Partial Poisson structure on Convenient manifolds

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0. Outline

1. Introduction

The concept of **Poisson structure** is a fundamental mathematical tool in Mathematical Physic and classical Mechanic (specially in finite dimension context) and, in an infinite dimensional context, in hydrodynamic framework, in mechanism for integrating some evolutionary PDE (for example Kdv), quantum mechanic.... In any of these situations, we have an algebra $\mathcal{A}$ of smooth functions on some manifold M (eventually infinite dimensional) which is provided with a Lie bracket $\{\ ,\ \}$ which satisfies the Leibniz property (called a **Poisson bracket**) and to the derivation $g \mapsto \{f, g\}$ in $\mathcal{A}$ we can associate a vector field X_f on M called a the **Hamiltonian vector field** of f.

1 : Continuation

In infinite dimension, when M is a Banach manifold and $\mathcal{A} = \mathcal{C}^{\infty}(M)$ such a framework was firstly defined and studied in a series of papers by A. Odzijewicz, T. Ratiu and their collaborators (2003-2009) (see for instance [6]) and we will see how this context is included in our presentation.
A more recent, approach was also proposed by K.H. Neeb, H. Sahlmann and T. Thiemann (**"Weak Poisson structures"** [4]) when M is a smooth manifold modelled on a l.c.t.v. space : they consider a subalgebra $\mathcal{A}$ of $C^{\infty}(M)$ which is provided with a Poisson bracket and so that the following separation assumption is satisfied :

- $\forall x \in M, \{\, d_x f(v) = 0,\ \forall f \in \mathcal{A}\} \implies \{v = 0\}$.

This condition implies that the Hamiltonian field X_f is defined for any $f \in \mathcal{A}$.

1 : Continuation

Our purpose is to propose, in an infinite dimensional context, a "Poisson framework" for which the Poisson bracket can be defined for some typical **local or global** smooth functions on M. Essentially we consider

- the algebra $\mathcal{A}(M)$ of smooth functions on M whose differential induces a section of a subbundle of $T'M$ of T^*M
- a bundle morphism $P : T'M \to TM$ such that :
- $\{f, g\}_P = dg(P(df))$ defines a Poisson bracket on $\mathcal{A}$.

Note that under the assumptions of "weak Poisson structures" the vector spaces Δ_x generates by $\{df(x), \; : \; f \in \mathcal{A}\}$ does not give rise to a subbundle of T^*M in general. However we have a well defined linear map $P_x : \Delta_x \to T_xM$ such that $P_x(df(x)) = X_f(x)$

2. Partial Poisson structure

We consider the Kriegl & Michor's convenient setting ([3]). For short, a convenient vector space E is a locally convex topological vector space (l.c.t.v.s) such that a curve $c : \mathbb{R} \longrightarrow E$ is smooth if and only if $\lambda \circ c$ is smooth for all continuous linear functionals λ on E. We get a second topology on E which is the final topology defined by the set of all smooth curves and called the c^∞-topology. This last topology may be different from the l.c.t.v.s topology and for the c^∞-topology E can be not a topological vector space. However Banach spaces and Fréchet spaces are convenient spaces and these two topologies coincide. A map $f : E \to \mathbb{R}$ is smooth if and only if $f \circ c : \mathbb{R} \to \mathbb{R}$ is a smooth map for any smooth curve c in E. Therefore we have an evident notion of **convenient manifold** modeled on the c^∞-topology of a convenient space.

2 : Continuation

Now let M be a convenient manifold modeled on convenient space $\mathbb{M}$. We denote by : $p_M : TM \to M$ its tangent bundle and by $p_M^* : T^*M \to M$ its cotangent bundle.
Consider

- a vector subbundle $p' : T'M \to M$ of $p_M^* : T^*M \to M$ such that $p' : T'M \to M$ is a convenient bundle
- a bundle morphism $P : T'M \to TM$ which is *skew-symmetric i.e.*

$$< \xi, P(\eta) >= - < \eta, P(\xi) >$$

for all sections ξ and η of $T'M$, where $< \, , \, >$ is the bilinear crossing between T^*M and TM.

2 : Continuation

If $\iota : T'M \to T^*M$ is the canonical injection, let $\mathcal{A}(M)$ be the set of smooth functions $f : M \to \mathbb{R}$ such that $df \circ \iota$ is a section of $p' : T'M \to M$.
So $\mathcal{A}(M)$ is a sub-algebra of the algebra $\mathcal{C}^\infty(M)$ of smooth functions on M. On $\mathcal{A}(M)$ we define :

$$\{f, g\}_P = < dg, P(df) > \tag{1}$$

In these conditions, the relation (1) defines a skew-symmetric bilinear map $\{\ ,\ \}_P : \mathcal{A}(M) \times \mathcal{A}(M) \to \mathcal{C}^\infty(M)$.

2 : Continuation

In fact the bilinear map $\{\ ,\ \}_P$ takes values in $\mathcal{A}(M)$ and satisfies the Leibniz property $\{f, gh\}_P = g\{f, h\}_P + h\{f, g\}_P$.

Definition

Let $p' : T'M \to M$ be a convenient subbundle of $p^*_M : T^*M \to M$ and $P : T'M \to TM$ a skew-symmetric morphism.
We say that $(M, \mathcal{A}(M), \{\ ,\ \}_P)$ is a ***partial Poisson structure*** *on M or $(M, \mathcal{A}(M), \{\ ,\ \}_P)$ is a* ***partial Poisson manifold*** *if*

- *the bracket $\{\ ,\ \}_P$ satisfies the Jacobi identity :*

$$\{f, \{g, h\}_P\}_P + \{g, \{h, f\}_P\}_P + \{h, \{f, g\}_P\}_P = 0;$$

2 : Continuation

If M is a *Hilbert (resp. Banach, resp. Fréchet)* manifold and if the subbundle $T'M$ is a Hilbert (resp. Banach, resp. Fréchet bundle), the partial Poisson manifold $(M, \mathcal{A}(M), \{\ ,\ \}_P)$ will be called a *partial Poisson Hilbert (resp. Banach, resp. Fréchet) manifold*.

From now the morphism P is fixed we simply denote by $\{\ ,\ \}$ the Poisson bracket $\{\ ,\ \}_P$. As classically, given a partial Poisson manifold $(M, \mathcal{A}(M), \{\ ,\ \})$, any function $f \in \mathcal{A}(M)$ is called a **Hamiltonian**, the associated vector field $X_f = P(df)$ is called a **Hamiltonian vector field**. In particular we have $\{f, g\} = X_f(g)$. Also we have $[X_f, X_g] = X_{\{f,g\}}$ which is equivalent to $P(d\{f, g\}) = [P(df), P(dg)]$

3. Some examples of partial Poisson manifolds

- A finite dimensional Poisson manifold $(M, \mathcal{A}(M), \{\ ,\ \})$ is a particular case of partial Poisson manifold.
  $P : T^*M \to TM$ is a obtain from the correspondence $f \mapsto X_f$
- The concept of Banach-Poisson manifold defined by A. Odzijewicz and T. Ratiu (cf [6]) corresponds to the case where M is a Banach manifold, $T'M = T^*M$, $\mathcal{A}(M) = \mathcal{C}^\infty(M)$ a Poisson bracket $\{\ ,\ \}$ on $\mathcal{C}^\infty(M)$ such $g \mapsto \{f, g\}$ defines a vector field X_f on M.
  Then $P : T^*M \to TM$ is a obtain from the correspondence $f \mapsto X_f$.

3 : Continuation

- A weak symplectic manifold is a convenient manifold M endowed with a closed 2-form ω such that the associated morphism $\omega^\sharp : X \mapsto \omega(X, \ )$ from $TM \to T^*M$ is injective. Therefore the bundle $T'M = \omega^\sharp(TM)$ and $P = (\omega^\sharp)^{-1}$
- Poisson brackets in Hydrodynamics (see Kolev [2])
Let $E \to M$ be a finite dimensional vector bundle and denote by $\mathcal{C}^\infty(M, E)$ the module of sections of this bundle provided with a Fréchet vector space structure. Given a smooth real function $F : \mathcal{C}^\infty(M, E) \to \mathbb{R}$, the directional derivative of F at $u \in \mathcal{C}^\infty(M, E)$ in the direction $X \in \mathcal{C}^\infty(M, E)$ is :
$D_X F(u) = \lim_{t \to 0} \frac{F(u + tX) - F(u)}{t}$

3 : Continuation

Assume that this directional derivative can be written

$$D_X F(u) = \int_M \frac{\delta F}{\delta u}(u).X dV \quad \forall X \in \mathcal{C}^\infty(M, E)$$

where $u \mapsto \frac{\delta F}{\delta u}(u)$ is a smooth map from $\mathcal{C}^\infty(M, E)$ into itself.
Note that $\frac{\delta F}{\delta u}$ is nothing more than a vector field on $\mathcal{C}^\infty(M, E)$ which is called the L^2 **gradient** of F.
When the manifold M is compact without boundary, if $\mathcal{A}$ is the set of such functionals, we can define a Poisson bracket $\{\ ,\ \}$ on $\mathcal{A}$ of type :

$$\{F, G\} = \int_M \frac{\delta F}{\delta u} \mathcal{D} \frac{\delta G}{\delta u} dV$$

where $\mathcal{D}$ is a linear differential operator.

3 : Continuation

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where $\mathcal{D}$ is a linear differential operator.
With adequate assumptions on $\mathcal{D}$, we get a partial Poisson manifold on $\mathcal{C}^\infty(M, E)$.

- Arnold bracket also gives rise to a partial Poisson structure in an analog way (see [2]).

4. Partial Banach Poisson manifolds and Banach Lie algebroid

Consider a Banach bundle $p_E : E \to M$ of typical fiber $\mathbb{E}$ over a manifold M modeled on a Banach space $\mathbb{M}$ and $p_{E^*} : E^* \to M$ the dual bundle of $p_E : E \to M$. For any section $s \in \Gamma(E)$, we associate the linear function Φ_s on E^* defined by $\Phi_s(\sigma) =< \sigma, s \circ p_{E^*}(\sigma) >$. Then $s \mapsto \Phi_s$ is injective.

We have the following properties (P. Cabau & F. P [1]) :

Proposition

The set

$$T'E^* = \bigcup_{\sigma \in E^*} \{\omega \in T^*_\sigma E^* : \omega = d(\Phi_s + f \circ p_{E^*})(\sigma),\ s \in \Gamma(E_U),\ f \in \mathcal{C}^\infty(U)\}$$

is a well defined subset of T^*E^* and if p' is the restriction of p_{E^*} to $T'E^*$ then

$$p' : T'E^* \to E^*$$

is a closed Banach subbundle of typical fiber $\mathbb{M}^* \times \mathbb{E}$. In particular, $T'E^* = T^*E^*$ if and only if $\mathbb{E}$ is reflexive.

4 : Continuation

We say that a function $\Phi : E^* \to \mathbb{R}$ is **linear** if the restriction of Φ to each fiber E^*_x over x is linear, $\forall x \in M$. It is clear that for any $s \in \Gamma(E)$ the associated function Φ_s is linear. Let $\mathcal{A}(E^*)$ be the subalgebra of $\mathcal{C}^\infty(E^*)$ of functions Φ such that $d\Phi \circ \iota$ is a section of $T'E^*$.

In fact for any open set U, $\mathcal{A}(E^*_{|U})$ is locally generated by functions of type Φ_s where s is a local section of E over U and functions of type $f \circ p_{E^*}$ for any function f on U.

If $P : T'E^* \to TE^*$ is a morphism which gives rise to a partial Poisson bracket $\{\ ,\ \}$ on $\mathcal{A}(E^*)$ we say that $\{\ ,\ \}$ is a **linear partial Poisson bracket** if $\{\Phi, \Psi\}$ is linear for any linear function Φ and Ψ on E^*. Recall that $p_E : E \to M$ has a **Banach Lie algebroid** structure $(E, M, \rho, [\ ,\ ]_\rho)$ if there exists a morphism (called anchor) $\rho : E \to TM$ and a (localizable) Lie bracket $[\ ,\ ]_\rho$ on $\Gamma(E)$ *i.e.*

- $[\ ,\ ]_\rho : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is bilinear ;
- $[s, fs']_\rho = df(\rho(s))\tau + f[s, s']_\rho,\ \forall s, \forall s' \in \Gamma(E),\ \forall f \in \mathcal{C}^\infty(M)$ (Leibniz property) ;
- $[s, [s', s"]_\rho]_\rho + [s', [s", s]_\rho]_\rho + [s", [s', s]_\rho]_\rho = 0,\ \forall s, s', s" \in \Gamma(E)$ (Jacobi property).

4 : Continuation

We have the following result (P Cabau & F. P [1])

Theorem 1

Let $P : T'E^* \to TE^*$ be a morphism which defines a linear partial Poisson bracket $\{\ ,\ \}$ on E^*. Then there exists a Banach Lie algebroid structure $(E, M, \rho, [\ ,\ ]_\rho)$ characterized by :

$$\Phi_{[s_1,s_2]_\rho} = \{\Phi_{s_1}, \Phi_{s_2}\},\ \forall s_1, s_2 \in \Gamma(E) \qquad (2)$$

$$\{\Phi_s, f \circ p_{E^*}\} = df(\rho(s)) \circ p_{E^*},\ \forall f \in \mathcal{C}^\infty(M),\ \forall s \in \Gamma(E). \qquad (3)$$

Conversely, for each Banach Lie algebroid structure $(E, M, \rho, [\ ,\ ]_\rho)$ there exists an unique antisymmetric morphism $P : T'E^* \to TE^*$ which defines a linear partial Poisson bracket on $\mathcal{A}(E^*)$ characterized by relations (2) and (3). Moreover $(E, M, \rho, [\ ,\ ]_\rho)$ is exactly the Banach Lie algebroid structure associated to P as in the first part.

5. Partial Poisson, integrability and perspectives

Consider a partial Poisson manifold $(M, \mathcal{A}(M), \{\ ,\ \})$ associated to some morphism $P : T'M \to TM$. Then $P(T'M)$ is a distribution on M called the **characteristic distribution**. According to the property of stability of Hamiltonian fields under Lie Bracket, we can look for conditions under which $P(T'M)$ is **integrable**

(*i.e.* for each $x \in M$, there exists a (convenient) submanifold L of M such that $x \in L$ and $T_yL = P(T'_yM)$ for all $y \in L$ and such a maximal submanifold L is called a leaf).

In this case, by same arguments used in the framework of Banach Lie structure (*cf.* A. Odzijewicz & T. Ratiu's results), it is easy to show that each leaf L can be provided with a weak symplectic structure "compatible" with the induced Poisson bracket on L.

4 : Continuation

In the Banach context as consequence of some Theorem of *Integrability of weak Banach distributions [5]* we obtain

Theorem 2

Let $(M, \mathcal{A}(M), \{\ ,\ \})$ be a Banach partial Poisson manifold $(M, \mathcal{A}(M), \{\ ,\ \})$ associated $P : T'M \to TM$. Assume that $P(T'M)$ is a closed distribution on M and for any $x \in M$, the vector space $\ker P_x$ is complemented in T'_xM. Then the characteristic distribution is integrable and each leaf can be provided with a weak symplectic structure "compatible" with the induces Poisson bracket on L.

4 : Continuation

In a work in the course of completion we have shown that,under adequate assumptions, projective limits and direct limits of Banach partial Poisson manifolds are convenient Partial Poisson manifolds. Note that the convenient setting is the good context in the case of direct limit of Banach partial Poisson since the natural topology on direct limit of an ascending sequence of finite dimensional manifolds is a convenient manifold modeled on the convenient space $\mathbb{R}^\infty$. On the other hand, many important examples of Fréchet manifolds are "described " as projective limit of Banach manifolds, so the "convenient framework" is well adapted. For direct limit and projective limit of Banach Lie algebroids we can prove a similar result of Theorem 1.

4 : Continuation

Our integrability criteria for weak Banach distributions allow us to get reasonable integrability criteria for direct limits of some particular Banach distributions. In fact, our proof is essentially based on the existence of a local flow for a vector field on Banach manifolds. Under adequate assumptions on ascending sequences of Banach manifold, this argument is still true on direct limit of Banach manifolds. In particular we obtain

Theorem 3

A direct limit of an ascending sequence of finite dimensional (partial) Poisson manifolds is a convenient (partial) Poisson manifold. The characteristic distribution on this limit is integrable. Each leaf is a convenient manifold and can be provided with a weak symplectic structure "compatible" with the induced Poisson bracket.

4 : Continuation

Unfortunately the same arguments are no longer valid for projective limits of Banach distributions without very strong assumptions. So we can prove a similar result to Theorem 2 for projective limit of Banach partial Poisson manifolds only under very strong assumptions. In particular we obtain a similar result to Theorem 2 in Fréchet framework ONLY under very strong assumptions which are rarely satisfied. Perhaps using other type of arguments such a result is true under weaker assumptions.....

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Thank you for your attention !