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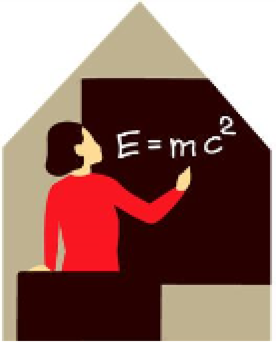

THE HAMILTON OPERATORS AND RELATED INTEGRABLE DIFFERENTIAL-ALGEBRAIC NOVIKOV-LEIBNIZ STRUCTURES

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As it is well known [25, 15, 13, 37], many of integrable Hamiltonian systems, discovered during the last decades, on the Lie-algebraic properties of their internal hidden symmetry structures. A first account of the Hamiltonian operators and related differential-algebraic structures, lying in the background of integrable systems was given by I. Gelfand and I. Dorfman [28, 22] and later was extended by B. Dubrovin and S. Novikov [23, 24], and also by A. Balinski and S. Novikov [9]. There were also devised some new special differential-algebraic techniques [41] for studying the Lax type integrability and the structure of related Hamiltonian operators for a wide class of the Riemann type hydrodynamic hierarchies. Just recently a lot of works appeared [4, 5, 6, 38] being devoted to the finite dimensional representations of the Novikov algebra. Their importance for constructing integrable multi-component nonlinear Camassa-Holm type dynamical systems on functional manifolds was demonstrated by I. Strachan and B. Szablikowski in [47], where there was suggested in part the Lie-algebraic imbedding of the Novikov algebra into the general Lie-Poisson orbits scheme of classification Lax type integrable Hamiltonian systems. It is also worth of mentioning the related work [29] by D. Holm-R. Ivanov where there were also constructed integrable multi-component nonlinear Camassa-Holm type dynamical systems on functional manifolds.

In our work we succeeded in formal differential-algebraic reformulating the classical Lie algebraic scheme and developed an effective approach to classification of the algebraic structures lying in the background of the integrable multicomponent Hamiltonian systems. In particular, we have devised a simple algorithm allowing to construct new algebraic structures within which the corresponding Hamiltonian operators exist and generate integrable multicomponent dynamical systems. We show, as examples, that the well known Novikov algebraic structure, obtained before in [28, 9] as a condition for a matrix differential expression to be Hamiltonian and in [12, 18, 30, 39] as that on a flat torsion free left-invariant affine connection on affine manifolds, affine structures and convex homogeneous cones, appears within the devised approach as a differentiation on the Lie-algebra, naturally associated with a suitably constructed differential loop algebra. As a direct generalization of this example it is obtained two new differentiations, whose background algebraic structures coincide, respectively, with the well known [2, 27] right Leibniz algebra, introduced in [16, 17, 33], and with a new so-called non-associative *"Riemann"* algebra, which naturally generate different Hamiltonian operators, describing a wide class of multi-component hierarchies [14, 41] of integrable multi-component hydrodynamic Riemann type systems. Their reductions appeared to be closely related both with the mentioned above integrable Camassa-Holm and with the Degasperis-Processi dynamical systems, and are of special interest from the equivalence transformation point of view, devised recently in [48]. A classical Poisson manifold approach, closely related with that analyzed in the work and allowing effectively enough to construct Hamiltonian operators, is also briefly revisited.

2. Hamiltonian operators and the related algebraic structures

Assume $(\mathbb{A};\circ)$ to be a finite dimensional algebra (in general noncommutive and nonassociative) over the closed field $\mathbb{K}$. In addition, we will endow the algebra $\mathbb{A}$ both with the natural Lie algebra extension $(\mathcal{L}_{\mathbb{A}};[\cdot,\cdot])$ by means of the usual commutator operation $[\cdot,\cdot]:\mathbb{A}\times\mathbb{A}\to\mathbb{A}$ and with the natural nondegenerate ad-invariant symmetric trace-like [10, 11, 45, 47] bilinear form $<\cdot,\cdot>:\mathcal{L}_{\mathbb{A}}\times\mathcal{L}_{\mathbb{A}}\to\mathbb{K}$:

$$<[a,b],c>=<a,[b,c]> \tag{2.1}$$

for any a, b and $c \in \mathcal{L}_{\mathbb{A}}$. Based on the algebra $\mathbb{A}$ one can construct the related loop algebra $\widetilde{\mathbb{A}}$ of smooth mappings $\mathbb{S}^1 \to \mathbb{A}$ and endow it with a suitably generalized commutator operation $[\cdot,\cdot] : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ subject to the natural pointwise multiplication operation $\circ : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$. The corresponding loop Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ is assumed to be naturally rigged with a generalized nondegenerate bilinear form $(\cdot,\cdot) : \mathcal{L}_{\widetilde{\mathbb{A}}} \times \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathbb{K}$, which is symmetric

$$(a, b) := \int_{\mathbb{S}^1} < a, b > dx = (b, a) \tag{2.2}$$

for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, and ad-invariant

$$([a, b], c) = (a, [b, c]) \tag{2.3}$$

for any a, b and $c \in \mathcal{L}_{\widetilde{\mathbb{A}}}$.

Remark 2.1. If the symmetric bilinear form (2.2) additionally satisfies the shifting property

$$(a \circ b, c) = (a, b \circ c) \tag{2.4}$$

for any a, b and $c \in \mathcal{L}_{\tilde{A}}$, then the ad-invariance condition (2.3) holds.

The form (2.2) makes it possible to construct the natural identification $\tilde{\mathbb{A}} \simeq \tilde{\mathbb{A}}^*$, in particular for a linear functional element $u \in \tilde{\mathbb{A}} \simeq \tilde{\mathbb{A}}^*$ we also will write, by definition, its left $L_u : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$, $L_u a := u \circ a$, and right $R_u : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$, $R_u a := u \circ a$, shift mappings on $\tilde{\mathbb{A}}$ for any $a \in \tilde{\mathbb{A}}$. Moreover, from (2.2) one easily follows that for an element $u \in \tilde{\mathbb{A}} \simeq \tilde{\mathbb{A}}^*$ one can naturally define the adjoint left $L_u^* : \tilde{\mathbb{A}}^* \to \tilde{\mathbb{A}}^*$ and right $R_u^* : \tilde{\mathbb{A}}^* \to \tilde{\mathbb{A}}^*$ shift mappings, satisfying the relationships

$$(L_u^* a,\ b) := (a, u \circ b), \quad (R_u^* a,\ b) := (a, b \circ u) \tag{2.5}$$

for any $a, b \in \tilde{\mathbb{A}}$. Based on properties listed above, one can naturally identify the space $\mathcal{L}^*_{\tilde{\mathbb{A}}}$, adjoint with respect to the ad-invariant form (2.2) to the loop Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, with the loop Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ itself and consider further the space of smooth scalar functions $\mathcal{D}(\mathcal{L}^*_{\tilde{\mathbb{A}}})$ on $\mathcal{L}^*_{\tilde{\mathbb{A}}}$ jointly with the related Lie-Poisson bracket on it:

$$\{f, g\}_0 := (u, [\nabla f(u), \nabla g(u)]) \tag{2.6}$$

for any $f, g \in \mathcal{D}(\mathcal{L}^*_{\tilde{\mathbb{A}}})$, where by definition, the weak gradient mapping $\nabla : \mathcal{D}(\mathcal{L}^*_{\tilde{\mathbb{A}}}) \to \mathcal{L}_{\tilde{\mathbb{A}}}$ is defined for any $h \in \mathcal{D}(\mathcal{L}^*_{\tilde{\mathbb{A}}})$ and all $\xi \in \mathcal{L}^*_{\tilde{\mathbb{A}}}$ at point $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$ as

$$(\xi, \nabla h(u)) := dh(u + \varepsilon\xi)/d\varepsilon|_{\varepsilon=0}. \tag{2.7}$$

Owing to the definition [1, 3, 13, 10, 25], the Lie-Poisson bracket (2.6) satisfies the classical Jacobi condition, thereby being a very powerful tool

for constructing the related Hamiltonian operators on the functional space $\mathcal{D}(\mathcal{L}^*_{\tilde{\mathbb{A}}})$. In particular, we will call, following [28, 36], a smooth mapping $\vartheta : \mathcal{L}^*_{\tilde{\mathbb{A}}} \to Hom(\mathcal{L}_{\tilde{\mathbb{A}}}; \mathcal{L}^*_{\tilde{\mathbb{A}}})$ a Hamiltonian operator if the related bracket

$$\{f, g\} := (\nabla f(u), \vartheta(u) \nabla g(u)) \tag{2.8}$$

is determined for any $f, g \in \mathcal{D}(\mathcal{L}^*_{\tilde{\mathbb{A}}})$ and satisfies the Jacobi identity.

Taking into account that the canonical Lie-Piosson bracket (2.6) does not involve essentially the loop Lie algebra structure of $\mathcal{L}_{\tilde{\mathbb{A}}}$, we will proceed further to a new Lie algebra structure on $\mathcal{L}_{\tilde{\mathbb{A}}}$ by means of its central extension. Namely, let $\mathcal{L}_{\hat{\mathbb{A}}} := \mathcal{L}_{\tilde{\mathbb{A}}} \oplus \mathbb{K}$ denote the centrally extended Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ endowed with the bracket

$$[(a;\alpha),(b;\beta)] := ([a,b];\omega_2(a,b)) \tag{2.9}$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ and $\alpha, \beta \in \mathbb{K}$, where the 2-cocycle $\omega_2 : \mathcal{L}_{\tilde{\mathbb{A}}} \times \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathbb{K}$ is a skew-symmetric bilinear form satisfying the Jacobi identity:

$$\omega_2([a,b],c) + \omega_2([b,c],a) + \omega_2([c,a],b) = 0 \tag{2.10}$$

for any a, b and $c \in \mathcal{L}_{\tilde{\mathbb{A}}}$. It is evidently that the existence of nontrivial central extensions on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ strongly depends on the algebraic structure of the algebra $\mathbb{A}$ lying in the background of the whole construction presented above. Yet there exist some general algebraic properties which allow to proceed further with success. Namely, assume that a smooth mapping $\mathrm{D} : \mathcal{L}^*_{\tilde{\mathbb{A}}} \to End\, \mathcal{L}_{\tilde{\mathbb{A}}}$ defines for $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$ a *weak differentiation* of the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, that is

$$(c, \mathrm{D}_u[a, b]) = (c, [\mathrm{D}_u a, b] + [a, \mathrm{D}_u b]) \tag{2.11}$$

for any a, b and $c \in \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$. Then the following important proposition [35, 45] holds.

Proposition 2.2. *Let a smooth mapping* $\mathrm{D}_u : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ *be for any* $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}}$ *a skew-symmetric differentiation of the Lie algebra* $\mathcal{L}_{\tilde{\mathbb{A}}}$. *Then the expression*

$$\omega_2(a, b) := (a, \mathrm{D}_u b) \tag{2.12}$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ *and* $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$ *defines a nontrivial 2-cocycle on the Lie algebra* $\mathcal{L}_{\tilde{\mathbb{A}}}$.

A proof is by means of direct checking the Jacobi identity (2.10) and is omitted.

There also exist other tools to construct *a priori* nontrivial differentiations on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$. For instance, as a simple consequence of Proposition 2.2 the following theorem [22, 35, 45] holds.

Theorem 2.3. *Let a nondegenerate linear skew-symmetric endomorphism* $\mathcal{R} : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ *satisfy the well known Yang-Baxter commutator condition:*

$$[\mathcal{R}a, \mathcal{R}b] = \mathcal{R}[\mathcal{R}a, b] + [a, \mathcal{R}b] \tag{2.13}$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ *. Then the inverse mapping* $\mathcal{R}^{-1} : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ *is a skew-symmetric differentiation of the Lie algebra* $\mathcal{L}_{\tilde{\mathbb{A}}}$ *and the expression*

$$\omega_2(a, b) = (a, \mathcal{R}^{-1}b) \tag{2.14}$$

defines for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ *a 2-cocycle on* $\mathcal{L}_{\tilde{\mathbb{A}}}$.

As an interesting and useful consequence of Theorem 2.3 consists in the fact that the following subspaces

$$\mathcal{L}^{\pm}_{\tilde{\mathrm{A}}} := \frac{1}{2}(\mathbb{I} \pm \mathcal{R})\mathcal{L}_{\tilde{\mathrm{A}}} \tag{2.15}$$

are Lie sub-algebras of $\mathcal{L}_{\tilde{\mathrm{A}}}$, splitting it into the direct sum $\mathcal{L}^{+}_{\tilde{\mathrm{A}}} \oplus \mathcal{L}^{-}_{\tilde{\mathrm{A}}} = \mathcal{L}_{\tilde{\mathrm{A}}}$. In particular, the $\mathcal{R}$-structures on the Lie algebra $\mathcal{L}_{\tilde{\mathrm{A}}}$ can be effectively exploited for constructing additional Hamiltonian operators on $\mathcal{L}^{*}_{\tilde{\mathrm{A}}}$.

To demonstrate this in more details, we will endow, following [28], the space $\mathcal{L}^*_{\widetilde{\mathbb{A}}}$ with the natural differential-algebraic structure assuming it to be a polynomial differential algebra $\widetilde{\mathbb{A}}(u)$, generated by an element $u \in \widetilde{\mathbb{A}}^*$ and its derivatives $u^{(j)} := D^j_x u, j \in \mathbb{Z}_+$, with respect to the standard differentiation $D_x := \partial/\partial x$, $x \in \mathbb{S}^1$, on $\widetilde{\mathbb{A}}$. On the algebra $\widetilde{\mathbb{A}}(u)$ one can naturally define the space of gradient wise differentiations $\Gamma_{\widetilde{\mathbb{A}}}(u)$ consisting of all linear uniform mapping $\xi : \widetilde{\mathbb{A}}(u) \to Der\ \widetilde{\mathbb{A}}(u)$, where by definition, $[\xi_h, D_x] = 0$ for any $h \in \widetilde{\mathbb{A}}(u)$ and the linear expression $\xi_h : \widetilde{\mathbb{A}}(u) \to \widetilde{\mathbb{A}}(u)$ acts on any element $f \in \widetilde{\mathbb{A}}(u)$ as

$$(\xi_h f)(u) := \frac{d}{d\varepsilon} f(u + \varepsilon h f)|_{\varepsilon=0} = f'(u) \circ h, \tag{2.16}$$

where $f'(u) : \widetilde{\mathbb{A}}(u) \to \widetilde{\mathbb{A}}(u)$ is the standard Frechet derivative on $\widetilde{\mathbb{A}}(u)$ at point $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$. Taking into account the action of the differentiations $-_{\widetilde{\mathbb{A}}}(u)$ on $\widetilde{\mathbb{A}}(u)$, one can rig it with a natural Lie algebra structure

$$[\xi_h, \xi_g] := \xi_{\{h,g\}}, \tag{2.17}$$

where, by definition, the element $\{h, g\} := g'(h) - h'(g) \in \widetilde{\mathbb{A}}(u)$. Following further [28], on the differential algebra $\widetilde{\mathbb{A}}(u)$ one can determine a space of scalar functionals $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ as the set of equivalent elements $f \sim h \in < \widetilde{\mathbb{A}}(u), 1 >$ for which $f - h \sim D_x g$ for some element $g \in < \widetilde{\mathbb{A}}(u), 1 >$, where $1 \in \mathbb{A} \cup \{1\}$ is the identity element, satisfying the conditions $1 \circ a = a \circ 1 = a$ for all $a \in \mathbb{A}$. The functional set $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ can be naturally identified with the set of scalar integrals $\{ \bar{f} := \int_{\mathbb{S}^1} < f(u), 1 > dx : f(u) \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)\}$, for which evidently $\bar{f} = \bar{h}$, if $f \simeq h \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. On the space of functionals $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ there exists a natural differential $\delta : \mathcal{F}_{\widetilde{\mathbb{A}}}(u) \to \Lambda^1(\widetilde{\mathbb{A}}(u))$ defined for any $\bar{f} \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ as

(2.18) $$\delta \bar{f}(\xi_h) := \int_{\mathbb{S}^1} < f^{\prime,*}(u)(1), h > dx,$$

where the conjugation mapping "$*$" is taken with respect to the bilinear form (2.2) on $\widetilde{\mathbb{A}}$ introduced before. Owing to the definition of the functional gradient $\nabla f(u) := f^{\prime,*}(u)(1)$ for all $u \in \widetilde{\mathbb{A}}$, the expression (2.18) can be equivalently rewritten as

(2.19) $$\delta \bar{f}(\xi_h) = (\nabla f(u), h).$$

Based now on the action (2.19), one can successively determine the whole Grassmann algebra $\Lambda(\widetilde{\mathbb{A}}(u))$ of differential forms on the differential algebra $\widetilde{\mathbb{A}}(u)$, generated by an element $u \in \widetilde{\mathbb{A}}$. In particular, if a closed nondegenerate differential 2-form $\omega^{(2)} \in \Lambda^2(\widetilde{\mathbb{A}}(u))$, $\delta\omega^{(2)} = 0$, is given on $\widetilde{\mathbb{A}}(u)$, then via the well known [1, 3, 13] differential geometric expression

$$-\omega^{(2)}(\xi_{\bar{f}}, \xi_{\bar{g}}) := \{\bar{f}, \bar{g}\}_{\omega^{(2)}}, \tag{2.20}$$

where, by definition, for any $\bar{f}, \bar{g} \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$

$$\delta\bar{f}(\cdot) := \omega^{(2)}(\xi_{\bar{f}}, \cdot) = (\delta u, \nabla\bar{f}(u)), \quad \delta\bar{g}(\cdot) := \omega^{(2)}(\xi_{\bar{g}}, \cdot) = (\delta u, \nabla\bar{g}(u)), \tag{2.21}$$

on the differential algebra $\widetilde{\mathbb{A}}(u)$ one determines for any $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$ the corresponding Hamiltonian operator $\vartheta(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}^*_{\widetilde{\mathbb{A}}}$ via the identification

$$\{\bar{f}, \bar{g}\}_{\omega^{(2)}} := (\vartheta(u)\nabla\bar{f}(u), \nabla\bar{g}(u)). \tag{2.22}$$

Within the notions introduced above one can easily state the following [35, 44, 45] proposition.

Proposition 2.4. *Assume that the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ allows a skew-symmetric nondegenerate $\mathcal{R}$-structure homomorphism $\mathcal{R} : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, satisfying the generalized Yang-Baxter condition*

$$[\mathcal{R}a, \mathcal{R}b] - \mathcal{R}([\mathcal{R}a, b] + [a, \mathcal{R}b]) = -\alpha[a, b] \tag{2.23}$$

for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ and $\alpha \in \mathbb{K}$. Then differential 2-forms $\omega_j^{(2)} \in \Lambda^2(\widetilde{\mathbb{A}}(u)), j = \overline{1, 2}$, on the algebra $\widetilde{\mathbb{A}}(u)$ defined as

$$\omega_1^{(2)}(\nabla_{\bar{f}}, \nabla_{\bar{g}}) := (\nabla\bar{f}(u), \mathcal{R}^{-1}\nabla\bar{g}(u)) \tag{2.24}$$

and

$$\omega_2^{(2)}(\nabla_{\bar{f}}, \nabla_{\bar{g}}) := (u, [\mathcal{R}\nabla\bar{f}(u), \mathcal{R}\nabla\bar{g}(u)]) \tag{2.25}$$

for any $\bar{f}, \bar{g} \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ are closed. Moreover, the corresponding Hamiltonian operators, determined from the relationships (2.24) and (2.25) via the identifications

$$\omega_1^{(2)}(\nabla_{\bar{f}}, \nabla_{\bar{g}}) := (\vartheta_1\nabla\bar{f}, \nabla\bar{g}), \quad \omega_2^{(2)}(\nabla_{\bar{f}}, \nabla_{\bar{g}}) := (\vartheta_2\nabla\bar{f}, \nabla\bar{g}), \tag{2.26}$$

are compatible, that is the sum $\lambda\vartheta_1 + \mu\vartheta_2 : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}^*_{\widetilde{\mathbb{A}}}$ for arbitrary $\lambda, \mu \in \mathbb{K}$ is also a Hamiltonian operator.

In a similar way as above one can state that the validity of the following [35, 31, 44, 45] so called "quadratic" compatibility proposition.

Proposition 2.5. *Let a skew-symmetric $\mathcal{R}$-structure $\mathcal{R}: \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ satisfy the Yang-Baxter condition (2.28). Then the following brackets*

$$\{\bar{f}, \bar{g}\}_1 := (u \circ \nabla \bar{f}(u), \mathcal{R}(u \circ \nabla \bar{g}(u))) - (\nabla \bar{f}(u) \circ u, \mathcal{R}(\nabla \bar{g}(u) \circ u)) \tag{2.29}$$

and

$$\{\bar{f}, \bar{g}\}_2 := (u, [\mathcal{R}\nabla \bar{f}(u), \nabla \bar{g}(u)] + [\nabla \bar{f}(u), \mathcal{R}\nabla \bar{g}(u)]) \tag{2.30}$$

defined for any $\bar{f}, \bar{g} \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, are Poissonian and compatible on $\tilde{\mathbb{A}}(u)$.

3. Algebraic structures related with Hamiltonian operators

We start with posing the following problem:

Problem: What conditions should an a priori taken algebra $\mathbb{A}$ *satisfy that the respectively constructed operator* $\vartheta(u) : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}^*_{\tilde{\mathbb{A}}}$, $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$, *appeared to be Hamiltonian?*

From the analysis provided above we know well that if this operator $\vartheta(u) : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$ corresponds to some 2-cocycle on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, then it will be *a priori* Hamiltonian. Moreover, owing to Theorem 2.3, if this 2-cocycle is generated by some differentiation $\mathrm{D}_u : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$, $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$, on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, the problem posed above reduces to direct checking the related Leibniz property (2.11) in $\mathcal{L}_{\tilde{\mathbb{A}}}$.

Example 1. To realize this scheme we proceed with considering for any $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$ a simple skew-symmetric differentiation $\mathrm{D}_u := L_u^* D_x + D_x L_u : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ acting as

$$(a, \mathrm{D}_u b) \quad : \quad = (a, (L_u^* D_x + D_x L_u) b) \; = \tag{3.1}$$

$$= \quad (u \circ a, D_x b) + (a, D_x u \circ b) + (a, u \circ D_x b)$$

for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, being parametrized by an arbitrary yet fixed element $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$ and modeling exactly the Hamiltonian operator analyzed before in [28, 9]. To verify that the expression (3.1) is a differentiation on the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, it is enough to check that the following three-linear *weak Leibniz relationship*

$$(a, \mathrm{D}_u[b, c]) = (a, [\mathrm{D}_u b, c] + [b, \mathrm{D}_u c]) \tag{3.2}$$

holds for any a, b and $c \in \mathcal{L}_{\tilde{\mathbb{A}}}$. As a result of simple enough calculations, taking into account that elements $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}}$ and $D_x u \in \mathcal{L}^*_{\tilde{\mathbb{A}}}$ are functionally independent, one obtains that the expression (3.1) is a skew-symmetric weak differentiation of the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ iff there are imposed on the algebra $\mathbb{A}$ the following two algebraic constraints:

$$L_{[a,b]} = [L_a, L_b], \quad [R_a, R_b] = 0, \tag{3.3}$$

where for any $a, b \in \mathbb{A}$ the expressions $L_a b := a \circ b$ and $R_a b := b \circ a$ denote, respectively, the induced left and right shifts on the algebra $\mathbb{A}$. The obtained commutator expressions (3.3) on the algebra $\mathbb{A}$ coincide exactly with those that determine [38] the *Novikov algebra* by means of the relationships

$$(3.4) \qquad (a \circ b) \circ c = (a \circ c) \circ b,$$

$$(a \circ b) \circ c - (b \circ a) \circ c = a \circ (b \circ c) - b \circ (a \circ c),$$

which hold for any a, b and $c \in \mathbb{A}$, and derived before in [28, 9].

Thus, we have stated once more that the expression (3.1) determines a Hamiltonian operator on the functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$, which *a priori* is compatible with that defined by the canonical Lie-Poisson bracket (2.6), owing to its central extension origin. The same scheme can be, evidently, applied to other skew-symmetric operators on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, being not necessary differential.

Remark 3.1. The same way one can construct a new dual Novikov algebra $\mathbb{A}$, related with the differentiation $\mathrm{D}_u = R_u^* D_x + D_x R_u : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$, for which the following relationships

$$R_{[a,b]} = [R_b, R_a], \quad [L_a, L_b] = 0, \tag{3.5}$$

hold for any a, b and $c \in \mathbb{A}$.

Example 2. We will check a skew-symmetric expression $\mathrm{D}_u := L_u^* D_x^{-1} + D_x^{-1} L_u : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ for $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$ on $\mathcal{L}_{\tilde{\mathbb{A}}}$, acting as

$$(a, \mathrm{D}_u b) := (u \circ a,\ D_x^{-1} b) + (a, D_x^{-1}(u \circ b)) \tag{3.6}$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$, where the *"inverse differentiation"* $D_x^{-1} : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ is naturally defined by means of the identity relationship $D_x \cdot D_x^{-1} = I$ on $\widetilde{\mathbb{A}}$. From (3.6) one easily obtains that

$$(a, \mathrm{D}_u b) = (u \circ a, D_x^{-1} b > +(a, u \circ D_x^{-1} b) - (a, D_x u \circ D_x^{-2} b) + \ldots \tag{3.7}$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$, being equivalent to a suitable pseudo-differential expression. Based on (3.7) one can easily enough derive that the expression (3.6) determines a skew-symmetric differentiation on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ iff the following algebraic constraints on the algebra $\mathbb{A}$

$$R_{b\circ a} = [R_a, R_b], \quad R_{a\circ b} + R_{b\circ a} = 0 \tag{3.8}$$

are satisfied for any $a, b \in \mathbb{A}$. The found above constraints (3.8) mean that the corresponding algebra $\mathbb{A}$ coincides exactly with the so called [33] *right Leibniz algebra*, defined by the following relationships

$$\begin{gathered} c\circ(b\circ a) = (c\circ a)\circ b - (c\circ b)\circ a, \\ c\circ(a\circ b) + c\circ(b\circ a) = 0 \end{gathered} \tag{3.9}$$

for any a, b and $c \in \mathbb{A}$, also intensively studied [2, 16, 17, 33, 27] in the literature.

Example 3. Now we will consider a skew-symmetric expression $\bar{\mathrm{D}}_u := D_x L_u^* D_x^{-1} - D_x^{-1} L_u D_x : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ for $u \in \tilde{\mathbb{A}}^* \simeq \tilde{\mathbb{A}}$ on $\mathcal{L}_{\tilde{\mathbb{A}}}$, acting as

$$(a, \bar{\mathrm{D}}_u\, b) \ : \ = (D_x u \circ a, D_x^{-1} b) + (a, D_x u \circ D_x^{-1} b) + (u \circ a, b) - (a, u \circ b) -$$

(3.10)

$$-(a, D_x^2 u \circ D_x^{-2}\, b) + (a, D_x^3 u \circ D_x^{-3} b) + \ldots$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$, being equivalent to a suitable pseudo-differential expression. Having checked the corresponding Leibniz property (3.2), we can obtain that (3.10) determines a skew-symmetric differentiation on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ iff the following algebraic constraints

(3.11) $$[R_a, R_b] = 0, \ L_{a \circ b} = R_{a \circ b} \ = L_{b \circ a}$$

hold for any $a, b \in \mathbb{A}$. The latter is equivalent to the following relationships on the algebra $\mathbb{A}$:

$$
\begin{aligned}
(c \circ a) \circ b &= (c \circ b) \circ a, \\
(a \circ b) \circ c &= c \circ (a \circ b) = (b \circ a) \circ c,
\end{aligned}
\tag{3.12}
$$

satisfied for any a, b and $c \in \mathbb{A}$. The obtained non-associative algebraic structure (3.12), called Riemann algebra owing to its applications to the integrable Riemann type multi-component hierarchies, can be obtained as a special reduction of the Novikov algebra (3.4) and characterized by the following Theorem.

Theorem 3.2. *Let an algebra $\mathbb{A}$ satisfy the algebraic conditions (3.12). Then the associated Lie algebra $\mathcal{L}_{\mathbb{A}} := (\mathbb{A}, [\cdot,\cdot])$ is nilpotent of length 2.*

Proof. Really, since relationships (3.12) entail that $[\mathbb{A},\mathbb{A}] \subset \mathbb{A}^2 \subset Z(\mathbb{A})$, where $Z(\mathbb{A})$ is the center of the algebra $\mathbb{A}$, the statement above holds. $\square$

It is here worthy to mention that similar four-dimensional over the complex numbers field $\mathbb{C}$ algebras, whose associated Lie algebras are n-th nilpotent, were before studied in [19].

Concerning the expression (3.10), we can easily observe that for the case of the loop algebra $\widetilde{\mathbb{A}}$ it is formally equivalent to the skew-symmetric differentiation $\bar{\mathrm{D}}_u := (L^*_{D_x u})D_x^{-1} + D_x^{-1}(L_{D_x u}) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, acting as

$$(b, \bar{\mathrm{D}}_u a) \quad : \quad = (D_x u \circ a, \;\; D_x^{-1} b) + (a, D_x u \circ D_x^{-1} b) +$$

(3.13)

$$- \quad < \;\; D_x^2 u, D_x^{-2} a \circ b > + < D_x^3 u, D_x^{-3} a \circ b > - ...$$

for any a, b and $c \in \mathcal{L}_{\widetilde{\mathbb{A}}}$. Really, owing to (2.6), the component $(u, [a, b]) = ([u, a], b) := (Ad_u a, b)$ for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ generates the trivial 2-cocycle $\bar{\omega}_2(a, b) := (Ad_u a, b)$, since the adjoint mapping $Ad_u : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ determines, evidently, a skew-symmetric differentiation of the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$.

It is noteworthy to remark here that simultaneously we have stated that the expressions (3.1), (3.6) and (3.13) determine Hamiltonian operators on a suitably constructed functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$, *a priori* compatible with that defined by the canonical Lie-Poisson bracket (2.6) owing to their central extension origin. In particular, following the algebraic scheme of the before mentioned work [47], based on the derived above right Lorenz algebra (3.9) and the new Riemann algebra (3.12) one can describe a wide class of multicomponent completely integrable dynamical systems containing, as follows from the recent results in [41], infinite hierarchies of the hydrodynamical Riemann type systems.

For instance, consider the generalized Riemann type dynamical system

$$D_t u_1 = u_2, D_t u_2 = u_3, \ldots, D_t u_N = 0, \tag{3.14}$$

where, by definition, $D_t := \partial/\partial t + u_1 D_x$, $u_j :=< u, e_j >, j = \overline{1, N}$, for $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$ and the set $\{e_j \in \mathbb{A} : j = \overline{1, N}\}$ for a fixed $N \in \mathbb{Z}_+$ is the basis for a suitable representation of the new N-dimensional non-associative Riemann algebra (3.12). The differentiation $\mathrm{D}_u : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, defined by (3.10) allows to calculate with respect to the mentioned above representation for cases $N = 2$ and $N = 3$ the corresponding Hamiltonian operators, coinciding with those constructed before in [41] modulo the trivial constant 2-cocycles on the loop Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$.

Example 4. Really, based on the relationships (3.12) for the case $N = 2$, one easily obtains the skew-symmetric two-dimensional matrix differentiation representation

$$\bar{\mathrm{D}}_u^{(2)} := \begin{pmatrix} 0 & u_{1,x} D_x^{-1} \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} \end{pmatrix}, \tag{3.15}$$

coinciding, modulo the trivial constant 2-cocycle

$$\bar{\omega}_2(a, b) := f_2(D_x^{-1} a, b), \tag{3.16}$$

determined for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ and a suitable symmetric bilinear form $f_2 : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \mathbb{K}$, with the Hamiltonian operator

$$\eta_2(u) = \begin{pmatrix} D_x^{-1} & u_{1,x} D_x^{-1} \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} \end{pmatrix}, \tag{3.17}$$

on the space $\widetilde{\mathbb{A}}(u)$ for the Riemann type dynamical system (3.14), whose Hamiltonian representation

$$\frac{d}{dt}(u_1, u_2)^\intercal = -\eta_2(u) \nabla H_2(u_1, u_2) \tag{3.18}$$

holds for the Hamiltonian function $H_2 \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, defined as

$$H_2 := \frac{1}{2} \int_0^{2\pi} (u_2 D_x u_1 - u_1 D_x u_2) dx. \tag{3.19}$$

Proceeding similarly to the case $N = 3$, one can easily obtain the skew-symmetric three-dimensional matrix differentiation $\bar{\mathrm{D}}_u^{(3)} : \mathcal{L}_{\tilde{\mathrm{A}}} \to \mathcal{L}_{\tilde{\mathrm{A}}}$ representation

$$\bar{\mathrm{D}}_u^{(3)} = \begin{pmatrix} 0 & u_{1,x}D_x^{-1} & 0 \\ D_x^{-1}u_{1,x} & u_{2,x}D_x^{-1} + D_x^{-1}u_{2,x} & D_x^{-1}u_{3,x} \\ 0 & u_{3,x}D_x^{-1} & 0 \end{pmatrix}, \tag{3.20}$$

coinciding, modulo the trivial constant 2-cocycle

$$\bar{\omega}_2(a,b) := f_3(D_x^{-1}a, b) \tag{3.21}$$

determined for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ and a suitable symmetric bilinear form $f_3 : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \mathbb{K}$, with the Hamiltonian operator

$$\eta_3(u) = \begin{pmatrix} D_x^{-1} & u_{1,x} D_x^{-1} & 0 \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} & D_x^{-1} u_{3,x} \\ 0 & u_{3,x} D_x^{-1} & 0 \end{pmatrix} \tag{3.22}$$

on the space $\widetilde{\mathbb{A}}(u)$ for the Riemann type dynamical system (3.14), whose Hamiltonian representation

$$\frac{d}{dt}(u_1, u_2, u_3)^{\intercal} = -\eta_3(u) \nabla H_3[u_1, u_2, u_3] \tag{3.23}$$

holds for a suitably constructed Hamiltonian function $H_3 \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$.

Example 5. Proceed now to an interesting observation concerning an infinite hierarchy [41] of the generalized Riemann type hydrodynamic systems

$$D_t u_1 = u_2, D_t u_2 = u_3, ..., D_t u_{N-1} = (D_x \bar{u}_N)^s, \ D_t \bar{u}_N = 0 \tag{3.24}$$

on the functional space $\tilde{\mathbb{A}}(u)$, where $s, N \in \mathbb{Z}_+$ and the algebra $\mathbb{A}$ is generated by the relationships (3.9). For the case $s = 2$ and $N = 3$ the found before skew-symmetric three-dimensional matrix differentiation $\bar{\mathrm{D}}_u^{(2|3)} : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ representation (3.20) in the form

$$\bar{\mathrm{D}}_u^{(2|3)}(u) = \begin{pmatrix} 0 & u_{1,x} D_x^{-1} & 0 \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} & D_x^{-1} \bar{u}_{3,x} \\ 0 & \bar{u}_{3,x} D_x^{-1} & 0 \end{pmatrix} \tag{3.25}$$

proves to coincide, modulo the trivial constant 2-cocycle

$$\bar{\omega}_{2,\bar{\eta}}(a,b) := f_{\bar{\eta}}(D_x^{-1}a,b) \tag{3.26}$$

determined for any $a,b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ and a suitable symmetric bilinear form $f_{\bar{\eta}}$: $\widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \mathbb{K}$, exactly with the Hamiltonian operator

$$\bar{\eta}_{2|3}\ (u) = \begin{pmatrix} D_x^{-1} & u_{1,x}D_x^{-1} & 0 \\ D_x^{-1}u_{1,x} & u_{2,x}D_x^{-1} + D_x^{-1}u_{2,x} & D_x^{-1}\bar{u}_{3,x} \\ 0 & \bar{u}_{3,x}D_x^{-1} & 0 \end{pmatrix} \tag{3.27}$$

on the space $\widetilde{\mathbb{A}}(u)$ for the Riemann type dynamical system (3.24), whose Hamiltonian representation

$$\frac{d}{dt}(u_1,u_2,\bar{u}_3)^{\intercal} = -\bar{\eta}_{2|3}\ (u)\nabla H_{\bar{\eta}_{2|3}}\ (u_1,u_2,\bar{u}_3) \tag{3.28}$$

holds for the Hamiltonian function $H_{\bar{\eta}_{2|3}} \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, defined as

$$H_{\bar{\eta}_{2|3}} := \frac{1}{2}\int_0^{2\pi}[2u_1(D_x\bar{u}_3)^2 - u_2^2 - u_1^2 D_x u_2]dx. \tag{3.29}$$

Moreover, one can additionally calculate such a constant 2-cocycle on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$

$$\bar{\omega}_{2,\bar{\vartheta}}(a,b) := f_{\bar{\vartheta}}^{(1)}(D_x^{-1}a,b) + f_{\bar{\vartheta}}^{(2)}(a,b),$$

determined for any $a,b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ by means of two suitably symmetric bilinear forms $f_{\bar{\vartheta}}^{(1)} : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \mathbb{K}$ and skew-symmetric bilinear forms $f_{\bar{\vartheta}}^{(2)} : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \mathbb{K}$, which naturally generates the compatible with (3.27) Hamiltonian operator

$$\bar{\vartheta}_{2|3}(u) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1/2D_x^{-1} \end{pmatrix} \tag{3.30}$$

on the space $\widetilde{\mathbb{A}}(u)$ for the same Riemann type dynamical system (3.24), whose Hamiltonian representation

$$\frac{d}{dt}(u_1, u_2, \bar{u}_3)^{\intercal} = -\bar{\vartheta}_{2|3}(u)\nabla H_{\bar{\vartheta}_{2|3}}(u_1, u_2, \bar{u}_3) \tag{3.31}$$

holds for the Hamiltonian function $H_{\bar{\vartheta}_{2|3}} \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, defined as

$$H_{\bar{\vartheta}_{2|3}} := \frac{1}{2}\int_0^{2\pi}[u_1\, D_x u_2 - u_2\, D_x u_1 - 2(D_x \bar{u}_3)^2]dx. \tag{3.32}$$

It is interesting to note here, as was already remarked in [42], that the generalized Riemann type hydrodynamic system (3.24) for $s = 3, N = 3$ reduces to the well-known integrable Degasperis-Processi dynamical system [21, 20] for the function $u := u_1$:

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{3.33}$$

Respectively, for $s = 2, N = 3$ the generalized Riemann type hydrodynamic system (3.24) for the function $u := u_1$ reduces to the well known [7] integrable Camassa-Holm dynamical system

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} + uu_{xxx} = 0, \tag{3.34}$$

whose different multi-component extensions were recently extensively studied in [26, 29, 8, 47]. As the mentioned above relationships present a nontrivial interest from mathematical point of view, they are worthy of further investigation. In particular, these reductions are of special interest from the equivalence transformation point of view, devised recently in [48].

4. The Poisson manifolds approach revisiting

It is interesting to look at the construction of the Hamiltonian operators presented above and revisit it from the standard point of view, considering them as those defined on the naturally associated [1, 3, 15, 13, 36, 37, 43] cotangent space $T^*(M)$ to some linear functional manifold $M \simeq \widetilde{\mathbb{A}}$. Then a Hamiltonian operator on M is defined [1] as smooth mapping $\vartheta : M \to Hom(T^*(M); T(M))$, such that for any fixed $u \in M$ the bracket

$$\{f, g\} := (\nabla f(u), \vartheta(u)\nabla g(u)), \tag{4.1}$$

where $f, g : M \to \mathbb{K}$ are arbitrary smooth mappings from the functional space

$\mathcal{D}(M) \simeq \mathcal{F}_{\tilde{\mathbf{A}}}(u)$, satisfies the Jacobi identity. The bracket (4.1) is determined on M by means of the natural convolution $(\cdot,\cdot)$ on the product $T^*(M) \times T(M)$, and respectively, the gradient $\nabla f(u) \in T^*(M)$ of a function $f \in \mathcal{D}(M)$ is calculated via the relationship

$$(\nabla f(u), h) := df[u + \varepsilon h]/d\varepsilon|_{\varepsilon=0} \tag{4.2}$$

for any $h \in T(M)$. It is well known [28, 32] that a linear operator $\vartheta(u) : T^*(M) \to T(M)$, determined at any point $u \in M$, is Hamiltonian iff the suitably defined Nijenhuis bracket

$$[[\vartheta(u), \vartheta(u)]] = 0 \tag{4.3}$$

identically on M. Namely, based on this condition (4.3) in the works [28, 46] there were formulated criteria for the operator $\vartheta(u) : T^*(M) \to T(M)$ to be

Hamiltonian on the functional manifold M. Yet these criteria appear to be very complicated and related with a large amount of cumbersome calculations even in the case of simple enough differential expressions. So, we have reanalyzed this problem from a slightly different point of view. First, recall that the Jacobi identity for the bracket (4.1) is completely equivalent to the fact that the bracket operator defined as $D_f(g) := \{f, g\}$ for a fixed $f \in \mathcal{D}(M)$ and arbitrary $g \in \mathcal{D}(M)$ acts as a differentiation on the space $(\mathcal{D}(M); \{\cdot, \cdot\})$:

$$D_f\{g, h\} = \{D_f(g), h\} + \{g, D_f(h)\}, \tag{4.4}$$

where $g, h \in \mathcal{D}(M)$ are taken arbitrary. The latter can be easily reformulated as follows: take any element $\varphi \in T^*(M)$, such that the Frechet derivative $\varphi'(u) = \varphi'^{,*}(u)$ at any $u \in M$ with respect to the convolution $(\cdot, \cdot)$ on $T^*(M) \times T(M)$, and construct at any $u \in M$ a vector field $K : M \to T(M)$ as

$$K(u) := \vartheta(u)\varphi(u). \tag{4.5}$$

Then the differentiation condition (4.4) can be equivalently rewritten [1, 36, 13, 37, 43] as the correspondingly vanishing Lie derivative of the operator $\vartheta(u) : T^*(M) \to T(M)$ along the vector field (4.5)

$$L_K\vartheta = \vartheta' \cdot K - \vartheta K'^{,*} - K'\vartheta = 0 \tag{4.6}$$

at any $u \in M$ for every "*self-adjoint*" element $\varphi \in T^*(M)$. Equivalently,

a taken linear operator $\vartheta(u) : T^*(M) \to T(M)$ is Hamiltonian iff the Lie derivative (4.6) weakly vanishes for all "self-adjoint" elements $\varphi \in T^*(M)$. Moreover, as it was observed in [34], it is enough to check the condition (4.6) only on the subspace of elements $\varphi \in T^*(M)$, satisfying the condition $\varphi'(u) = 0$ for any $u \in M$.

In particular, it is enough to check that a skew-symmetric matrix-differential operator on M of the form

$$\vartheta(u) := a(u)D_x + D_x a^{\intercal}(u), \tag{4.7}$$

where, by definition, an n-dimensional squared matrix $a(u) := (\sum_{s=1}^{n} u_s a_{ij}^s : i,j = \overline{1,n}, n \in \mathbb{Z}_+)$, $u \in M$, satisfies the condition (4.6) iff the linearly independent elements from $\underset{\mathbb{K}}{span}\{e_j \in \mathbb{A} : j = \overline{1,n}\}$ generate the finite dimensional non-associative Novikov algebra (3.4) and satisfy the conditions $e_i \circ e_j = \sum_{s=1}^{n} a_{ij}^s e_s$ for all $i,j = \overline{1,n}$. Similarly, one can verify that the skew-symmetric inverse-differential operator

$$\vartheta(u) := a(u)D_x^{-1} + D_x^{-1}a^{\intercal}(u), \tag{4.8}$$

where, as above $a(u) := (\sum_{s=1}^{n} u_s a_{ij}^s : u \in M, i,j = \overline{1,n}, n \in \mathbb{Z}_+)$, the sign " $\intercal$ " means the usual matrix transpotion, is Hamiltonian iff the basic non-associative algebra $\mathbb{A} := \underset{\mathbb{K}}{span}\{e_j : j = \overline{1,n}\}$ coincides with the right Leibniz algebra (3.9) and the conditions $e_i \circ e_j = \sum_{s=1}^{n} a_{ij}^s e_s$ for any $i,j = \overline{1,n}$ hold. The skew-symmetric inverse-differential operator (4.8) can be naturally generalized to the expression

$$\vartheta(u) := D_x a(u) D_x^{-1} - D_x^{-1} a^\intercal(u) D_x, \tag{4.9}$$

which can be rewritten equivalently as

$$\vartheta(u) = a(D_x u) D_x^{-1} + D_x^{-1} a^\intercal(D_x u) + a(u) - a^\intercal(u). \tag{4.10}$$

The condition (4.6) for the operator (4.10) to be Hamiltonian reduces to the constraints on the related non-associative algebra $\mathbb{A} := span_{\mathbb{K}}\{e_j : j = \overline{1,n}\}$ exactly coinciding with (3.12), and analyzed in some details in Section 3.

As was already mentioned, based on the matrix representations of the right Leibniz algebra (3.9) and the new non-associative Riemann algebra (3.12) one can construct differentiations $D_u : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ on the associated Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ generating many different Hamiltonian operators suitable for describing a wide class of multi-component hierarchies [14, 41, 42] of integrable hydrodynamic Riemann type systems and their different reductions.

5. Conclusion

In this work we succeeded in formal differential-algebraic reformulating the criteria [28, 46, 34] for a given differential expression to be Hamiltonian and developed an effective approach to classification of the algebraic structures lying in the background of the integrable multicomponent Hamiltonian systems. We have devised a simple algorithm allowing to construct new algebraic structures within which the corresponding Hamiltonian operators exist and generate integrable multicomponent dynamical systems. We showed, as examples, that the well known Novikov algebraic structure, obtained before in [28, 9] as a condition for a matrix differential expression to be Hamiltonian, appears within the devised approach as a differentiation on the Lie-algebra, naturally associated with a suitably constructed differential loop algebra. By means of a direct generalization of this example it is obtained new differentiations, whose background algebraic structures coincide, respectively, with the right Leibniz algebra, introduced in [16, 17, 33] and with a new "Riemann" non-associative algebra, which generate Hamiltonian operators, describing a wide class of multi-component hierarchies [14, 41] of integrable multicomponent hydrodynamic Riemann type systems. Their reductions appeared to be closely related both to the integrable Camassa-Holm and with the Degasperis-Processi dynamical systems, and are of special interest from the equivalence transformation point of view, devised recently in [48]. We also briefly revisited a classical Poisson manifold approach to constructing Hamiltonian operators on functional manifolds.

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AGH

Dziękuję serdecznie za uwagę!

Thanks for your attention!