Hamiltonian systems and Maslov indices, corresponding to Schrödinger operators with delta-potentials

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July 8, 2016

Outline

Statement of the problem

Spectral problem for the Schrödinger operator with δ-potential.
M — Riemannian manifold, $\mathrm{dim} M \leq 3$.

$$\hat{H} = -\frac{h^2}{2}\Delta + \alpha\delta_P$$

Δ is the Beltrami – Laplace operator.
Problem: asymptotics of the spectrum as $h \to 0$.

R. de L.Kronig, W.G. Penney, Proc. R. Soc. Lond. Ser. A, 130:814, 499-513, 1931.

F.A. Berezin, L.D. Faddeev "Remarks on the Schrödinger equation with singular potential. Doklady Math., 1961, v.131, pp 1011 - 1014.

S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden. Solvable models in quantum mechanics. Providence: AMS Chelsea Publishing, 2005.

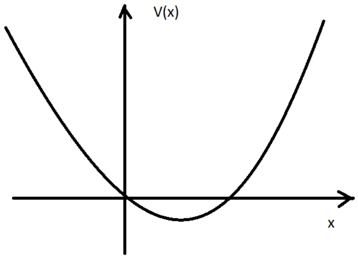
S. Albeverio, P. Kurasov. Singular perturbations of differential operators. Cambridge: Cambridge University Press, 2000.

Smooth potential

Let $M = \mathbb{R}$,

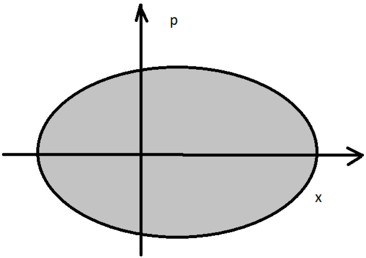
$$\hat{H} = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x),$$

$V(x) \to +\infty, \quad |x| \to \infty.$


V(x)
x

Λ — curve on the phase plane.

$$\frac{1}{2}p^2 + V(x) = E.$$



Theorem

Let E be solution of the Bohr — Sommerfeld equation

$$\frac{1}{2\pi h}\int_{\Lambda} p dx + \frac{1}{2} = m \in \mathbb{Z}.$$

Then there exists an eigenvalue λ of $\hat{H}$:

$$\lambda = E + o(h).$$

δ-potential

$$\hat{H} = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x) + \alpha\delta(x - x_0).$$

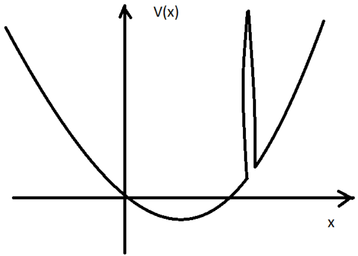
Formal definition:

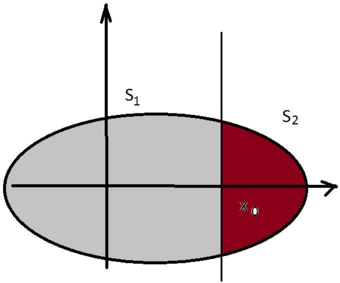
$$\hat{H}_0 = -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R}\backslash x_0.$$

Boundary conditions

$$\psi(x_0 + 0) = \psi(x_0 - 0),$$

$$\psi'(x_0 + 0) - \psi'(x_0 - 0) = \frac{2\alpha}{h^2}\psi(x_0).$$


V(x)
x


S_1
S_2
x_0

Theorem

Let E be solution of the equation

$$\cos\left(\frac{1}{2h}(S_1+S_2)\right) = \frac{\alpha}{hp(x_0)}\Big(\sin\left(\frac{1}{2h}(S_1+S_2)\right)-\cos\left(\frac{1}{2h}(S_1-S_2)\right)\Big).$$

Then there exists an eigenvalue λ of $\hat{H}$:

$$\lambda = E + o(h).$$

Limit cases

$\frac{\alpha}{h} \to 0$,

$$\frac{S_1 + S_2}{2\pi h} + \frac{1}{2} = m \in \mathbb{Z},$$

$\frac{\alpha}{h} \to \infty$,

$$\frac{S_1}{2\pi h} + \frac{1}{4} = m_1 \in \mathbb{Z}, \quad \frac{S_2}{2\pi h} + \frac{3}{4} = m_2 \in \mathbb{Z}.$$

Maslov theory for smooth potentials.
$\hat{H} = -\frac{h^2}{2}\Delta + V(x)$.
Let Λ be compact invariant manifold of the classical Hamilton system on T^*M with the Hamilton function $H = \frac{1}{2}|p|^2 + V(x)$.

Theorem

(V.P. Maslov) Let Λ satisfies quantization condition

$$\frac{1}{2\pi h}[\theta] + \frac{1}{4}[\mu] \in H^1(\Lambda, \mathbb{Z})$$

and let $\hat{H}$ be self-adjoint. Then there exists a point λ of the spectrum, such that

$$\lambda = H|_\Lambda + O(h^2).$$

$\theta = \sum_j p_j dx_j$.

$$\frac{1}{2\pi h}\int_{\gamma}\theta+\frac{1}{4}\mu(\gamma)=m\in\mathbb{Z}.$$

μ — Maslov index. $\pi : T^*M \to M$ — natural projection, Σ — cycle of singularities of π.

$$\mu(\gamma)=\gamma\circ\Sigma.$$

Example: integrable Hamiltonian system, $H = \frac{1}{2}|p|^2 + V(x)$.
Λ — Liouville tori, I — action variables. Quantization conditions

$$\frac{1}{h} I_j + \frac{1}{4}\mu_j = m_j \in \mathbb{Z}.$$

Definition of the operator with delta-potential δ_P (Berezin, Faddeev). 2 properties

- $\hat{H}$ is self-adjoint;
- If $\psi(P) = 0$, then $\hat{H}\psi = -\frac{h^2}{2}\Delta\psi$.

Formal definition. $\hat{H}_0 = -\frac{h^2}{2}\Delta|_{\psi \in H^2(M), \psi(P)=0}$.
$\hat{H}$ is a self-adjoint extension of $\hat{H}_0$.

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Explicit description of the domain.
For $\psi \in D(\hat{H})$ we have a decomposition

$$\psi = aF(x) + b + o(1),$$

$$F = -\frac{1}{4\pi d(x,P)}, \quad \dim M = 3, \quad F = \frac{1}{2\pi} \log d(x,P), \quad \dim M = 2.$$

Boundary condition

$$a = \frac{2\alpha}{h^2} b.$$

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Symmetric manifold

Let M be 2D surface of revolution or 3D spherically symmetric manifold, $M \cong S^2$ or $M \cong S^3$.

$$M \subset \mathbb{R}^3, \quad y = (f(z)\cos\varphi, f(z)\sin\varphi f(z), z)$$

or

$$M \subset \mathbb{R}^4, \quad y = (f(z)\cos\theta\cos\varphi, f(z)\cos\theta\sin\varphi f(z), f(z)\sin\theta, z)$$

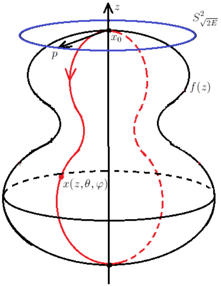
$z \in [z_1, z_2]$,
$f = \sqrt{(z - z_1)(z_2 - z)}w(z)$, w — analytic.

Result: Lagrangian manifold

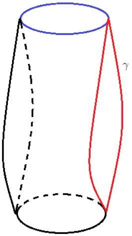
$\Lambda_0 : p \in T^*_P M, \quad |p| = 2E$, $\Lambda = \bigcup_t g_t \Lambda_0$, g_t — geodesic flow.

$\Lambda \cong T^2, \quad \dim M = 2$, $\Lambda \cong S^2 \times S^1, \quad \dim M = 3$.

Trajectories



Lagrangian manifold



Result: eigenvalues

Theorem

Let E be solution of the equation

$$\tan\left(\frac{1}{2h}\oint_{\gamma}(p,dx)\right) = \frac{2}{\pi}\left(\log\left(\frac{\sqrt{2E}}{h}\right) + \frac{\pi h^2}{\alpha} + c\right), \quad n = 2,$$

c is Euler constant.

$$\tan\left(\frac{1}{2h}\oint_{\gamma}(p,dx)\right) = \frac{2h^3}{\sqrt{2E}\alpha}, \quad n = 3.$$

Theorem

Here γ is closed geodesic.
There exists an eigenvalue λ of $\hat{H}$, such that

$$\lambda = E + o(h).$$

Critical values of α.

2D-case. Let

$$\frac{\alpha \log 1/h}{h^2} \to 0 \quad or \quad \frac{\alpha \log 1/h}{h^2} \to \infty.$$

Then E up to small terms satisfies

$$\frac{1}{2\pi h} \int_\gamma (p, dx) + \frac{1}{2} = m \in \mathbb{Z}.$$

Critical value

$$\alpha \sim \frac{h^2}{\log(1/h)}.$$

Critical values of α.

3D case.
Let $\alpha/h^3 \to 0$. Then E satisfies

$$\frac{1}{2\pi h}\int_\gamma (p, dx) + \frac{1}{2} = m \in \mathbb{Z}.$$

Let $\alpha/h^3 \to \infty$. Then E satisfies

$$\frac{1}{2\pi h}\int_\gamma (p, dx) = m \in \mathbb{Z}.$$

Critical value $\alpha \sim h^3$.

Jump of the Maslov index

In 3D case the analog of the Maslov index jumps as α passes through the critical value. $\Lambda_0 : p \in T^*_P M, |p| = 2E$,
$F : \Lambda_0 \to \Lambda_0, F(p) = -p$
General formula for big α

$$\frac{1}{2\pi h}\int_\gamma (p, dx) + \frac{1}{4}(\mu(\gamma) + (\deg F - 1)) = m \in \mathbb{Z}.$$

THANK YOU
FOR YOUR ATTENTION!