Normal forms for germs of vector fields with quadratic leading part. The complete classification.

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Complex plane vector fields with zero linear part

$$\dot{x} = \alpha x^2 + \beta xy + \gamma y^2 + \dots, \qquad \dot{y} = \delta x^2 + \zeta xy + \eta y^2 + \dots$$

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Euler vector field

$$\boldsymbol{E} = x\partial_x + y\partial_y$$

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$$\boldsymbol{V}_0 + \boldsymbol{W}$$

$$(\mathrm{Ad}_{\exp \boldsymbol{Z}})_* \boldsymbol{V} = \boldsymbol{V} + [\boldsymbol{Z}, \boldsymbol{V}] + \ldots = \boldsymbol{V} - \mathrm{ad}_{\boldsymbol{V}} \boldsymbol{Z} + \ldots$$

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The homological operator $\mathrm{ad}_{\boldsymbol{V}_0}$ is split into two homological operators:

$$f \longmapsto C(\boldsymbol{V}_0)f := \boldsymbol{V}_0(f), \quad f \longmapsto D(\boldsymbol{V}_0)f := \boldsymbol{V}_0(f) - \mathrm{div}\,\boldsymbol{V}_0 \cdot f.$$

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Let

$$\mathcal{F} = \mathbb{C}[[x,y]], \quad \mathcal{Z} = \{\boldsymbol{Z} = z_1(x,y)\partial_x + z_2(x,y)\partial_y : z_i \in \mathbb{C}[[x,y]]\}$$

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We put

$$\begin{aligned}
\mathrm{ad}_{\boldsymbol{V}}\boldsymbol{Z} &= [\boldsymbol{V},\boldsymbol{Z}], \\
A(\boldsymbol{V})f &= f \cdot \boldsymbol{V}, \\
B(\boldsymbol{V})\boldsymbol{Z} &= \boldsymbol{V} \wedge \boldsymbol{Z}/\partial_x \wedge \partial_y, \\
C(\boldsymbol{V})f &= \boldsymbol{V}(f) = \partial f/\partial \boldsymbol{V}, \\
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The operators $C(\boldsymbol{V})$, $\mathrm{ad}_{\boldsymbol{V}}$ and $D(\boldsymbol{V})$ are called the **homological operators**

Commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{F} & \xrightarrow{A(\boldsymbol{V})} & \mathfrak{Z} & \xrightarrow{B(\boldsymbol{V})} & \mathfrak{F} & \longrightarrow & 0 \\
 & & \downarrow C(\boldsymbol{V}) & & \downarrow \mathrm{ad}_{\boldsymbol{V}} & & \downarrow D(\boldsymbol{V}) & & \\
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ker $C(\boldsymbol{V})=$ { **First Integrals** (**FI**s) of $\boldsymbol{V}$}

ker $D(\boldsymbol{V})=${ **Inverse Integrating Multipliers** (**IIM**s) of $\boldsymbol{V}$: M such that $\mathrm{div} M^{-1}\boldsymbol{V} \equiv 0$.}

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$$a(u)\frac{\mathrm{d}\tilde{f}}{\mathrm{d}u} = db(u)\tilde{f} + \tilde{g},$$
$$a(u)\frac{\mathrm{d}\tilde{f}}{\mathrm{d}u} = [db(u) - c(u)]\,\tilde{f} + \tilde{g}$$

The solutions are of the form

$$\tilde{f}(u) = \mathrm{const} \cdot u^{\alpha} (u-1)^{\beta} \int^{u} \tau^{-\alpha-1} (\tau - 1)^{-\beta-1} \tilde{g}(\tau) \mathrm{d}\tau,$$
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if $\alpha, \beta \notin \mathbb{Z}$ (respectively, $\gamma, \delta \notin \mathbb{Z}$):

$$\Omega_C(g) = \mathrm{P.V.} \int_0^1 \omega_C(g), \quad \omega_C = u^{-\alpha-1} (u-1)^{-\beta-1} \tilde{g}(u) \mathrm{d}u,$$
$$\Omega_D(g) = \mathrm{P.V.} \int_0^1 \omega_D(g), \quad \omega_D = u^{-\gamma-1} (u-1)^{-\delta-1} \tilde{g}(u) \mathrm{d}u.$$

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$$\mathrm{Im} C(\boldsymbol{V}_0) = \{\Omega_C = 0\}, \quad \mathrm{Im} D(\boldsymbol{V}_0) = \{\Omega_D = 0\}$$

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If $\mathrm{lin}{=}3$ then $\boldsymbol{V}_0$ has the Darboux **First Integral** (**FI**) of the form $x^a y^b (y-x)^c$ and

$$\boldsymbol{V}_0 = x\left[(b+c)y - bx\right]\frac{\partial}{\partial x} + y\left[(a+c)x - ay\right]\frac{\partial}{\partial y}.$$

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$a + b + c = 1, \ a \notin \mathbb{Q}, \ bc \neq 0;$

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3. **Rational PFI without Polynomial IIMs Case**:

$$a = p, \; b = q, \; -c = p, \; s \neq 0, \; \gcd(p, s) = 1 \; (\textbf{Subcase 1}),$$
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4. **Rational PFI with 1-Factor IIM Case**:

$$a=b=1,\ -c=r\geq 3,\ r\neq 4\ (\textbf{Subcase 1}),$$
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$$a=1,\ -b=q,\ -c=r,\ 1\leq r\leq q>1\ (\textbf{Subcase 1}),$$
$$a=1,\ b=c=1\ (\textbf{Subcase 2}),$$
$$a=1,\ b=0,\ -c=r\geq 2\ (\textbf{Subcase 3});$$

6. **Non-resonant with CF Case**:

$$\boldsymbol{V}_0 = (y - x)\,(bx\partial_x - ay\partial_y) = G\boldsymbol{X},$$

$$a \notin \mathbb{Q},\;\; a + b = 1,\;\; c = 0;$$

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8. **Rational PFI without Polynomial IIMs and with CF Case**:

$$a = p,\ \ -b = q,\ \ c = 0,\ \ 1 < p < q,\ \ \gcd(p, q) = 1,\ \ G = y - x;$$

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$$a = p,\ \ -b = q,\ \ c = 0,\ \ 1 < p < q,\ \ \gcd(p, q) = 1,\ \ G = y - x;$$

9. **Quadratic CF Case**:

$$a = c = 0,\ \ b = 1,\ \ G = x(y - x);$$

10. **Double Invariant Line Case**:

$$\boldsymbol{V}_0 = xy\partial_y + (ax + y)\boldsymbol{E},$$
$$a \neq 0, 1, 1/2, 1/3, \ldots, \ (\textbf{Subcase 1}),$$
$$a = 1/n, \ \ n = 1, 2, \ldots \ (\textbf{Subcase 2});$$

10. **Double Invariant Line Case**:

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$$\boldsymbol{V}_0 = x^2\partial_y + (ax+y)\boldsymbol{E};$$

12. **Double Invariant Line with CF Case**:

$$\boldsymbol{V}_0 = (bx+y)\left(x\partial_y + a\boldsymbol{E}\right), \ \ a \neq 0, \ (\textbf{Subcase 1}),$$
$$\boldsymbol{V}_0 = x\left(x\partial_y + a\boldsymbol{E}\right), \ a \neq 0, \ (\textbf{Subcase 2}).$$

The Rational PFI with 1-Factor IIM Case

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$$F = \frac{xy}{(y-x)^r},$$

we have the vector field

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with $a = b = 1$, $c = -r$ and $r \neq 4$.

The first level analysis

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Lemma

We have $\ker C_d(\boldsymbol{V}_0) = 0$ *for any* d, $\ker D_d(\boldsymbol{V}_0) = 0$ *if* $d \neq r+1$ *and* $\ker D_{r+1}(\boldsymbol{V}_0) = \mathbb{C} \cdot M$, *where*

$$M = (2-r)(y-x)^{r+1}$$

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We have $\boldsymbol{X}_F = (y-x)^{-r-1}\boldsymbol{V}_0$ with

$$\boldsymbol{V}_0 = x\,((1-r)\,y - x)\,\partial_x + y\,((1-r)\,x - y)\,\partial_y$$

Lemma

We have $M = B(\boldsymbol{V}_0)\boldsymbol{T}$, *moreover*

$$\mathrm{ad}_{\boldsymbol{V}_0}\boldsymbol{T} = 2(4-r)\,(y-x)^{r-2}\,\boldsymbol{V}_0$$

(which is nonzero for $r \neq 4$*).*

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In the first level analysis of the homological operators we use only the operators associated with $\boldsymbol{V} = \boldsymbol{V}_0$. Firstly we localize the subspaces $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$ complementary to $\mathrm{Im}\,C_d$ and $\mathrm{Im}\,D_d$, where we have the following first level homological operators:

$$C_d(\boldsymbol{V}_0), D_d(\boldsymbol{V}_0) : \mathcal{F}_d \longmapsto \mathcal{F}_{d+1},$$

i.e., the restrictions of $C(\boldsymbol{V})$, $D(\boldsymbol{V})$ to the $d+1$ dimensional spaces of homogeneous polynomials.

With $a = b = 1$, $c = -r$ and $s = 2 - r$ we have

$$\alpha = -d\frac{1}{r-2},\ \beta = d\frac{r}{r-2},\ \gamma = -(d-3)\frac{1}{r-2} + 1,\ \delta = (d-3)\frac{r}{r-2} + 1,$$
$$\alpha + \beta = d + d\frac{1}{r-2},\quad \gamma + \delta = d - 1 + (d-3)\frac{1}{r-2}.$$

in

$$\tilde{f}(u) = \mathrm{const} \cdot u^{\alpha}(u-1)^{\beta} \int^{u} \tau^{-\alpha-1}(\tau-1)^{-\beta-1}\tilde{g}(\tau)\mathrm{d}\tau,$$
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(solutions in (x, u) variables of homological equations

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With $a = b = 1$, $c = -r$ and $s = 2 - r$ we have

$$\alpha = -d\frac{1}{r-2},\ \beta = d\frac{r}{r-2},\ \gamma = -\left(d-3\right)\frac{1}{r-2} + 1,\ \delta = \left(d-3\right)\frac{r}{r-2} + 1,$$
$$\alpha + \beta = d + d\frac{1}{r-2},\quad \gamma + \delta = d - 1 + (d-3)\frac{1}{r-2}.$$

in

$$\tilde{f}(u) = \mathrm{const} \cdot u^{\alpha}\left(u-1\right)^{\beta}\int^{u}\tau^{-\alpha-1}\left(\tau-1\right)^{-\beta-1}\tilde{g}(\tau)\mathrm{d}\tau,$$
$$\tilde{f}(u) = \mathrm{const} \cdot u^{\gamma}\left(u-1\right)^{\delta}\int^{u}\tau^{-\gamma-1}\left(\tau-1\right)^{-\delta-1}\tilde{g}(\tau)\mathrm{d}\tau$$

(solutions in (x, u) variables of homological equations

$$C(\boldsymbol{V}_0)f = g,\quad D(\boldsymbol{V}_0)f = g)$$

Recall that the solutions $\tilde{f}(u)$ should be polynomial; otherwise, the corresponding polynomial g lies outside $\mathrm{Im}\,C(\boldsymbol{V}_0)$ or $\mathrm{Im}\,D(\boldsymbol{V}_0)$.

We put

$$\begin{aligned} g^C &= x^{d+1}, \\ g^D &= x^{d-1}y(y-x) \text{ if } d \neq r+1, \\ g_0^D &= y^{r+1}\left[(1-r)\,x - y\right], \;\; g_1^D = x^{r+1}\left[(1-r)\,y - x\right], \end{aligned}$$

if $d = r+1$, as a potential bases for $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$.

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We get the forms

$$\omega_C(g^C) = \frac{\mathrm{d}u}{u^{\alpha+1}\,(u-1)^{\beta+1}}, \quad \omega_D(g^D) = \frac{\mathrm{d}u}{u^{\gamma}\,(u-1)^{\delta}}$$

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$$

For $d/(r-2)$ non-integer, its period

$$
\Omega_C\left(g^C\right) = \mathrm{const} \cdot B(\alpha, \beta) \neq 0
$$

If $d/(r-2) = m \in \mathbb{Z}$ then we get the function

$$\tilde{f}^C = \frac{1}{2-r} \frac{(u-1)^{mr}}{u^m} \int^u \frac{\tau^{m-1} \mathrm{d}\tau}{(\tau-1)^{mr+1}}$$

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If $(d-3)/(r-2) \notin \mathbb{Z}$ then the corresponding period

$$\Omega_D\left(g^D\right) \neq 0$$

If $d-3=m(r-2)$, $m\in\mathbb{Z}$, then the solution of homological equation is

$$\tilde{f}^D=\frac{1}{2-r}\frac{(u-1)^{mr+1}}{u^{m-1}}\int^u\frac{\tau^{m-2}}{(\tau-1)^{mr+2}}\mathrm{d}\tau$$

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But for $m=1$, i.e., $d=r+1$, we have two generators, g_0^D and g_1^D, and we define two periods

$$\Omega_D^{0,1}(g_j^D)=\mathrm{Res}_{u=0,1}\omega_D(g_j^D),\quad j=0,1$$

We have

$$\omega_D\left(g_0^D\right) = \frac{(1-r)u-1}{u(u-1)^{r+2}}\mathrm{d}u, \ \omega_D\left(g_1^D\right) = \frac{u^r(u+r-1)}{(u-1)^{r+2}}\mathrm{d}u$$

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and we define the **period matrix**

$$\begin{pmatrix} \Omega_D^0\left(g_0^D\right) & \Omega_D^0\left(g_1^D\right) \\ \Omega_D^1\left(g_0^D\right) & \Omega_D^1\left(g_1^D\right) \end{pmatrix}$$

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Since $\Omega_D^0\left(g_0^D\right) = (-1)^{r-1} \neq 0$, $\Omega_D^1\left(g_1^D\right) = 1 \neq 0$ and $\Omega_D^0\left(g_1^D\right) = 0$, this matrix has triangular form, with nonzero entries on the diagonal, and hence is nondegenerate.

$$\ker D_{r+1}(\boldsymbol{V}_0) = \mathbb{C} \cdot (2 - r)\,(y - x)^{r+1}$$

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$$\mathrm{ad}_{\boldsymbol{V}_0} \boldsymbol{T} = 2(4 - r)\,(y - x)^{r-2}\, \boldsymbol{V}_0 \neq 0 \quad \text{for} \quad r \neq 4$$

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The corresponding period

$$\Omega_C(g) = \mathrm{P.V.} \int_0^1 u^{-\alpha-1}\,(u-1)^{-\beta-1}\, du \neq 0$$

where $\alpha = -(r-3)/(r-2)$, $\beta = r\,(r-3)\,/\,(r-2)\,,\ \alpha+\beta \notin \mathbb{Z}$, therefore $g = (y-x)^{r-2} \notin \mathrm{Im}\, C_d\,(\boldsymbol{V}_0)\,,\ d = r-3$.

Theorem

The first level normal forms in the **Rational PFI with 1-Factor IIM Case**:

$$(1+\psi(x))\left(\boldsymbol{V}_0+\mathbf{U}+\varphi(x)\boldsymbol{E}\right),$$
$$\boldsymbol{U}=ay^r\partial_x+bx^r\partial_y,$$
$$\mathcal{I}\left(\varphi\right)=\mathbb{Z}_{\geq 2},\quad \mathcal{I}\left(\psi\right)=\mathbb{Z}_{\geq 1};$$

or

$$(1+\psi(x))\,\boldsymbol{V}_0,$$
$$\mathcal{I}\left(\psi\right)=\mathbb{Z}_{\geq 1}\diagdown\left\{r-2\right\};$$

where $\varphi=\sum_{i\in\mathcal{I}(\varphi)}a_ix^i$ *and* $\psi=\sum_{i\in\mathcal{I}(\psi)}b_ix^i$ *(the latter form is complete).*

The second level analysis

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$$\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1, \qquad \deg \boldsymbol{V}_1 > 1$$

We have to consider a few possibilities:

$$\boldsymbol{V}_1 = ax^k \boldsymbol{E} \ \text{ or } \ \boldsymbol{V}_1 = ay^r \partial_x + bx^r \partial_y$$

(from the orbital normal form) or

$$\boldsymbol{V}_1 = cx^l \boldsymbol{V}_0$$

(associated with the orbital factor).

Our analysis is reduced to the operator $D(\boldsymbol{V})$ acting on functions of the form $\xi M + f$, where $\xi \in \mathbb{C}$ and $M = (y - x)^{r+1}$ is the generator of $\ker D_{r+1}(\boldsymbol{V}_0)$.

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We get the following second level homological operator:

$$\begin{gathered} \widetilde{D}(\boldsymbol{V}) : \mathbb{C} \oplus \mathcal{F}_d \longmapsto \mathcal{F}_{d+1}, \\ (\xi, f) \longmapsto \xi D(\boldsymbol{V}_1) M + D(\boldsymbol{V}_0) f, \end{gathered}$$

where $d = k + r = \deg \boldsymbol{V}_1 + r$.

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where $d = k + r = \deg \boldsymbol{V}_1 + r$.

Lemma

We have:
(a) $D\left(x^k \boldsymbol{E}\right) M = (r-1-k)\, x^k M,$
(b) $D\left(ay^r \partial_x + bx^r \partial_y\right)(y-x)^{r+1} = (r+1)\left(bx^r - ay^r\right)(y-x)^r.$

The case $\boldsymbol{V}_1 = ax^k \boldsymbol{E}$

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$k \neq r - 1$, $d = k + r > r + 1$ and $\operatorname{codim} \operatorname{Im} D_d(\boldsymbol{V}_0) = 1$

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In order to demonstrate the surjectivity of the operator $\widetilde{D}(\boldsymbol{V})$, it is sufficient to show that
$D(\boldsymbol{V}_1) M = \mathrm{const} \cdot x^k (y-x)^{r+1} \notin \operatorname{Im} D_d(\boldsymbol{V}_0)$.

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$D(\boldsymbol{V}_1)\, M = \mathrm{const} \cdot x^k (y-x)^{r+1} \notin \operatorname{Im} D_d(\boldsymbol{V}_0)$.

$$\omega_D(g) = u^{\vartheta}(u-1)^{\theta} \mathrm{d}u, \qquad \text{for} \qquad g = x^k (y-x)^{r+1}$$

with $\vartheta = \frac{k+r-3}{r-2} - 2 > 0$ and $\theta = r - 1 - (k+r-3)\frac{r}{r-2}$

If $\frac{d-3}{r-2} = \frac{k+r-3}{r-2}$ is not integer then

$$\Omega_D(g) = \mathrm{const} \cdot B(\vartheta + 1, \theta + 1) \neq 0$$

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$$\Omega_D\left(g\right) = \mathrm{const} \cdot B\left(\vartheta + 1, \theta + 1\right) \neq 0$$

Otherwise, either

$$\Omega_D(g) = \mathrm{Res}_{u=1}\omega_D(g) \neq 0$$

or

$$\Omega_D(g) = \int_{\infty}^{0} \omega_D(g) \neq 0$$

Theorem

The complete normal form in the **Rational PFI with 1-Factor IIM Case** *with* $\boldsymbol{V}_1 = ax^k\boldsymbol{E}$ *is:*

either

$$(1+\psi(x))\left(\boldsymbol{V}_0 + \boldsymbol{V}_1 + bx^r\partial_y + \varphi(x)\boldsymbol{E}\right), \quad k < r-1,$$
$$\mathfrak{I}(\psi) = \mathbb{Z}_{\geq 1}, \quad \mathfrak{I}\left(\varphi\right) = \mathbb{Z}_{>k} \diagdown \{r-1, k+r+1\}$$

or

$$(1+\psi(x))\left(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}\right), \quad k > r-1,$$
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The case $\boldsymbol{V}_1 = ay^r\partial_x + bx^r\partial_y$

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Here $d = 2r - 1 \neq r - 1$ and $(a, b) \neq (0, 0)$. By considering a suitable period $\Omega_D(g)$ for $g = (bx^r - ay^r)(y - x)^r$ we have

$$\tilde{f}^D = \mathrm{const} \cdot \frac{(u-1)^{r+1}}{u} \int^u \frac{(a - b\tau^r)}{(\tau - 1)^{r+2}} \mathrm{d}\tau.$$

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The form $\omega_D(g) = (a - bu^r)(u - 1)^{-r-2}\mathrm{d}u$ has trivial residua at $u = 0, 1$, so the only period is

$$\Omega_D(g) = \int_\infty^0 \omega_D(g) = \frac{1}{r+1}\left(a + (-1)^{r+1} b\right)$$

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If $a + (-1)^{r+1}b \neq 0$ then we are done. Otherwise, we can use the IIM M either to cancel another term from the orbital normal form (provided it is $\neq \boldsymbol{V}_0 + \boldsymbol{V}_1$) or to improve the orbital factor.

Theorem

The second level normal form in the **Rational PFI with 1-Factor IIM Case** *with* $\boldsymbol{V}_1 = ay^r\partial_x + bx^r\partial_y$ *is: either*

$$(1+\psi(x))\left(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}\right),$$
$$a + (-1)^{r+1}b \neq 0,$$
$$\mathcal{I}(\varphi) = \mathbb{Z}_{\geq r} \diagdown \{2r-3\}, \;\; \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1},$$

(this form is complete) or

$$(1+\psi(x))\left(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}\right),$$
$$\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y),$$
$$\mathcal{I}(\varphi) = \mathbb{Z}_{\geq r}, \;\; \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1},$$

or

$$(1+\psi(x))\left(\boldsymbol{V}_0 + \boldsymbol{V}_1\right),$$
$$\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y),$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq 1} \diagdown \{r-2\},$$

(this form is complete).

The case $\boldsymbol{V}_1 = cx^l \boldsymbol{V}_0$

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Here the orbital normal form is $\boldsymbol{V}_0$, so M is used to reduce only one term from the series $\psi(x)$. The result is like in the first level normal form.

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Third level

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Here the orbital normal form is $\boldsymbol{V}_0$, so M is used to reduce only one term from the series $\psi(x)$. The result is like in the first level normal form.

Third level

We consider homological operators associated with the vector fields $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2$ such that $\boldsymbol{V}_0 + \boldsymbol{V}_1$ has nontrivial IIM, i.e.,

$$\boldsymbol{V}_1 = a(y^r \partial_x + (-1)^r x^r \partial_y)$$

The case $\boldsymbol{V}_2 = cx^l \boldsymbol{E}$

The case $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$

Here $c \neq 0$ and $l \geq r$. As before we use $D(\boldsymbol{V}_2)M \notin \mathrm{Im} D(\boldsymbol{V}_0)$ to reduce the term $x^{l+r}\boldsymbol{E}$ from the orbital normal form.

The case $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$

Here $c \neq 0$ and $l \geq r$. As before we use $D\left(\boldsymbol{V}_2\right)M \notin \mathrm{Im}D\left(\boldsymbol{V}_0\right)$ to reduce the term $x^{l+r}\boldsymbol{E}$ from the orbital normal form.

Theorem

The complete normal form in the **Rational PFI with 1-Factor IIM Case** *with* $\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y)$ *and* $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$ *is:*

$$(1+\psi(x))\left(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2 + \varphi(x)\boldsymbol{E}\right),$$
$$\mathcal{I}\left(\varphi\right) = \mathbb{Z}_{>l}\diagdown\left\{l+r\right\}, \quad \mathcal{I}\left(\psi\right) = \mathbb{Z}_{\geq 1}.$$

The main theorem

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Theorem

The list of complete and non-equivalent normal forms for vector fields $\dot{x} = \alpha x^2 + \beta xy + \gamma y^2 + \dots, \quad \dot{y} = \delta x^2 + \zeta xy + \eta y^2 + \dots$ *with the leading parts* $\boldsymbol{V}_0$ *is the following.*

For the **Generic Non-resonant Case**, *the* **Rational PFI without Polynomial IIMs Case**, *the* **Double Invariant Line Case (Subcase 1)**, *the* **Triple Invariant Line Case**, *the* **Non-resonant with CF Case**, *the* **Rational PFI without Polynomial IIMs and with CF Case**, *and the* **Double Invariant Line with CF (Subcase 1)**

$$\{1 + \psi(x)\}\{\boldsymbol{V}_0 + \varphi(x)\boldsymbol{E}\},$$
$$\mathfrak{I}(\varphi) = \mathbb{Z}_{>1}, \quad \mathfrak{I}(\psi) = \mathbb{Z}_{>0};$$

Theorem - cont.

for the **Double Invariant Line Case (Subcase 2)**

$$\{1+\psi(x)\}\left\{\boldsymbol{V}_0 + a_{n+1}x^{n+2}\partial_y + \varphi(x)\boldsymbol{E}\right\},$$
$$\mathfrak{I}(\varphi) = \mathbb{Z}_{>1}\diagdown\{n+1\};$$

for the **Double Invariant Line with CF Case (Subcase 2)**

$$\boldsymbol{V}_0 + \varphi(y)\partial_x + \psi(y)\partial_y,$$
$$\mathfrak{I}(\varphi) = \mathbb{Z}_{>2}, \quad \mathfrak{I}(\psi) = \mathbb{Z}_{>2}.$$

For the **Rational PFI with 1-Factor IIM Case** *we have:*
either

$$\{1+\psi(x)\}\left\{\boldsymbol{V}_0 + \boldsymbol{V}_1 + bx^r\partial_y + \varphi(x)\boldsymbol{E}\right\},$$
$$\boldsymbol{V}_1 = ax^k\boldsymbol{E}, \quad k < r-1,$$
$$\mathfrak{I}(\psi) = \mathbb{Z}_{\geq 1}, \quad \mathfrak{I}(\varphi) = \mathbb{Z}_{>k}\diagdown\{r-1, k+r+1\},$$

or

Theorem - cont.

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\varphi(x)\boldsymbol{E}\},$$
$$\boldsymbol{V}_1 = ax^k\boldsymbol{E},\ k>r-1,$$
$$\mathfrak{I}(\psi)=\mathbb{Z}_{\geq 1},\ \ \mathfrak{I}(\varphi)=\mathbb{Z}_{>k}\diagdown\{k+r+1\},$$

or

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\varphi(x)\boldsymbol{E}\},$$
$$\boldsymbol{V}_1 = ay^r\partial_x+bx^r\partial_y,\ a+(-1)^{r+1}b\neq 0,$$
$$\mathfrak{I}(\varphi)=\mathbb{Z}_{\geq r}\diagdown\{2r-3\},\ \ \mathfrak{I}(\psi)=\mathbb{Z}_{\geq 1},$$

or

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\varphi(x)\boldsymbol{E}\},$$
$$\boldsymbol{V}_1 = a(y^r\partial_x+(-1)^r x^r\partial_y),\ \boldsymbol{V}_2 = cx^l\boldsymbol{E},$$
$$\mathfrak{I}(\varphi)=\mathbb{Z}_{>l}\diagdown\{l+r\},\ \ \mathfrak{I}(\psi)=\mathbb{Z}_{\geq 1},$$

or

Theorem - cont.

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\}$$
$$\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r \partial_y),$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}\diagdown\{r-2\},\ (\textbf{Subcase 1}),$$

or

$$\{1+cx^j+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\},$$
$$\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r \partial_y),\ r=4$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{>j}\diagdown\{j+2\},$$

or

$$\{1+\psi(x)\}\,\boldsymbol{V}_0,$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}\diagdown\{r-2\}\quad (\textbf{Subcase 1}),$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}\quad (\textbf{Subcase 2}),$$

or

$$\boldsymbol{V}_0+\boldsymbol{V}_1,$$
$$\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r \partial_y),\ r=4$$

Theorem - cont.

For the **Rational PFI with 2-Factor IIM Case** *we have: either*

$$\{1+\psi(x)\}\{\mathbf{V}_0+\mathbf{V}_1+\mathbf{U}+\varphi(x)\mathbf{E}\},$$
$$\mathbf{V}_1 = ax^k\mathbf{E},\ k<q+r,$$
$$\mathbf{U} = ax^{q+r}\mathbf{E}+by^{q+r+1}\partial_x,$$
$$\mathcal{I}(\varphi)=\mathbb{Z}_{\geq k}\setminus\{q+r,k+q+r+2\},\ \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1},$$

or

$$\{1+\psi(x)\}\{\mathbf{V}_0+\mathbf{V}_1+\varphi(x)\mathbf{E}\},$$
$$\mathbf{V}_1 = ax^k\mathbf{E},\ k>q+r,$$
$$\mathcal{I}(\varphi)=\mathbb{Z}_{\geq k}\setminus\{k+q+r+2\},\ \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1},$$

or

$$\{1+\psi(x)\}\{\mathbf{V}_0+\mathbf{V}_1+\mathbf{V}_2+\varphi(x)\mathbf{E}\},$$
$$\mathbf{V}_1 = ax^{q+r}\mathbf{E}+bx^{q+r+1}\partial_y,\ \mathbf{V}_2 = cx^l\mathbf{E},$$
$$\mathcal{I}(\varphi)=\mathbb{Z}_{\geq l}\setminus\{l+q+r-1\},\ \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1},$$

or

Theorem - cont.

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\},$$
$$\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + bx^{q+r+1}\partial_y,$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}\setminus\{q+r-1\}, \text{ (**Subcases 1** and **3**)},$$

or

$$\left\{1+cx^j+\psi(x)\right\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\},$$
$$\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + bx^{q+r+1}\partial_y,$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq j+2}, \text{ (**Subcase 2**)},$$

or

$$\left\{1+cx^j+ex^m+\psi(x)\right\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\},$$
$$\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + bx^{q+r+1}\partial_y,$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{\geq m+2}, \text{ (**Subcase 2**)},$$

or

$$\{1+cx\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\}, \text{ (**Subcase 2**)}.$$

Theorem - cont.

For the **Polynomial PFI Case** *(with three subcases) we have: either*

$$\{1+\psi(x)+\chi(y)\}\boldsymbol{V}_0,$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus I_0,\ \ \mathcal{I}(\chi)=I_1,\ (\textbf{Subcases 1 and 2}),$$

or

$$\{1+\psi(x)\}\boldsymbol{V}_0,$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0},\ (\textbf{Subcase 3}),$$

or

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+[\varphi(x)+\phi(y)]\boldsymbol{E}\},$$
$$\boldsymbol{V}_1=ax^j\boldsymbol{E}\neq 0,\quad 2\leq j\neq 1 \pmod{s},$$
$$\mathcal{I}(\varphi)=\mathbb{Z}_{>j}\setminus I_j,\ \ \mathcal{I}(\phi)=\mathbb{Z}_{>j}\cap I_1,$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus(\mathbb{Z}_{>j}\cap I_j),\ \ \mathcal{I}(\chi)=I_1,$$

or

Theorem - cont.

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+[\varphi(x)+\phi(y)]\boldsymbol{E}\},$$
$$\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E},$$
$$a\Delta_n\neq 0,$$
$$\mathcal{I}(\varphi)=\mathbb{Z}_{>2ns+1}\diagdown I_1,\quad \mathcal{I}(\phi)=(\mathbb{Z}_{>2ns+1}\cap I_1)\diagdown\{3ns+1\},$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0}\diagdown(\mathbb{Z}_{>ns+1}\cap I_1),\quad \mathcal{I}(\chi)=I_1,$$

where $\Delta_n=\Delta_n(a,b,c,d)=\det\left(\Omega_D^i\left(g_j^D\right)\right)$ *or*

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+[\varphi(x)+\phi(y)]\boldsymbol{E}\},$$
$$\boldsymbol{V}_1=by^{ns+1}\boldsymbol{E},\quad \boldsymbol{V}_2=(cx^{2ns+1}+ey^{ls+1})\boldsymbol{E},$$
$$bc\neq 0,$$
$$\mathcal{I}(\varphi)=\mathbb{Z}_{>2ns+1}\diagdown\{3ns+1\},\quad \mathcal{I}(\phi)=\varnothing,$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0},\quad \mathcal{I}(\chi)=\mathbb{Z}_{\leq ns+1}\cap I_1,$$

or

Theorem - cont.

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2\},$$
$$\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E},$$
$$a\neq 0=\Delta_n,$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus(Z_{>ns+1}\cap I_1)\setminus\{ns\},\quad \mathcal{I}(\chi)=I_1,\ \textbf{(Subcases 1, 2)},$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{\le n+1},\quad \mathcal{I}(\chi)=\mathbb{Z}_{>0}\setminus\{n\},\ \textbf{(Subcase 3)},$$

or

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2\},$$
$$\boldsymbol{V}_1=by^{ns+1}\boldsymbol{E}\neq 0,\quad \boldsymbol{V}_2=ey^{2ns+1}\boldsymbol{E},$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus\{ns\},\quad \mathcal{I}(\chi)=\mathbb{Z}_{\le ns+1}\cap I_1,\ \textbf{(Subcases 1, 2)},$$
$$\mathcal{I}(\psi)=\mathbb{Z}_{>0},\quad \mathcal{I}(\chi)=\mathbb{Z}_{\le n+1}\setminus\{n\},\ \textbf{(Subcase 3),}$$

or

Theorem - cont.

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+[\varphi(x)+\phi(y)]\boldsymbol{E}\},$$
$$\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_3=hx^k\boldsymbol{E},$$
$$\boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E}\text{ and }\boldsymbol{V}_2\equiv 0\text{ if }k<2ns+1,$$
$$ah\neq 0=\Delta_n,\ ns+1<k\not\equiv 1\pmod{s},$$
$$\mathfrak{I}(\varphi)=\mathbb{Z}_{>k}\setminus I_1\setminus\{k+ns\},\quad \mathfrak{I}(\phi)=\mathbb{Z}_{>k}\cap I_1,$$
$$\mathfrak{I}(\psi)=\mathbb{Z}_{>0}\setminus(\mathbb{Z}_{>ns+1}\cap I_1),\quad \mathfrak{I}(\chi)=I_1,$$

or

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+[\varphi(x)+\phi(y)]\boldsymbol{E}\},$$
$$\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_3=hy^{ls+1}\boldsymbol{E},$$
$$\boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E}\text{ and }\boldsymbol{V}_2\equiv 0\text{ if }l<2n,$$
$$ah\neq 0=\Delta_n,\ n<l\neq 2n,,$$
$$\mathfrak{I}(\varphi)=\mathbb{Z}_{>ls+1}\setminus I_1,\quad \mathfrak{I}(\phi)=(\mathbb{Z}_{>ls+1}\cap I_1)\setminus\{(l+n)s+1\},$$
$$\mathfrak{I}(\psi)=\mathbb{Z}_{>0}\setminus(\mathbb{Z}_{>ns+1}\cap I_1),\quad \mathfrak{I}(\chi)=I_1,$$

Theorem - cont.

or

$$\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+[\varphi(x)+\phi(y)]\boldsymbol{E}\},$$
$$\boldsymbol{V}_1 = by^{ns+1}\boldsymbol{E} \neq 0, \quad \boldsymbol{V}_3 = hx^k\boldsymbol{E} \neq 0,$$
$$\boldsymbol{V}_2 = ey^{2ns+1}\boldsymbol{E} \text{ and } \boldsymbol{V}_2 \equiv 0 \text{ if } k < 2ns+1,$$
$$ns+1 < k \neq 2ns+1,,$$
$$\mathcal{I}(\varphi) = \mathbb{Z}_{>k} \setminus \{(k+n)\,s+1\}, \quad \mathcal{I}(\phi) = \varnothing,$$
$$\mathcal{I}(\psi) = \mathbb{Z}_{>0}, \quad \mathcal{I}(\chi) = \mathbb{Z}_{\leq ns+1} \cap I_1.$$

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THANK YOU.