and $0 \le p < q$, we have

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \le s \right) = F_2(s)$$

uniformly for σ in a compact subset of $(0, 1)$.

Remarks:

When $p = 0$ (only jumps to the left, $\gamma = 1$) the model is called TASEP for *totally* asymmetric .... TASEP is a determinantal process whereas ASEP is not. The above limit law for TASEP was proved by Johannson in 2000.

KPZ & Stochastic Heat Equation

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \lambda \left(\frac{\partial h}{\partial x} \right)^2 + W$$

Problem term

Bertini & Giacomin (1997) two essential insights:

- Define the solution to the KPZ equation via a Hopf-Cole transformation:

$$h(t,x) = -\log Z(t,x)$$

where Z=Z(t,x) satisfies the stochastic heat equation

$$\frac{\partial Z}{\partial t} = \frac{1}{2} \frac{\partial^2 Z}{\partial x^2} - Z(t,x)W$$

- Z(t,x) is obtained from ASEP in a particularly delicate asymptotic limit called WASEP (weakly asymmetric simple exclusion process)

◆ For wedge initial conditions (droplet), S.Sasamoto & H. Spohn and independently G.Amir, I. Corwin & J. Quastel carried out this program which required new theorems about the relation between KPZ and the stochastic heat equation. Both groups used the ASEP results of Widom & C.T. which required a very delicate asymptotic analysis of the TW formula.

◆ Later nonrigorous methods (replica method) reproduced these results and extended them to the flat initial condition case. This was carried out by V. Dotsenko and independently by P. Calabrese, P. Le Doussal & A. Rosso.

◆ A. Borodin & I. Corwin in their paper "Macdonald Processes" have a rigorous version of the replica method.

Theorem. For any $T > 0$ and $X \in \mathbb{R}$, the Hopf-Cole solution to KPZ with narrow wedge initial data, given by $H(T, X) = -\log Z(T, X)$ with initial data $Z(0, X) = \delta_{X=0}$, has the following probability distribution

$$\mathbb{P}(H(T, X) - \frac{X^2}{2T} - \frac{T}{24} \geq -s) = F_T(s)$$

where $F_T(s)$ deos not depend upon X and is given by

$$F_T(s) = \int_C \frac{d\mu}{\mu} e^{-\mu} \det(I - K_{\sigma_T,\mu})_{L^2(\kappa_T^{-1} s, \infty)}$$

where $\kappa_T = 2^{-1/3} T^{1/3}$, C is a contour positively oriented and going from $+\infty + \epsilon i$ around $\mathbb{R}^+$ to $+\infty - \epsilon i$, and K_σ is an operator given by its integral kernel

$$\begin{aligned} K_\sigma(x, y) &= \int_{-\infty}^{\infty} \sigma(t) \mathrm{Ai}(x + t) \mathrm{Ai}(y + t)\, dt \\ \sigma_{T,\mu} &= \frac{\mu}{\mu - e^{-\kappa_T t}} \end{aligned}$$

Corollary. The Hopf-Cole solution to the KPZ equation with narrow wedge initial data has the following long-time and short-time asymptotics

$$F_T(2^{-1/3}T^{1/3}s) \longrightarrow F_2(s), \quad T \to \infty$$
$$F_T(2^{-1/2}\pi^{1/4}T^{1/4}(s - \log\sqrt{2\pi T}) \longrightarrow G(s), \quad T \to 0$$

The KPZ equation is in the KPZ Universality Class!

References to Sasamoto/Spohn & Amir/Corwin/Quastel Work:

1. G. Amir, I. Corwin, J. Quastel, Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions, *Commun. Pure Appl. Math.* **64**:466–537 (2011).

2. T. Sasamoto and H. Spohn, One-dimensional KPZ equation: an exact solution and its universality, *Phys. Rev. Lett.* **104**:23 (2010).

3. T. Sasamoto and H. Spohn, The crossover regime for the weakly asymmetric simple exclusion process, *J. Stat. Phys.* **140**:209–231 (2010).

4. I. Corwin, The Kardar-Parisi-Zhang equation and universality class, *Random Matrices: Theory and Applications* 01:1130001 (2012).

Thank you for your attention!