Białowieża 2016

Geometric structures canonically related to a W^*-algebra

Anatol Odzijewicz
Institute of Mathematics
University in Białystok

REFERENCES:

1. A. Odzijewicz, T. Ratiu. *Banach Lie-Poisson spaces and reduction*, Comm. Math. Phys., 243 (2003) 1-54
2. A. Odzijewicz, A. Sliżewska. *Banach Lie groupoids associated to W^*-algebra*, to appear in J. Sympl. Geom.
3. A. Odzijewicz, G. Jakimowicz, A. Sliżewska. *Banach-Lie algebroids associated to the groupoid of partially invertible elements of a W^*-algebra*. J.Geom.Phys., 95 (2015) 108-126
4. A. Odzijewicz, G. Jakimowicz, A. Sliżewska. *Fibre-wise linear Poisson structures related to W^*-algebra* (in preparation)

P - Banach manifold (modeled on non-refleksive Banach space in general)

- $\pi \in \Gamma^\infty(\bigwedge^2 T^{**}P)$
- $\Gamma^\infty(T^*P) \ni \alpha \mapsto \#\alpha := \pi(\alpha, \cdot) \in \Gamma^\infty(\bigwedge^2 T^{**}P)$

$$\begin{array}{ccc} T^*P & \xrightarrow{\#} & T^{**}P \\ \uparrow & & \uparrow \\ T^\flat P & \xrightarrow{\#} & TP \end{array} \quad , \tag{1}$$

- $T^*P \supset T^\flat P$ - quasi Banach vector subbundle (without Banach complement in general)
- $\#T^\flat P = TP$
- $\mathcal{P}^\infty(P) := \{f \in C^\infty(P) : \ \#df \in \Gamma^\infty TP\}$

$\mathcal{P}^\infty(P)$ - Poisson algebra with respect to $\{f, g\} := \pi(df, dg)$

$\mathcal{G}, B$ - Banach manifold with Hausdorff underlying topology
Banach-Lie groupoid $\mathcal{G} \rightrightarrows B$:

1. **source map** $\mathbf{s} : \mathcal{G} \to B$ and **target map** $\mathbf{t} : \mathcal{G} \to B$ - submersions
2. **product** $m : \mathcal{G}^{(2)} \to \mathcal{G}$

$$m(g,h) =: gh,$$

   defined on **the set of composable pairs**

$$\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} : \ \mathbf{s}(g) = \mathbf{t}(h)\},$$

3. **identity section** $\varepsilon : B \to \mathcal{G}$ - immersion
4. **inverse map** $\iota : \mathcal{G} \to \mathcal{G}$, $\iota \circ \iota = id$,

which satisfy the following conditions:

$$\mathbf{s}(gh) = \mathbf{s}(h), \quad \mathbf{t}(gh) = \mathbf{t}(g), \tag{2}$$

$$k(gh) = (kg)h, \tag{3}$$

$$\varepsilon(\mathbf{t}(g))g = g = g\varepsilon(\mathbf{s}(g)), \tag{4}$$

$$\iota(g)g = \varepsilon(\mathbf{s}(g)), \qquad g\iota(g) = \varepsilon(\mathbf{t}(g)), \tag{5}$$

where $g, k, h \in \mathcal{G}$.
The shorter definition: A groupoid is a small category with invertible morphisms.

A Banach-Lie **algebroid** on Banach manifold M is a Banach vector bundle $q : A \to M$ together with:

1. $a : A \to TM$ (anchor map)
2. $[\ ,\ ] : \Gamma A \times \Gamma A \to \Gamma A$ (Lie bracket) such that

$[X, fY] = f[X,Y] + a(X)(f)Y$
$a([X,Y]) = [a(X), a(Y)]$
for all $X, Y \in \Gamma A,\ f \in C^\infty(M)$.

Fibre-wise linear Poisson structure
One takes as a (sub) Poisson manifold $P = E$, where E is the total space of a Banach vector bundle $E \stackrel{q}{\to} M$ such that

- $\mathcal{P}^\infty(E) \supset P^\infty_{lin}(E)$ - fibre-wise linear
- $\mathcal{P}^\infty(E) \supset P^\infty_{B}(E)$ - constant on the fibres of q
- $\{P^\infty_B(E), P^\infty_B(E)\} = 0$
- $\{P^\infty_B(E), P^\infty_{lin}(E)\} \subset P^\infty_B(E)$
- $\{P^\infty_{lin}(E), P^\infty_{lin}(E)\} \subset P^\infty_{lin}(E)$

W^*-algebra

A C^*-algebra $\mathfrak{M}$ is called W^*-**algebra** (von Neumann algebra) if there exists a Banach space $\mathfrak{M}_*$ such that

$$(\mathfrak{M}_*)^* = \mathfrak{M}, \tag{6}$$

$\mathfrak{M}_*$ - **predual Banach space of** $\mathfrak{M}$, $\quad \mathfrak{M}_* \subset \mathfrak{M}^*$
$\sigma(\mathfrak{M}, \mathfrak{M}_*)$ - topology on $\mathfrak{M}$

- $\mathfrak{M}_* \ni \rho \geq 0$ and $\|\rho\| = 1$

ρ - state (normal) of the quantum system

- $\mathcal{L}(\mathfrak{M}) \ni p \;\Leftrightarrow\; p^2 = p = p^* \in \mathfrak{M}$

$\mathcal{L}(\mathfrak{M})$ - lattice complete in $\sigma(\mathfrak{M}, \mathfrak{M}_*)$-topology
$\mathcal{L}(\mathfrak{M})$ - „quantum logic” (propositions calculus)

- morphism of lattices $\Sigma : \mathcal{B} \to \mathcal{L}(\mathfrak{M}) \;\equiv\;$ quantum observables

Example: $\quad \mathfrak{M} = L^\infty(\mathfrak{M}) \qquad \mathfrak{M}_* = L^1(\mathfrak{M})$ - standard model of quantum mechanics

- $Df(\rho), Dg(\rho) \in \mathfrak{M}$ for $f, g \in C^\infty(\mathfrak{M}^*)$
- $ad^*\mathfrak{M}_* \subset \mathfrak{M}_*$, $\quad ad^* : \mathfrak{M}^* \to \mathfrak{M}^*$
  and $\quad ad_X Y := [X, Y] \;$ for $\; X, Y \in \mathfrak{M}$
- $(\mathfrak{M}, [\cdot,\cdot])$- Banach-Lie algebra

Hence one has Lie-Poisson bracket

$$\{f, g\}_{LP}(\rho) := \langle \rho, [Df(\rho), Dg(\rho)]\rangle$$

Hamiltonian equation

$$\frac{d}{dt}\rho = ad^*_{DH(\rho)}\rho$$

for a Hamiltonian $H \in C^\infty(\mathfrak{M}_*)$
Example: $\quad \mathfrak{M} = L^\infty(\mathfrak{M}) \qquad \mathfrak{M}_* = L^1(\mathfrak{M})$
(1) $\frac{d}{dt}\rho = [DH(\rho), \rho]$ - non-linear von Neumann-Liouville equation
(2) The cases of the infinite Toda lattice and the non-linear Schrödinger can be also written in this way.

Left support $l(x) \in \mathcal{L}(\mathfrak{M})$ (**right support** $r(x) \in \mathcal{L}(\mathfrak{M})$) of $x \in \mathfrak{M}$ is the least projection in $\mathfrak{M}$, such that

$$l(x)x = x \qquad (\text{resp. } x\, r(x) = x). \tag{7}$$

If $x \in \mathfrak{M}$ is selfadjoint, then **support** $s(x)$

$$s(x) := l(x) = r(x).$$

Polar decomposition for $x \in \mathfrak{M}$

$$x = u|x|, \tag{8}$$

where $u \in \mathfrak{M}$ is partial isometry and $|x| := \sqrt{x^*x} \in \mathfrak{M}^+$, such that

$$l(x) = s(|x^*|) = uu^*, \qquad r(x) = s(|x|) = u^*u.$$

Let $G(p\mathfrak{M}p)$ be the group of all invertible elements in W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$.
We define the set $\mathcal{G}(\mathfrak{M})$ of **partially invertible** elements in $\mathfrak{M}$

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M}; \quad |x| \in G(p\mathfrak{M}p), \text{ where } p = s(|x|)\}$$

Remark: $\mathcal{G}(\mathfrak{M}) \subsetneq \mathfrak{M}$ in a general case.

Proposition

The set $\mathcal{G}(\mathfrak{M})$ with

1. the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$
$$\mathbf{s}(x) := r(x), \qquad \mathbf{t}(x) := l(x),$$
2. the product defined as the product in $\mathfrak{M}$ on the set
$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}); \;\; \mathbf{s}(x) = \mathbf{t}(y)\},$$
3. the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ as the inclusion map,
4. the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by
$$\iota(x) := |x|^{-1} u^*,$$

is a groupoid over $\mathcal{L}(\mathfrak{M})$.

Groupoid of partially isometries of W^*-algebra $\mathfrak{M}$

Proposition

The set of partial isometries $\mathcal{U}(\mathfrak{M}) \subset \mathfrak{M}$ is the wide subgroupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Orbits

• Inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$

$$I_x p := x p \iota(x), \quad \mathsf{s}(x) = p$$

on the lattice $\mathcal{L}(\mathfrak{M})$ gives the equivalence relation:

$$p \sim q \quad \Leftrightarrow \quad q \in \mathcal{O}_p.$$

• The equivalence class $[p]$ of p in sense of Murray-von Neumann is the orbit $\mathcal{O}_p$ of $p \in \mathcal{L}(\mathfrak{M})$.

Remark

The Murray-von Neumann classification of W^*-algebras directly corresponds to the classification of orbits of the inner action of $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ or $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on the lattice of projections $\mathcal{L}(\mathfrak{M})$.

For $p \in \mathcal{L}(\mathfrak{M})$ let us define the subset of $\mathcal{L}(\mathfrak{M})$

$$\Pi_p := \{q \in \mathcal{L}(\mathfrak{M}) : \quad \mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M}\}$$

then $\qquad p = x_p - y_p \; \in \; q\mathfrak{M}p \oplus (1-p)\mathfrak{M}p.$

The above defines the bijection $\varphi_p : \Pi_p \xrightarrow{\sim} (1-p)\mathfrak{M}p$ and section $\sigma_p : \Pi_p \to \mathbf{t}^{-1}(\Pi_p)$ by

$$\sigma_p(q) := x_p, \qquad \varphi_p(q) := y_p.$$

In order to find the transition maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'})$$

in the case $\Pi_p \cap \Pi_{p'} \neq \emptyset$ one has for $q \in \Pi_p \cap \Pi_{p'}$ the splittings

$$\begin{aligned} \mathfrak{M} &= q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M} \\ \mathfrak{M} &= q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}. \end{aligned} \tag{9}$$

and we obtain

$$y_{p'} = (\varphi_{p'} \circ \varphi_p^{-1})(y_p) = (b + dy_p)\iota(a + cy_p),$$

where $a = p'p$, $b = (1-p')p$, $c = p'(1-p)$ and $d = (1-p')(1-p)$.

Theorem

The family of maps

$$(\Pi_p, \varphi_p) \quad p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on a $\mathcal{L}(\mathfrak{M})$. This atlas is modeled by the family of Banach spaces $(1-p)\mathfrak{M}p$, where $p \in \mathcal{L}(\mathfrak{M})$.

Fact: If $p' \in \mathcal{O}_p$ then $(1-p)\mathfrak{M}p \cong (1-p')\mathfrak{M}p'$.

For projections $\tilde{p}, p \in \mathcal{L}(\mathfrak{M})$ we define the set

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p)$$

and the map $\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ in the following way

$$\psi_{\tilde{p}p}(x) := (\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))) = (y_{\tilde{p}}, z_{\tilde{p}p}, y_p)\,.$$

Transition maps

$$\begin{aligned} y_{p'} &= (b + dy_p)\iota(a + cy_p), \\ z_{p'\tilde{p}'} &= (a + cy_p)z_{p\tilde{p}}\iota(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}}) \\ \tilde{y}_{\tilde{p}'} &= (\tilde{b} + \tilde{d}\tilde{y}_{\tilde{p}})\iota(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}}), \end{aligned} \tag{10}$$

where $(y_{p'}, z_{p'\tilde{p}'}, \tilde{y}_{\tilde{p}'}) = (\psi_{p'\tilde{p}'} \circ \psi^{-1}_{p\tilde{p}})(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$
$\tilde{a} = \tilde{p}'\tilde{p}$, $\tilde{b} = (1-\tilde{p}')\tilde{p}$, $\tilde{c} = \tilde{p}'(1-\tilde{p})$ and $\tilde{d} = (1-\tilde{p}')(1-\tilde{p})$.

$\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a complex Banach-Lie groupoid

Theorem

The family of maps

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p}) \quad \tilde{p}, p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on the groupoid $\mathcal{G}(\mathfrak{M})$. The complex Banach manifold structure of $\mathcal{G}(\mathfrak{M})$ has type $\mathfrak{G}$, where $\mathfrak{G}$ is the set of Banach spaces

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

indexed by the pair of equivalent projections of $\mathcal{L}(\mathfrak{M})$.

Proposition

The complex Banach groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ endowed with the underlying topology is a separable (Hausdorff) topological groupoid.

In order to investigate the Banach-Lie groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ it is enough to restrict ourselfs to

$$\mathcal{L}_{p_0}(\mathfrak{M}) := \{p \in \mathcal{L}(\mathfrak{M}) : \ p \sim p_0\} = \mathcal{O}_{p_0}$$

$$\mathcal{G}_{p_0}(\mathfrak{M}) := \mathbf{t}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap \mathbf{s}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})).$$

Proposition

The Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a disjoint union of Banach-Lie subgroupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $p_0 \in \mathcal{L}(\mathfrak{M})$, which are its closed-open Banach subgroupoids.

Principal bundle P_0

We consider $P_0 := \mathbf{s}^{-1}(p_0)$ as the total space of the G_0-principal bundle $\pi_0 := \mathbf{t}|_{P_0} : P_0 \to \mathcal{L}_{p_0}(\mathfrak{M})$, where G_0 is the Banach-Lie group $G(p_0\mathfrak{M}p_0)$ of the invertible elements of the W^*-subalgebra $p_0\mathfrak{M}p_0$. The free right actions of G_0 on P_0 and on $P_0 \times P_0$ are defined by

$$\kappa : P_0 \times G_0 \ni (\eta, g) \mapsto \eta g \in P_0 \tag{11}$$

and by

$$\kappa_2 : P_0 \times P_0 \times G_0 \ni (\eta, \xi, g) \mapsto (\eta g, \xi g) \in P_0 \times P_0, \tag{12}$$

respectively. The above allows us to define the quotient groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ of the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$, which by definition is the **gauge groupoid** associated to the G_0-principal bundle $\pi_0 : P_0 \to P_0/G_0 \cong \mathcal{L}_{p_0}(\mathfrak{M})$.

The complex analytic maps

$$\phi : \frac{P_0 \times P_0}{G_0} \ni \langle \eta, \xi \rangle \mapsto \eta \xi^{-1} \in \mathcal{G}_{p_0}(\mathfrak{M}) \tag{13}$$

$$\varphi : P_0/G_0 \ni \langle \eta \rangle \mapsto \eta \eta^{-1} \in \mathcal{L}_{p_0}(\mathfrak{M}) \tag{14}$$

define the isomorphism

$$\begin{tikzcd} \frac{P_0 \times P_0}{G_0} \arrow[r, "\phi"] \arrow[d, shift right] \arrow[d, shift left] & \mathcal{G}_{p_0}(\mathfrak{M}) \arrow[d, shift right, "\mathbf{t}"'] \arrow[d, shift left, "\mathbf{s}"] \\ P_0/G_0 \arrow[r, "\varphi"] & \mathcal{L}_{p_0}(\mathfrak{M}) \end{tikzcd} , \tag{15}$$

of Banach-Lie groupoids.

One has

$$\begin{tikzcd} \mathcal{J}(\mathfrak{M}) \arrow[r, "\hookrightarrow"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G}(\mathfrak{M}) \arrow[r, "(\mathsf{t},\mathsf{s})"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{L}(\mathfrak{M}) \times_{\mathcal{R}} \mathcal{L}(\mathfrak{M}) \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}), \end{tikzcd} \quad (16)$$

where

- $\mathcal{J}(\mathfrak{M}) := ker\ (\mathsf{t}, \mathsf{s}) = \{x \in \mathcal{G}(\mathfrak{M}); \quad \mathsf{t}(x) = \mathsf{s}(x)\}$ is the inner (totaly intransitive) subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$,
- $\mathcal{L}(\mathfrak{M}) \times_{\mathcal{R}} \mathcal{L}(\mathfrak{M}) \ni (q, p)$ iff $q \sim p$.

Using the above functorial correspondence we obtain the short exact sequence of algebroids

$$\begin{tikzcd} \mathcal{A}\mathcal{J}(\mathfrak{M}) \arrow[r, "\iota"] \arrow[d, "T\mathbf{t}"'] & \mathcal{A}\mathcal{G}(\mathfrak{M}) \arrow[r, "a"] \arrow[d, "T\mathbf{t}"'] & T\mathcal{L}(\mathfrak{M}) \arrow[d] \\ \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}). \end{tikzcd} \tag{17}$$

Atiyah sequence of algebroids

Proposition

The Atiyah sequence (17) is isomorphic to

$$\begin{tikzcd} \mathcal{A}(\mathfrak{M}) \arrow[r, "\iota"] \arrow[d] & \mathcal{M}^L(\mathfrak{M}) \arrow[r, "a"] \arrow[d] & \mathcal{T}(\mathfrak{M}) \arrow[d] \\ \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}), \end{tikzcd} \tag{18}$$

where

$$\mathcal{A}(\mathfrak{M}) := \{(x,q) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad x \in q\mathfrak{M}q\}$$

$$\mathcal{M}^L(\mathfrak{M}) := \{(x,q) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad x \in \mathfrak{M}q\}$$

$$\mathcal{T}(\mathfrak{M}) := \{(x,q) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad x \in (1-q)\mathfrak{M}q\}.$$

Atiyah sequence of algebroids

Proposition

The short exact sequence

$$
\begin{tikzcd}
\mathcal{A}_{p_0}(\mathfrak{M}) \arrow[r, "\iota"] \arrow[d] & \mathcal{M}^L_{p_0}(\mathfrak{M}) \arrow[r, "a"] \arrow[d] & \mathcal{T}_{p_0}(\mathfrak{M}) \arrow[d] \\
\mathcal{L}_{p_0}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}_{p_0}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}_{p_0}(\mathfrak{M}),
\end{tikzcd}
\tag{19}
$$

of Banach-Lie algebroids, defined as the restriction of (18) to $\mathcal{L}_{p_0}(\mathfrak{M})$, is isomorphic to the Atiyah sequence

$$
\begin{tikzcd}
p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} P_0 \arrow[r, "a"] \arrow[d] & TP_0/G_0 \arrow[r, "\iota"] \arrow[d] & T(P_0/G_0) \arrow[d] \\
P_0/G_0 \arrow[r, "\sim"] & P_0/G_0 \arrow[r, "\sim"] & P_0/G_0
\end{tikzcd}
\tag{20}
$$

Proof: The maps

$$I_{\mathcal{A}} : p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} P_0 \ni \langle x, \eta\rangle \mapsto (\eta x\eta^{-1}, \eta\eta^{-1}) \in \mathcal{A}_{p_0}(\mathfrak{M}) \quad (21)$$

$$I_{\mathcal{M}} : \frac{\mathfrak{M}p_0 \times P_0}{G_0} \ni \langle \vartheta, \eta\rangle \mapsto (\vartheta\eta^{-1}, \eta\eta^{-1}) \in \mathcal{M}^L_{p_0}(\mathfrak{M}) \quad (22)$$

$$I_{\mathcal{T}} : \frac{\mathfrak{M}p_0 \times P_0}{TG_0} \ni \langle\langle \vartheta, \eta\rangle\rangle \mapsto ((1-\eta\eta^{-1})\vartheta\eta^{-1}, \eta\eta^{-1}) \in \mathcal{T}_{p_0}(\mathfrak{M}) \quad (23)$$

define isomorphisms between the corresponding vector bundles appearing in the diagrams (20) and (19) and they commute with the horizontal arrows of these diagrams. We also have $\mathcal{L}_{p_0}(\mathfrak{M}) \cong P_0/G_0$.

Coordinate description

$(v,\eta)\in\mathfrak{M}p_0\times P_0\cong TP_0 \qquad v=\frac{d}{dt}\eta(t)|_{t=0}$

One has:

$$\eta=(p+y_p)z_{pp_0}$$

$$v=[a_p+(p+y_p)b_p]z_{pp_0}$$

where

$$a_p=\frac{d}{dt}y_p(t)|_{t=0},\qquad b_p=\frac{d}{dt}z_{pp_0}(t)|_{t=0}z_{pp_0}^{-1}$$

$$a_p=(v-\eta(p\eta)^{-1}v)(p\eta)^{-1} \tag{24}$$

$$b_p=pv(p\eta)^{-1} \tag{25}$$

$$y_p=\eta(p\eta)^{-1}-p \tag{26}$$

$$z_{pp_0}=p\eta. \tag{27}$$

Algebroid bracket and anchor map

Proposition

(i) The anchor map $a : \mathcal{AL}(\mathfrak{M}) \to T\mathcal{L}(\mathfrak{M})$ acts on $\mathfrak{X} = a_p \frac{\partial}{\partial y_p} + b_p \frac{\partial}{\partial z_{pp_0}} \in \Gamma^\infty \mathcal{M}^L(\mathfrak{M})$ as follows

$$a(\mathfrak{X}) = a_p \frac{\partial}{\partial y_p}; \quad (28)$$

(ii) The vertical part of $\mathfrak{X}$ is given by $b_p \frac{\partial}{\partial z_{pp_0}}$;

Proposition cont.

(iii) The Lie bracket of $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma^\infty \mathcal{M}^L(\mathfrak{M})$ assumes the form

$$[\mathfrak{X}_1, \mathfrak{X}_2] = a_p \frac{\partial}{\partial y_p} + b_p \frac{\partial}{\partial z_{pp}}, \tag{29}$$

where

$$a_p = \left\langle \frac{\partial a_{2p}}{\partial y_p}, a_{1p} \right\rangle - \left\langle \frac{\partial a_{1p}}{\partial y_p}, a_{2p} \right\rangle$$

and

$$b_p = \left\langle \frac{\partial b_{2p}}{\partial y_p}, a_{1p} \right\rangle - \left\langle \frac{\partial b_{1p}}{\partial y_p}, a_{2p} \right\rangle + [b_{2p}, b_{1p}].$$

$\mathfrak{M} = (\mathfrak{M}_*)^*$
$\mathcal{A}_*(\mathfrak{M}) \subset \mathcal{A}^*(\mathfrak{M}) \quad \longleftrightarrow \quad (q\mathfrak{M}q)^* \supset (q\mathfrak{M}q)_* \cong q\mathfrak{M}_*q$
$\mathcal{A}_*\mathcal{G}(\mathfrak{M}) \subset \mathcal{A}^*\mathcal{G}(\mathfrak{M}) \quad \longleftrightarrow \quad (\mathfrak{M}q)^* \supset (q\mathfrak{M})_* \cong \mathfrak{M}_*q$
$T_*\mathcal{L}(\mathfrak{M}) \subset T\mathcal{L}^*(\mathfrak{M}) \quad \longleftrightarrow \quad ((1-q)\mathfrak{M}q)^* \supset ((1-q)\mathfrak{M}q)_* \cong q\mathfrak{M}_*(1-q)$
where for every $x \in \mathfrak{M}$

$$\begin{array}{l} \langle R_a^*\varphi,\ x\rangle := \langle \varphi,\ ax\rangle \\ \langle L_a^*\varphi,\ x\rangle := \langle \varphi,\ xa\rangle \end{array}, \tag{30}$$

Remark: $R_a^*\mathfrak{M}_* \subset \mathfrak{M}_*, \qquad L_a^*\mathfrak{M}_* \subset \mathfrak{M}_*.$

So, one has

$$\begin{tikzcd} T_*\mathcal{L}(\mathfrak{M}) \arrow[r, "a_*"] \arrow[d] & \mathcal{A}_*\mathcal{G}(\mathfrak{M}) \arrow[r, "\iota_*"] \arrow[d] & \mathcal{A}_*\mathcal{J}(\mathfrak{M}) \arrow[d] \\ \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}(\mathfrak{M}) \end{tikzcd} \tag{31}$$

The short exact sequence

$$\begin{tikzcd}
T_*\mathcal{L}_{p_0}(\mathfrak{M}) \arrow[r, "a_*"] \arrow[d] & \mathcal{A}_*\mathcal{G}_{p_0}(\mathfrak{M}) \arrow[r, "\iota_*"] \arrow[d] & \mathcal{A}_*\mathcal{J}_{p_0}(\mathfrak{M}) \arrow[d] \\
\mathcal{L}_{p_0}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}_{p_0}(\mathfrak{M}) \arrow[r, "\sim"] & \mathcal{L}_{p_0}(\mathfrak{M})
\end{tikzcd} \tag{32}$$

is isomorphic to the predual Atiyah sequence

$$\begin{tikzcd}
T_*(P_0/G_0) \arrow[r, "a_*"] \arrow[d] & T_*P_0/G_0 \arrow[r, "\iota_*"] \arrow[d] & p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0^*}} P_0 \arrow[d] \\
P_0/G_0 \arrow[r, "\sim"] & P_0/G_0 \arrow[r, "\sim"] & P_0/G_0,
\end{tikzcd} \tag{33}$$

Fibre-wise linear sub Poisson structure of the predual Atiyah sequence

- Weak symplectic structure of $T_*P_0 \cong p_0\mathfrak{M}_* \times P_0$

$(\varphi, \eta) \in T_*P_0, \qquad \xi_{(\varphi,\eta)} = (\varphi, \eta, \theta, \vartheta)$

$$\omega_{(\varphi,\eta)}(\xi^1_{(\varphi,\eta)}, \xi^2_{(\varphi,\eta)}) = \langle\theta_1, \vartheta_2\rangle - \langle\theta_2, \vartheta_1\rangle. \tag{34}$$

By

$$\omega_{(\varphi,\eta)}(\xi_{(\varphi,\eta)}, \cdot) : T_{(\varphi,\eta)}(p_0\mathfrak{M}_* \times P_0) \to T^*_{(\varphi,\eta)}(p_0\mathfrak{M}_* \times P_0) \tag{35}$$

one defines the bundle morphism

$$\flat : T(T_*P_0) \hookrightarrow T^*(T_*P_0)$$

where

$$T^\flat(T_*P_0) := \flat(T(T_*P_0)) \subsetneq T^*(T_*P_0)$$

is a proper Banach vector subbundle.

- **Momentum map** $J_0 : T_* P_0 \to p_0 \mathfrak{M}_* p_0$ such that

$$\omega(\xi^x, \cdot) = -d\langle \gamma, \xi^x \rangle = -d\langle J_0, x \rangle \tag{36}$$

where

$$\xi^x(f) = \frac{d}{dt} f(exp(-tx)\varphi, \eta \; exp(tx))|_{t=0}$$

for $x \in p_0 \mathfrak{M} p_0$.
One has

$$J_0(\varphi, \eta) = \varphi\eta \quad \text{i.e.} \quad \langle J_0(\varphi, \eta), x \rangle = \langle \varphi, \eta x \rangle$$

for any $x \in p_0 \mathfrak{M} p_0$.

• For $f \in C^\infty(T_*P_0)$ one has
$\frac{\partial f}{\partial \eta}(\varphi, \eta) \in (\mathfrak{M}p_0)^*$ and $\frac{\partial f}{\partial \varphi}(\varphi, \eta) \in (p_0\mathfrak{M}_*)^* = \mathfrak{M}p_0$.
• Thus for $f, g \in C^\infty(T_*P_0)$one defines the bracket

$$\{f, g\} = \langle \frac{\partial g}{\partial \eta}, \frac{\partial f}{\partial \varphi} \rangle - \langle \frac{\partial f}{\partial \eta}, \frac{\partial g}{\partial \varphi} \rangle \tag{37}$$

which is bilinear, anti-symmetric and satisfies the Leibniz property but not satisfies the Jacobi identity for arbitrary smooth functions.
• Therefore we define

$$\mathcal{P}^\infty(T_*P_0) := \left\{ f \in C^\infty(T_*P_0) : \frac{\partial f}{\partial \eta}(\varphi, \eta) \in (\mathfrak{M}p_0)_* \subset (\mathfrak{M}p_0)^* \right\}.$$

• Since $T^\flat(T_*P_0) \subsetneq T^*(T_*P_0)$ the bundle map $\# : T^\flat(T_*P_0) \to T(T_*P_0)$, the inverse to $\flat : T(T_*P_0) \hookrightarrow T^*(T_*P_0)$, is not defined on the whole of $T^*(T_*P_0)$, it will be called a sub Poisson morphism.

• $\mathcal{P}^\infty(T_*P_0) \supset \mathcal{P}^\infty_{G_0}(T_*P_0)$ - the Poisson subalgebra of G_0-invariant functions

$$\mathcal{P}^\infty_{G_0}(T_*P_0) \cong \mathcal{P}^\infty(T_*P_0/G_0)$$

$(\mathcal{P}^\infty(T_*P_0/G_0), \{\cdot,\cdot\}_{G_0})$ - a Poisson algebra

$$\{F, G\}_{G_0} := \{F \circ \pi_{*G_0}, G \circ \pi_{*G_0}\}, \tag{38}$$

where $\pi_{*G_0} : T_*P_0 \to T_*P_0/G_0$.

• Lie-Poisson bracket of $F, G \in C^\infty(p_0\mathfrak{M}_*p_0)$, $\frac{\partial F}{\partial \beta}(\beta) \in p_0\mathfrak{M}p_0$

$$\{F, G\}_{LP}(\beta) := \left\langle \beta, \left[\frac{\partial F}{\partial \beta}(\beta), \frac{\partial G}{\partial \beta}(\beta)\right]\right\rangle \tag{39}$$

Proposition

(i) One has the surjective Poisson submersions:

$$\begin{tikzcd}
& T_*P_0 \arrow[dl, "\pi_{*G_0}"'] \arrow[dr, "J_0"] & \\
T_*P_0/G_0 & & p_0\mathfrak{M}_*p_0
\end{tikzcd} \tag{40}$$

of the weak symplectic manifold (T_*P_0, ω) on the sub Poisson manifold $(T_*P_0/G_0, \{\cdot,\cdot\}_{G_0})$ and the Banach Lie-Poisson space $(p_0\mathfrak{M}p_0, \{\cdot,\cdot\}_{LP})$.

(ii) The Poisson subalgebras $J_0^*(C^\infty(p_0\mathfrak{M}_*p_0)$ and $\pi_{*G_0}^*(\mathcal{P}^\infty(T_*P_0/G_0)) = \mathcal{P}_{G_0}^\infty(T_*P_0)$ of $\mathcal{P}^\infty(T_*P_0)$ are polar one to another with respect to the weak symplectic form ω.

Theorem

(i) The Banach vector bundles map $\iota_* : T_*P_0/G_0 \to p_0\mathfrak{M}_*p_0 \times_{Ad^*_{G_0}} P_0$ is a Poisson submersion.

(ii) One has $ker\ \iota_* = J_0^{-1}(0)/G_0$, where $J_0^{-1}(0)/G_0$ is the weak symplectic leaf in T_*P_0/G_0 obtained by the Marsden-Weinstein symplectic reduction procedure. The predual anchor map $a_* : T_*(P_0/G_0) \hookrightarrow T_*P_0/G_0$ is an immersion which defines the bundle isomorphism $T_*(P_0/G_0) \cong J_0^{-1}(0)/G_0$, where the precotangent bundle $T_*(P_0/G_0)$ is endowed with the canonical weak symplectic structure.

Theorem

All groupoids in the front of the short exact sequence

$$
\begin{tikzcd}
T_*\left(\frac{P_0\times P_0}{G_0}\right) \arrow[r] \arrow[drr, "a_2^*"] \arrow[dd, shift left] \arrow[dd, shift right] & \frac{P_0\times P_0}{G_0} \arrow[drr, equal] \arrow[dd, shift left] \arrow[dd, shift right] & & & & \\
& & \frac{T_*P_0\times T_*P_0}{G_0} \arrow[r] \arrow[drr, "\iota_2^*"] \arrow[dd, shift left] \arrow[dd, shift right] & \frac{P_0\times P_0}{G_0} \arrow[drr, equal] \arrow[dd, shift left] \arrow[dd, shift right] & & \\
T_*P_0/G_0 \arrow[r] \arrow[drr, "id"] & \frac{P_0}{G_0} \arrow[drr, equal] & & & \frac{P_0\times p_0\mathfrak{M}_* p_0\times P_0}{G_0} \arrow[r] \arrow[dd, shift left] \arrow[dd, shift right] & \frac{P_0\times P_0}{G_0} \arrow[dd, shift left] \arrow[dd, shift right] \\
& & T_*P_0/G_0 \arrow[r] \arrow[drr, "{[\pi^*]}"] & \frac{P_0}{G_0} \arrow[drr, equal] & & \\
& & & & P_0/G_0 \arrow[r] & \frac{P_0}{G_0},
\end{tikzcd}
$$

are the sub Poisson $\mathcal{VB}$-groupoids and the corresponding horizontal arrows of its define Poisson morphisms.

THANK YOU