Geometry and Topology on the Bethe Ansatz for the Periodic ASEP model

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Before I forget ...

- Thank you for letting me speak!
- Joint work with Eric Brattain (UC Davis) and Norman Do (Monash University).
- The Completeness of the Bethe Ansatz for the Periodic ASEP (*arXiv:1511.03762v1*).

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- The Completeness of the Bethe Ansatz for the Periodic ASEP (*arXiv:1511.03762v1*).

Theorem (E. Brattain, N. Do, A.S. 2015)

The Bethe ansatz is complete for the periodic ASEP model.

Outline

I will introduce the components of Theorem 1 :

- ASEP Model
  - Continuous Markov Process.
- Bethe Ansatz
  - Educated guess of the solution.
- Proof of Theorem 1 (Sketch!)
  - Counting all the solutions given by the ansatz.

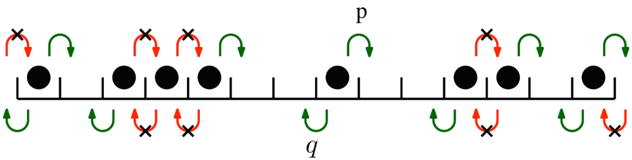

Figure : ASEP on a line.

A **asymmetric**

S **simple**

E **exclusion**

P **process**

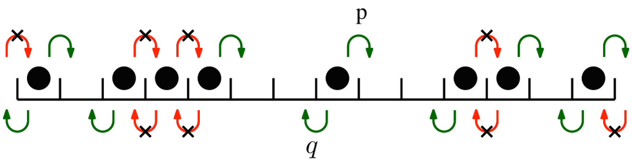

Figure : ASEP on a line.

A **asymmetric**: non-equilibrium statistical mechanics.

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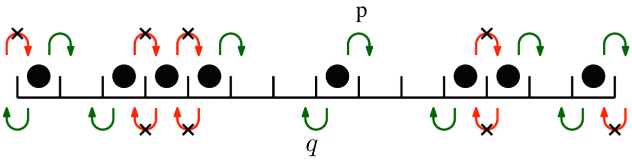
E **exclusion**: no two particles can be on the same position.

P **process**: continuous Markov process.



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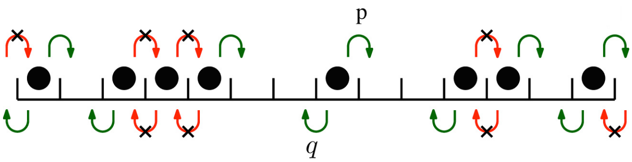
The advantage of this model is that while in spite of its simplicity, it carries a rich structure. Namely, it's an *Exactly Solvable Model* through the Bethe ansatz. This has made this model the poster child for non-equilibrium mechanics (e.g. driven lattice gases with hard core repulsion) and the KPZ universality class (e.g. $1 + 1$ random growth interfaces).



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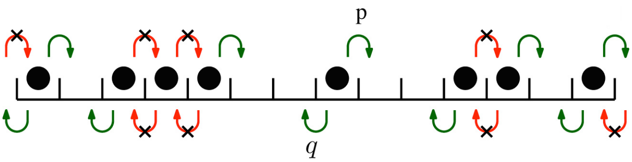
- L sites.
- N particles.
- Particles jump with probability p to the right and probability q to the left.
- The configuration of the system is represented by the position of the particles.



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- L sites.
- N particles.
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- The configuration of the system is represented by the position of the particles.
  - i.e.$\langle 1, 3, 4, 5, 8, 11, 12, 14|$

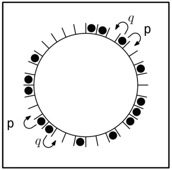

Figure : ASEP on a ring.

Our state space on a ring of length L is $\bigotimes_{i=1}^{L} \mathbb{C}_i^2$ with a basis $\{e_S | S \subset [L]\}$, where $\mathbb{C}_i^2 = \langle e_i^+, e_i^- \rangle$ and

$$e_S = \left(\bigotimes_{i \in S} e_i^+ \right) \otimes \left(\bigotimes_{i \in [L] \setminus S} e_i^- \right)$$

We have conservation of particles. We fix N and consider $span\{u_S || S| = N\}$ the space of N particles on a ring of length L. We use the notation

$$\langle x_1, \ldots, x_N | = e_{\{x_1, \ldots, x_N\}}$$

to make the coordinate dependence more explicit.

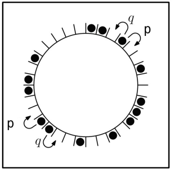

Figure : ASEP on a line.

- The transition rate matrix governing the process is a sum of local jumps operators

$$M = \sum_{i=1}^{L} M_i, \qquad M_i : \mathbb{C}_i^2 \otimes \mathbb{C}_{i+1}^2 \to \mathbb{C}_i^2 \otimes \mathbb{C}_{i+1}^2$$

$$M_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The fact that the sum of each column is zero represents the conservation of probability.

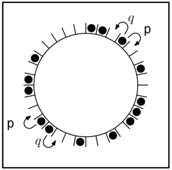

Figure : ASEP on a ring.

Given a state $\sum_{\vec{x}} \langle x_1, \ldots, x_N | u(x_1, \ldots, x_N; t)$ at time t, the state at time $t + dt$ is given by

$$\left(\sum_{\vec{x}} \langle x_1, \ldots, x_N | u(x_1, \ldots, x_N; t) \right) M dt = \sum_{\vec{x}} \langle x_1, \ldots, x_N | du(x_1, \ldots, x_N; t + dt)$$

The coefficients $u(x_1, \ldots, x_N; t)$ can be thought of as the probability of being in state $\langle x_1, \ldots, x_N |$ at time t given some initial conditions determined by $u(x_1, \ldots, x_N; 0)$.

The transition rate matrix translates into the *master equation* on the coefficients

$$\frac{du}{dt} = \sum_{i=1}^{N} \left[pu(x_i - 1)\tilde{\delta}_{i,i-1} + qu(x_i + 1)\tilde{\delta}_{i+1,i} - pu(x_i)\tilde{\delta}_{i+1,i} - qu(x_i)\tilde{\delta}_{i,i-1} \right]$$

- $u(x_i \pm 1)$ is a shorthand for $u(x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N; t)$.
- Define $\tilde{\delta}_{i,j} = 0$ if $x_i = x_j + 1$ and $\tilde{\delta}_{i,j} = 1$ otherwise.
- Periodic boundary conditions impose:

$$u(x_1, \ldots, x_N; t) = u(x_2, \ldots, x_N, x_i + L; t).$$

- The initial state $Y = (y_1, \ldots, y_N)$ imposes:

$$u(x_1, \ldots, x_n) = \delta_{X,Y}.$$

Figure : Hans Bethe

We use the Bethe ansatz to solve the eigenvalue problem.

$$\frac{d}{dt}\sum_{\vec{x}}\langle\vec{x}|u(\vec{x};t) = E\sum_{\vec{x}}\langle\vec{x}|u(\vec{x};t) = \left(\sum_{\vec{x}}\langle\vec{x}|u(\vec{x};t)\right)M.$$

Method introduced to by H. Bethe in 1931 to solve the 1D Heisenberg XXZ spin-chain model. Since, it has been used in many 1D models that are restricted by some boundary conditions (e.g. periodic boundaries, reservoirs at the boundaries, monodromies).

- If all the particles are far apart from each other, there is no interactions. The master equation is:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \left[pu(x_i - 1)\tilde{\delta}_{i,i-1} + qu(x_i + 1)\tilde{\delta}_{i+1,i} - pu(x_i)\tilde{\delta}_{i+1,i} - qu(x_i)\tilde{\delta}_{i,i-1} \right]$$
$$= \sum_{i=1}^{N} [pu(x_i - 1) + qu(x_i + 1) - pu(x_i) - qu(x_i)]$$
$$= \sum_{i=1}^{N} [pu(x_i - 1) + qu(x_i + 1) - u(x_i)]$$

- The interactions are recorded by the boundary conditions:

$$pu(x_i, x_i) + qu(x_i + 1, x_i + 1) - u(x_i, x_i + 1) = 0$$

The Bethe ansatz proposes solutions of the form:

$$u_{\vec{z}}(x_1, \ldots, x_N) = \sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}.$$

- S_N is the symmetric group on N elements.
- The solutions are parametrized by vectors $\vec{z} \in \mathbb{C}^N$.
- The coefficients $A_\sigma \in \mathbb{C}$ are set to satisfy the boundary conditions.

Given the Bethe ansatz solutions:

$$u_{\vec{z}}(x_1, \dots, x_N) = \sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}.$$

Recall, the boundary conditions:

$$pu(x_i, x_i) + qu(x_i + 1, x_i + 1) - u(x_i, x_i + 1) = 0$$

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Recall, the boundary conditions:

$$pu(x_i, x_i) + qu(x_i + 1, x_i + 1) - u(x_i, x_i + 1) = 0$$

Then:

$$A_\sigma(z_1, \ldots, z_N) = \prod_{\text{inversions } (i,j)} -\frac{p + qz_iz_j - z_i}{p + qz_iz_j - z_j}$$

- An inversion of a permutation σ is a pair (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$.

Given the Bethe ansatz solutions:

$$u_{\vec{z}}(x_1, \ldots, x_N) = \sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i},$$

and the coefficients:

$$A_\sigma(z_1, \ldots, z_N) = \prod_{\text{inversions } (i,j)} -\frac{p + q z_i z_j - z_i}{p + q z_i z_j - z_j}$$

The periodicity constraint:

$$u(x_1, \ldots, x_N) = u(x_2, \ldots, x_N, x_i + L).$$

Bethe Ansatz

Given the Bethe ansatz solutions:

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and the coefficients:

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The periodicity constraint:

$$u(x_1, \dots, x_N) = u(x_2, \dots, x_N, x_i + L).$$

Then, we obtain the *Bethe ansatz equations*:

$$z_j^L = (-1)^{N-1} \prod_{i=1}^{N} \frac{p + qz_jz_i - z_j}{p + qz_jz_i - z_i} \text{ for } j = 1, 2, \dots, N.$$

Theorem (E. Brattain, N. Do, A.S. 2015)

The Bethe ansatz is complete on the periodic ASEP model for generic parameters L, N, and p.

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The Bethe ansatz is complete on the periodic ASEP model for generic parameters L, N, and p.

- We treat p as a complex number.
- The generic condition states that p lies in a Zariski open set, and no conditions on L or N.
- We show that there exists $\binom{L}{N}$ non-trivial solutions to the Bethe ansatz.
- The solutions are linearly independent.

Two Particles

Lets consider the two particle case (i.e. $N = 2$) to highlight our expectations and possible difficulties. The proposed solutions are:

$$u(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1},$$

with the coefficients related by:

$$A_{12} = -A_{21} \frac{p + qz_1 z_2 - z_1}{p + qz_1 z_2 - z_2},$$

and the Bethe equations:

$$z_1^L = -\frac{p + qz_1 z_2 - z_1}{p + qz_1 z_2 - z_2}, \qquad z_2^L = -\frac{p + qz_1 z_2 - z_2}{p + qz_1 z_2 - z_1}.$$

Note that if $z = z_1 = z_2$

- $A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1} = (A_{12} + A_{21}) z^{x_1 + x_2}$.
- $A_{12} = -A_{21} \frac{p + q z_1 z_2 - z_1}{p + q z_1 z_2 - z_2} = -A_{21}$.
- So, $u(x_1, x_2) = 0$.

This is what we call an *inadmissible* solution. In the two particle case, we show that all the inadmissible solutions are of the form $z_1 = z_2$, and in for general L and N, we classify all the *inadmissible* conditions.
Back in the $N = 2$, case

$$(z_1 z_2)^L = \left(-\frac{p + q z_1 z_2 - z_1}{p + q z_1 z_2 - z_2} \right) \left(-\frac{p + q z_1 z_2 - z_2}{p + q z_1 z_2 - z_1} \right) = 1.$$

Thus, in solving the Bethe equations, we will let $z_2 = \epsilon z_1^{-1}$ for some L^{th} root of unity.

Given the Bethe equations:

$$z_1^L = -\frac{p + qz_1z_2 - z_1}{p + qz_1z_2 - z_2}, \qquad z_2^L = -\frac{p + qz_1z_2 - z_2}{p + qz_1z_2 - z_1},$$

and $z_2 = \epsilon z_1^{-1}$ for some L^{th} root of unity. We have:

$$z_1^L = -\frac{p + q\epsilon - z_1}{p + q\epsilon - \epsilon z_1^{-1}},$$

which is equivalent to the degree L polynomial

$$(p + \epsilon q)z_1^L - \epsilon z_1^{L-1} - z_1(p + \epsilon q) = (z_1 \pm \epsilon^{1/2})f(z_1, p).$$

The factor $(z_1 \pm \epsilon^{1/2})$ correspond to the *inadmissible* solutions and the $\pm$ is determined by the parity of L.

In order to make the previous argument precise we recall:

Theorem (Lefschetz)

Given two (differentiable) functions $\psi, \phi : C \to X$ on compact orientable manifolds. The number of solutions (up to multiplicity) of the equation $\psi(pt) = \phi(pt)$ is given by the Lefschetz coincidence number:

$$\lambda(\psi, \phi) = \sum_{n=0}^{\dim_{\mathbb{R}} X} (-1)^n Tr(\psi_n \phi^n).$$

where $\psi_n : H_n(C) \to H_n(X)$ is the pushforward in homology and $\phi^n : H_n(X) \to H_n(C)$ is the Poincaré dual of the pullback in cohomology.

Example: Lefschetz Theorem on Two Particles

Let $X = C = \mathbb{C}^3$ and the define $\psi, \phi : C \to X$ by

$$\phi(z_1, z_2, w_{12}) = (z_1^L, z_2^L, w_{12})$$
$$\psi(z_1, z_2, w_{12}) = (w_{12}, w_{12}^{-1}, -\frac{p + qz_1 z_2 - z_1}{p + qz_1 z_2 - z_2}).$$

The coincidence equation $\phi(pt) = \psi(pt)$ becomes

$$z_1^L = w_{12}$$
$$z_2^L = w_{12}^{-1}$$
$$w_{12} = -\frac{p + qz_1 z_2 - z_1}{p + qz_1 z_2 - z_2},$$

equivalent to the Bethe equations

$$z_1^L = -\frac{p + qz_1 z_2 - z_1}{p + qz_1 z_2 - z_2}, \qquad z_2^L = -\frac{p + qz_1 z_2 - z_2}{p + qz_1 z_2 - z_1}.$$

Example: Lefschetz Theorem on Two Particles

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$$\psi(z_1, z_2, w_{12}) = (w_{12}, w_{12}^{-1}, -\frac{p + qz_1z_2 - z_1}{p + qz_1z_2 - z_2}).$$

The Lefschetz Theorem doesn't quite apply since $\mathbb{C}^3$ is not compact and the third component of ϕ is not well-defined on the locus:

$$p + qz_1z_2 - z_2 = 0 = p + qz_1z_2 - z_1.$$

We fix this by letting $\mathbb{C}^3 \hookrightarrow (\mathbb{CP}^1)^3 = X$ and $C = Blow(X)$.

Example: Lefschetz Theorem on Two Particles

The next step is to compute the trace of the maps

$$\psi_n : H_n(C) \to H_n(X)$$
$$\phi^n : H_n(X) \to H_n(C).$$

In this case, we have $H_*(X) = \bigoplus_{n=0}^{6} H_n(X) \cong \mathbb{C}^8$ and $\bigoplus_{n=0}^{6} H_n(C) \cong H_*(X) \oplus H$ and the Lefschetz coincidence $\lambda(\psi, \phi)$ is the trace of an 8 by 8 matrix. In the end, we establish a bijection

$$\lambda(\psi, \phi) = \#\{\text{Rooted Planted Forest}\}.$$

Along with combinatorics, this establishes Theorem 1.

Thank you for your attention!

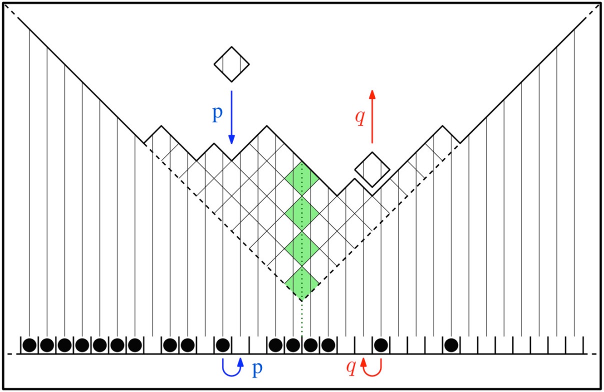

Figure : Random Growth.

- Occupied position is to slope $-\frac{1}{2}$ as vacant position is to slope $\frac{1}{2}$.
- Height function $h(x, t)$.
- Initial Conditions.