

XXXV Workshop on Geometric Methods in Physics

Biały Las, Poland, June 27-July 3, 2010

Elliptic and Hyperelliptic Fibrations in Integrability

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“It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of the existence of laws of nature and of the human mind’s capacity to divine them. The observation which comes closest to an explanation for the mathematical concepts’ cropping up in physics which I know is Einstein’s statement that the only physical theories which we are willing to accept are the beautiful ones.”

Eugene Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, in Communications in Pure and Applied Mathematics, vol. 13, 1960 1-14

“The aspect of mathematics which fascinates me most is the rich interaction between its different branches, the unexpected links, the surprises, and my aim will be to illustrate this by considering some simple problems.”

M.F. Atiyah, *The Unity of Mathematics*, Bull. London Math. Soc. (1978) 10 (1): 69-76.

"Algebraic" Integrability: A Two-way street

One Direction

Korteweg and de Vries modeled shallow-water waves

$$u_t + 6uu_x + u_{xxx} = 0 \quad (\text{KdV})$$

(after suitably changing scales M. Toda, Nonlinear Waves and Solitons Ch. 5)

A stationary wave propagating with velocity c

depends on $X = x - ct$. After two integrations,

$$\frac{1}{2}(u_X)^2 = -u^3 + \frac{c}{2}u^2 + Du + E \quad \begin{matrix} D, E \text{ constants of} \\ \text{integration} \end{matrix}$$

When the RHS has repeated roots as a polynomial in u ,
the solution is an elementary function, e.g. $D = E = 0$

$$u = \frac{c}{2} \operatorname{sech}^2 \left[\frac{\sqrt{c}}{2} (X + \delta) \right] \quad S \text{ const. of integration}$$

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2 \kappa (x - ct + \delta) \quad c = 4\kappa^2$$

But when $E - U(u) = (u - c_1)(u - c_2)(c_3 - u)$ with
 $= \frac{1}{2}(u_X)^2$

distinct c_i , $u = u_\infty + A \operatorname{cn}^2 \kappa X$ is periodic, with modulus k ,

$$A = 2\kappa^2 k^2, \quad u_\infty = \frac{1}{6}c_2, \quad \kappa^2 = \frac{1}{12}(c_3 - c_1), \quad A = \frac{1}{6}(c_3 - c_2)$$

The Other Direction

$$\phi^{(\pm)} = -\frac{d^2}{dz^2} \ln \sigma(z), \quad (\phi')^2 = 4\phi^3 - g_2\phi - g_3$$

(KdV) $\Leftrightarrow (D_x D_t + D_x^4) F \cdot F = 0$ for Hirota's bilinear operator

$$D_{t_n} F \cdot F := \left(\frac{\partial}{\partial t_n} - \frac{\partial}{\partial t'_n} \right) F(t) F(t') \Big|_{t=t', t=(t_1, t_2, \dots)}$$

$$\mu = 2 \frac{\partial^2}{\partial x^2} \ln F$$

$$y^2 = x^{2g+1} + \lambda_{2g+1} x^{2g} + \dots + \lambda_0 f(x) \text{ hyperelliptic curve } X$$

e.g., $g=3$

$$(D_3^4 - 4\lambda_6 D_3^2 - 4D_3 D_2 - 4\lambda_5 \lambda_7) \sigma \cdot \sigma = 0$$

H.F. Baker, On the hyperelliptic sigma functions, Amer. J. Math 20 (1898), 301-384

Hyperelliptic sigma functions

$$\omega_i = \frac{x^{i-1} dx}{2y} \quad i=1, \dots, g \quad 1^{\text{st}} \text{ kind}$$

$$\eta_j = \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j) \lambda_{k+1+j} x^k dx \quad j=1, \dots, g \quad 2^{\text{nd}} \text{ kind}$$

1/2 period matrices

$$\omega' = \frac{1}{2} \oint_{\alpha_j} \omega_i \quad \omega' = \frac{1}{2} \oint_{\beta_j} \omega_i$$

$$\eta' = \frac{1}{2} \oint_{\alpha_j} \eta_i \quad \eta' = \frac{1}{2} \oint_{\beta_j} \eta_i$$

(α_j, β_j) symplectic homology basis

The generalized Legendre relation holds:

$$M \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix} M^t = \frac{i\pi}{2} \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$$

The theta function with 'modulus' Ω and characteristics $\Omega a + b$ $a, b \in \mathbb{C}^g$ is:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z; \Omega) := \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left[\frac{1}{2} {}^t(n+a)\Omega(n+a) + {}^t(n+a)(z+b) \right]$$

and the "Kleinian" σ function is

$$\sigma(u) := \gamma \exp \left(-\frac{1}{2} {}^t u \eta' \omega'^{-1} u \right) \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \left(\frac{1}{2} \omega'^{-1} u; \Omega \right)$$

with $\Omega \delta'' + \delta'$ the Riemann constant, and γ a non-zero constant (depending on $\lambda_0, \dots, \lambda_{2g+1}$)

N.B. σ is a modular function under the action of $Sp(2g, \mathbb{Z})$

By analogy with $g=1$, $\delta_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u)$, $\xi_i = \frac{\partial}{\partial u_i} \ln \sigma(u)$

Moreover, let

$$\sigma_{\#}(u) = \left(\prod_i \frac{\partial}{\partial u_i} \right) \sigma(u) \quad \begin{array}{ll} i=3, 4, \dots, g-1 & \text{if } g \text{ odd} \\ i=2, 4, \dots, g & \text{if } g \text{ even} \end{array}$$

$$\sigma_b(u) = \left(\prod_i \left(\frac{\partial}{\partial u_i} \right) \right) \sigma(u) \quad \begin{array}{ll} i=3, 5, \dots, g-1 & \text{if } g \text{ even} \\ i=3, 5, \dots, g & \text{if } g \text{ odd} \end{array}$$

One Direction

Toda ("Vibration of a Chain with Nonlinear Interaction",
 J. Phys. Soc. Japan 22 (1967) 431-436) following
 computer simulation derived Hamiltonian for a chain of
 particles with nearest-neighbor interaction, exponential potential
 and found elliptic-function solutions. In terms of Weierstrass' \wp ,
 differentiate twice

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\sigma(u-v)}{[\sigma(u)\sigma(v)]^2}$$

let $v=u_0$ a constant, $V_n(t) = -\wp(nu_0 + t + t_0)$ any integer n ,

t_0 a constant, $V_c = \wp(u_0)$, $q_n = -\ln [V_n(t) - V_c]$, then

$$-\frac{d^2}{dt^2} q_n = e^{-Q_{n+1}} - 2e^{-Q_n} + e^{-Q_{n-1}} \quad \text{or}$$

$$-\frac{d^2}{dt^2} Q_n = e^{Q_n - Q_{n+1}} - e^{Q_{n-1} - Q_n}, \quad q_n = Q_n - Q_{n-1}$$

N.B. Toda used Jacobi's elliptic functions, $y^2 = (x-e_1)(x-e_2)(x-e_3)$

modulus k , $k^2 = (e_2 - e_3)(e_1 - e_3)$, let $t_0 = -\omega''$

$$\wp(t + nu_0 - \omega'') - \wp(t + n u_0) = (e_1 - e_3) n s^2((e_1 - e_3)^{1/2}(t + n u_0)) + e_3.$$

The Other Direction

Y. Kodama S. Matsutani 6.P.
Annales de l'Inst. Fourier (2013) 635-688

For $(x_i, y_i) \in X$ $i=1 \dots g$, $(x', y') \in X$, $u = w((x_1, y_1), \dots, (x_g, y_g))$

where $w((x_1, y_1), \dots, (x_g, y_g)) = \sum_{i=1}^g \int_{\infty}^{(x_i, y_i)} \begin{bmatrix} \omega_i \\ \omega_g \end{bmatrix}$ Abel map,

define the operator

$$D = \sum_{i=1}^g x'^{i-1} \frac{\partial}{\partial u_i}$$

the point $c = 2w(x', y')$,

$$\tilde{D} = \sigma_b(c) D,$$

the numbers $P(u) = P(u, c) = \sum_{i=1}^g \sum_{j=1}^g f_{ij}(u) x'^{i+j-2}$

$$P_c(c) = f_{1,2}(x')$$

$$f_{1,2}(x) = \frac{\partial_x^2 f(x)}{2 f(x)} - \sum_{i=0}^g \left(i^2 \lambda_{2i+1} x^{2i-1} + i(i-1) \lambda_{2i} x^{2i} \right)$$

the vector
 $t = (t_1, \dots, t_g)$ $t_j = (x')^{1-j} \sum_{i=1}^g \int_{\infty}^{(x_i, y_i)} \omega_j$ $j=1 \dots g$

and t^\perp such that $u = nc + t^\perp + t$

$$\text{then } -D^2 \ln (P(t+nc+t^\perp) - P_c(c)) = P(t+(n+1)c+t^\perp) - 2P(t+nc+t^\perp) + P(t+(n-1)c+t^\perp)$$

and the Toda-lattice equation

$$-D^2 q_n(t) = e^{-q_{n+1}} - 2e^{-q_n} + e^{-q_{n-1}}$$

$$\text{for } q_n(t) = -\ln (P_n(t+t^\perp) - P_c(c)), \text{ for } P_n(t+t^\perp) = -P(t+nc+t^\perp)$$

What about integrability?

M. Kac and P. van Moerbeke "A complete solution of the periodic Toda problem"
 Proc. Nat. Acad. Sci. U.S.A. 72 (1975) 2879-2880

a completely integrable Hamiltonian system

$$H = \frac{1}{2} \sum_{k=1}^N p_k^2 + \sum_{k=1}^N \exp(Q_k - Q_{k+1})$$

where $P_k = P_{k+N}$, $Q_k = Q_{k+N}$, in Flaschka coordinates

$$\frac{d}{dt} a_k = a_k(b_{k+1} - b_k), \quad k=1, \dots, N$$

$$\frac{d}{dt} b_k = a_k - a_{k-1}$$

Let $\sigma^{(n)}(t, t^\perp) = \sigma(t + nc + t^\perp)$, $\sigma^{(c)} = \sigma_b(c)$

$$\xi^{(n)}(t, t^\perp) = \sum_{i=1}^n x'^{i-1} \xi_i(t + nc + t^\perp), \quad \xi^{(c)} = \frac{1}{2} \tilde{D} \ln \sigma_b(c)$$

$$f^{(n)}(t, t^\perp) = \sum_{i,j=1}^n x'^{i+j-2} f_{ij}(t + nc + t^\perp), \quad f^{(c)} = t^\perp = f_{1,2}(x')$$

then $a_n = f^{(n)}(t, t^\perp) - f^{(c)}(t^\perp)$

$$b_{n-1} = D \left[\frac{\ln[\sigma^{(n)}(t, t^\perp)]}{\sigma^{(n-1)}(t, t^\perp)} \right] - \xi_c$$

satisfy the Kac-van Moerbeke periodic Toda system provided

$$P(n) = P(n + Nc)$$

(a condition in terms of "division polynomials", meromorphic on X ,

example $X: y^2 = x^3 - x$, $x' = 1 + \sqrt{2}$, $N=4$)

Kac-van Moerbeke spectral curve: $\det(L - z) = 0$

$$L = \begin{bmatrix} b_1 & 0 & \cdots & q_N w^{-1} \\ a_1 b_2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & q_{N-1} b_N & \end{bmatrix} \quad \text{has genus } N-1$$

Remarks:

differential-difference Toda flow linear on both Jacobians
 branch points of X_{N-1} are rational functions of x', y'

Why a fibration? An analogy

W. Barth and J. Michel ("Modular curves and Poncelet polygons" Math. Ann. 295 (1993) 25-49) define the modular curve

$$X_{\infty}(k, 2) = \overline{H/M_{\infty}(k) \cap \Gamma(2)} \quad (\text{level-2 str. and point of order } k)$$

in terms of the Poncelet condition in Cayley form ($\det M_k = 0$ where M_k is a matrix whose entries are the coefficients of the formal power series

$$\sqrt{h(u)}, \quad v^2 = h(u) \text{ a cubic}$$

N. Hitchin ("Poncelet polygons and the Painlevé equations" in Geometry and Analysis, Tata Inst. Fund. Res., 1995, 151-185) shows that it is birational to the Painlevé curve

$$\text{PVI} \quad \frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \left(\frac{dy}{dx} \right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \frac{dy}{dx}$$

$$+ \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left(\alpha + \beta \frac{x}{y} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right)$$

corresp. to $(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8} \right)$ and dihedral $D_k \subset SU(2)$

V. Dragovic and V. Shramchenko arXiv:math.AG/1506.06301

higher dimensional Poncelet \leftrightarrow isomonodromy for Schlesinger system

V. P. Burgos and A. S. Zheglov "On Dirichlet, Poncelet and Abel problems" arXiv: 0903.253

$$P_n = (x_n, x_n^2, 1) \in C \quad n=1, 2, \dots, N$$

Satisfies the periodic Toda lattice

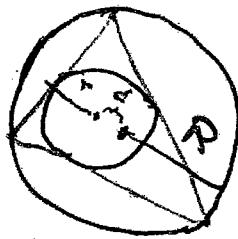
$$x_n = f((n-1)x_0 + t)$$

$$C: y^2 = xz \quad A \\ D: [xy^2] \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

$$E: w^2 = \frac{1}{a_2 + a_4} [xx^2] A \begin{bmatrix} x^2 \\ z^2 \end{bmatrix} = 1$$

QRT maps correspond to an elliptic surface

T.W. Chaundy conditions. 1923 Proc. London Math. Soc.



$$R^2 - d^2 = 2Rr \quad (\text{Euler})$$

condition for n -sided polygon:

Pencil of conics

$$S: x^2 + y^2 + z^2 = 0$$

$$S_\lambda: a_\lambda^2 x^2 + b_\lambda^2 y^2 + c_\lambda^2 z^2 = 0$$

$$a_\lambda^2 - b_\lambda^2, b_\lambda^2 - c_\lambda^2, c_\lambda^2 - a_\lambda^2 \text{ equal } a^2 - b^2, b^2 - c^2, c^2 - a^2$$

$$\text{let } a_\lambda = i \sin u_\lambda, b_\lambda = c \nu u_\lambda, c_\lambda = k^{-1} \ln u_\lambda$$

$$k^2 = (a^2 - b^2) / (a^2 - c^2)$$

$$a_\lambda^2 = f(u_\lambda) - e_1, b_\lambda^2 = f(u_\lambda) - e_2, c_\lambda^2 = f(u_\lambda) - e_3$$

Th. If a variable n -agon be always inscribed in S and $n-1$ of its sides envelop u_1, \dots, u_{n-1} , then n th side envelopes one of the conics $\pm u_1, \pm u_2, \dots, \pm u_{n-1}$ ($\rightarrow \pm u \dots \pm u$ closed polygon $A_0, \dots, A_{n-1}, A_0 \Leftrightarrow u = (n-1)u_1$.

$$\text{If } a_n^2 - a^2 = b_n^2 - b^2 = c_n^2 - c^2 = \lambda_n,$$

addition formula

$$a_{m+n} = \frac{b_n c_n a_m - b_m c_m a_n}{\lambda_n - \lambda_m}$$

$$a_{m-n} = \frac{b_n c_n a_m + b_m c_m a_n}{\lambda_n - \lambda_m}$$

$$\text{e.g. } a_{2n} = \frac{-b_n^2 c_n^2 + c_n^2 a_n^2 + a_n^2 b_n^2}{2 a_n b_n c_n}$$

P[24]

$$[R^2 - d^2 + 2r\sqrt{Rd}] [(R-d)r + (R+d)\sqrt{(r-R+d)(r+R-d)}]$$

$$= 2r(R^2 - d^2) [4Rd(r-R+d)(r+R-d)]^{1/4}$$