EXPONENTIAL OPERATORS
in physical theories

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Abstract

In the present day quantum theories the finite or infinite products of the exponential operations $e^{a_1}e^{a_2}\dots e^{a_n}$ (and their continuous equivalents) are of the known importance, but the problem of how to represent them by a single exponential operation e^{Ω} where Ω is the "phase operator" presents some combinatorial difficulties. The report below presents the algorithms which make this task significantly easier. In some cases like the 1D oscillator with time dependent elastic force they lead to interesting exact solutions. In some other more dimensional cases they traduce themselves into the important non-linear matrix equations.

Contents

INTRODUCTION

In the customary calculation trends of QM some fine algebraic problems for non-commuting exponents very seldom appear. The report below is dedicated to the exponential functions $e^a e^b$ for the non-commuting a, b, also for multiple equivalents $e^{a_1} e^{a_2} \dots e^{a_n}$ where the $a_k = -iH_k\delta_k$ represent distinct, non-commuting evolution steps, as well as to their limiting case i.e. the evolution generated by the $-iH(t)dt$ where $H(t)$ is a time dependent Hamiltonian. In all these cases, the question is, how to express it by the single exponential $e^{\Omega(t)}$, where $\Omega(t)$ is the "phase operator"?

The attempts to solve the last problem by iterating the non-linear equation for Ω failed due to the fast increasing complication of each step. (Magnus writes about the "combinatorial mess"). However, who is interested in a symbolic but simple solution of the problem (giving explicatively all the approximation steps in form of multicommutator expressions), can see it in Section 2.3 of this report.

In cases when the operation exponents represent the finite dimensional Lie algebra, the terms of the infinite multiple commutator series start to repeat themselves, summing up to some closed matrix expressions. So it happens for quantum systems with Hamiltonians quadratic in the canonical operators $q_1, ..., q_n, p_1, ...p_n$. In the simplest case of 1D time dependent oscillator Hamiltonians $H(t) = \frac{p^2}{2} + \beta(t)\frac{q^2}{2}$ with the variable elastic force, the evolution can be represented by the time dependent 2×2 symplectic matrix which in the symmetry intervals of β allows the explicit solutions, offering the soft imitations of the oscillator δ-kicks or the distorted cases of the free evolution (see Section 3.2-3.4 with the corresponding stability maps). Certain many dimensional models lead also to some interesting nonlinear matrix equations, with possible physical importance, though in general, they cannot be resolved without the help of computers. In several places our report offers no details, but only hints and references for interested readers.

1 THE DISCRETE CASE

1.1 An auxiliary space

A simple exponential structure was considered in 70-tieth by Jerzy Plebański, who was interested in the evolution operators generated by a continuous family of time dependent Hamiltonians, each two commuting to a number. Surprisingly, one of the most naive solution was obtained during the discussions at the Warsaw Institute of Theoretical Physics in 1957, by considering just two exponents commuting to a number. The solution was obtained a bit mysteriously, by repeating the same problem in two identical copies of the Hilbert space.

Suppose, we have two operators in a Hilbert space $\mathcal{H}$, commuting to a number $\alpha \in \mathbb{C}$. The problem is to express $e^a e^b$ as a single exponential e^Ω. Then consider a twin copy $\mathcal{H}'$ of the same Hilbert space with the twin copies a', b' of the operators a, b commuting to the same number $[a', b'] = \alpha$. Now define the tensor product $\mathcal{H} \otimes \mathcal{H}'$.

The operators a, b, a', b' can be interpreted as operators acting in whole $\mathcal{H} \otimes \mathcal{H}'$, both a, b transforming only the component $\mathcal{H}$ without affecting $\mathcal{H}'$ and inversely, a', b' transforming $\mathcal{H}'$ without affecting $\mathcal{H}$. Hence, the a, b and their functions commute with a', b'. It is also essential that in the commutator $[a+b', b+a']$ two commutators $[a, b] = \alpha$ and $[b', a'] = -\alpha$ cancel, so the $(a+b')$ and $(b+a')$ commute.

Varsovia 1957:

H

a, b

$[a, b] = \alpha$

$\otimes$

Auxiliary space:

H'

a', b'

$[a', b'] = \alpha$

Now consider the exponential products:

$$e^{a+b}\, e^{a'+b'} = e^{a+b+a'+b'} =$$
$$= e^{(a+b')+(b+a')} =$$
$$= e^{(a+b')}\, e^{(b+a')} =$$
$$= e^{a} e^{b'} e^{b} e^{a'} =$$
$$= e^{a} e^{b} \cdot e^{b'} e^{a'}$$

By multiplying both sides by $e^{-(a+b)}$ from the left and by $e^{-a'}e^{-b'}$ from the right one obtains:

$$\Rightarrow e^{-(a+b)} e^{a} e^{b} = e^{(a'+b')} e^{-a'} e^{-b'}$$

$$\shortparallel$$

number $= \varkappa \quad \Rightarrow e^{a} e^{b} = e^{i\varphi} e^{a+b}$

The only operator which acts in $\mathcal{H}$ without touching $\mathcal{H}'$ but simultaneously acts in $\mathcal{H}'$ without affecting $\mathcal{H}$ is just a number. Hence:

$$e^{-(a+b)}e^{a}e^{b} = \kappa \in \mathbb{C}.$$

If however e^a and e^b generate the unitary operations in $\mathcal{H}$, then κ must also be a unitary operator, implying $\kappa = e^{i\phi}$. Hence:

$$e^{a}e^{b} = e^{i\phi}e^{a+b}.$$

The result is easily (though just symbolically) extended to any number of operators or to the quantum evolution generated by the time dependent Hamiltonians H(t) commuting to numbers $[H(t), H(t')] = \alpha(t, t')$.

It is worth noticing that the auxiliary structures were used as a legitimate tool to prove some mathematical facts not only for the exponential multiplication. The analogous techniques are recently used by S.L. Woronowicz by associating the Heisenberg with 'anti-Heisenberg' descriptions in his research on quantum groups.

Plebański, meanwhile, considered the argument extremely peculiar, and he wanted more security. We have shown the lemma to Iwo Białynicki-Birula, who found that the solution though strange, was correct. But later on, he found also that the whole result was just an incomplete form of the very old problem of Baker–Campbell–Hausdorff (BCH) formula. So it was indeed, and to check it for a pair (or family) of operators commuting to a number, a simple differential equation works the best.

1.2 The differential equation

Suppose a, b are elements of a certain topological algebra $\mathcal{A}$ with some elements a, b commuting to a number $[a, b] = \alpha \in \mathbb{C}$. Then consider the 1-parameter families $e^{\lambda a}$ and $e^{\lambda b}$ ($\lambda \in \mathbb{R}$). Assume they are continuous and differentiable. Now, apply the derivative $\frac{d}{d\lambda}$ assuming that it is linear and with ordinary properties when acting on products. Both $e^{\lambda a}$ and $e^{\lambda b}$ are differentiable, obeying the obvious rules $\frac{d}{d\lambda}e^{\lambda a} = ae^{\lambda a} = e^{\lambda a}a$ (similarly for b). Now consider the product

$$U(\lambda) = e^{\lambda a} e^{\lambda b} e^{-\lambda(a+b)}.$$

Its derivative is:

$$\frac{dU}{d\lambda} = e^{\lambda a}(a+b)e^{\lambda b}e^{-\lambda(a+b)} - e^{\lambda a}e^{\lambda b}(a+b)e^{-\lambda(a+b)}. \quad (1.1)$$

This can be simplified by an obvious lemma. The formula of Baker [1]:

$$e^{\lambda B} A e^{-\lambda B} = A + \lambda[B, A] + \frac{\lambda^2}{2!}[B, [B, A]] + \dots$$

is obtained by developing formally the left side into the Taylor series in λ. It becomes specially elementary if only few multicommutators don't vanish. This happens precisely for $B = b$ and $A = a$, when already the first commutator is a number $[b, a] = -\alpha$ and commutes with all the rest. So:

$$e^{\lambda b} a e^{-\lambda b} = a - \lambda\alpha$$

implying the permutation rules

$$e^{\lambda b} a = (a - \lambda\alpha)e^{\lambda b}$$

(and the similar one by interchanging a and b and changing the sign of α). By employing it to the second part of the formula (1.1) one can interchange $e^{\lambda b}$ with $(a+b)$

obtaining to the right of $e^{\lambda a}$ the sum of 5 terms in which $a+b$ cancels with $-(a+b)$ leaving in place only the numeral term $-\lambda\alpha$, commuting with everything. Hence:

$$\frac{dU(\lambda)}{d\lambda} = \lambda\alpha U(\lambda)$$

is operator valued differential equation which can be easily solved:

$$U(\lambda) = e^{\frac{\lambda^2}{2}\alpha}U(0) = e^{\frac{\lambda^2}{2}[a,b]}U(0)$$

implying:

$$e^{\lambda a}e^{\lambda b} = e^{\frac{\lambda^2}{2}\alpha}e^{\lambda(a+b)} = e^{\lambda(a+b)+\frac{\lambda^2}{2}[a,b]},$$

i.e. the first approximation step for the general Baker-Campbell-Hausdorff formula — explaining the exact value of the phase factor $i\phi$ in our previous argument. The generalization for the multiple or continuous exponents commuting always to the numbers can be readily obtained.

In the similar formal way, one can show also that even if $[a,b]$ is not a number, but both double commutators are: $[a,[a,b]] \in \mathbb{C}$ and $[b,[b,a]] \in \mathbb{C}$, then the multiplication formula becomes:

$$e^{\lambda a}e^{\lambda b} = e^{\lambda(a+b)+\frac{\lambda^2}{2}[a,b]+\frac{\lambda^3}{12}([a,[a,b]]+[b,[b,a]])}.$$

The cases when all n-th order commutators are numbers, or a general exponent in form of an infinite multicommutator series require already the use of the "polarization derivative" of Hausdorff [2, 1, 3]. The related challenge is the composition of the continuous exponential products.

As interesting might be *a dual Zassenhaus problem* [4] of how to decompose the exponential $e^{\lambda(a+b)}$ into the product of simpler ones. One of the proposed decompositions is

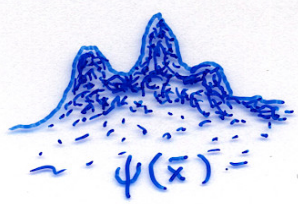
$$e^{\lambda(a+b)} = e^{\lambda a} e^{\lambda b} e^{-\frac{\lambda^2}{2}[a,b]} e^{\frac{\lambda^3}{6}(2[b,[a,b]]+[a,[a,b]])} \dots .$$

The same result of Friedrichs [6] implies as well that all increasing order exponents in this infinite product are Lie elements (and so, can be always written e.g. in Dynkin's multicommutator notation).

WHY ALL THIS CAN BE OF INTEREST FOR PHYSICS?

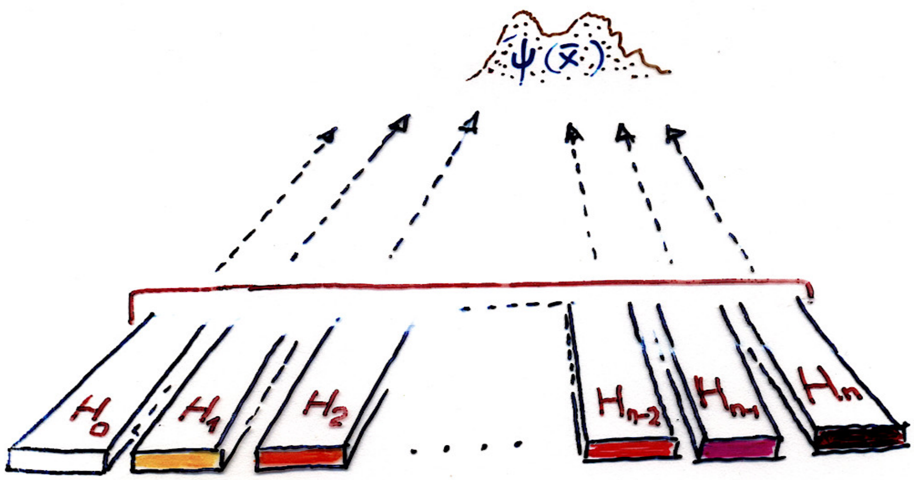
If the physical theories exist at all, it is only because...

...the physical objects:



exist in variable surrounding...

ALGEBRAIC TOOLS:



Press two keys:

$$U = e^{-i\tau_2 H_2} e^{-i\tau_1 H_1} = e^a e^b$$

Baker-Campbell-Hausdorff (1904-1906):

$$e^a e^b = e^{a+b+\frac{1}{2}[a,b]+\frac{1}{12}\left([a,a,b]+[b,[b,a]]\right)+}$$

Press more keys:

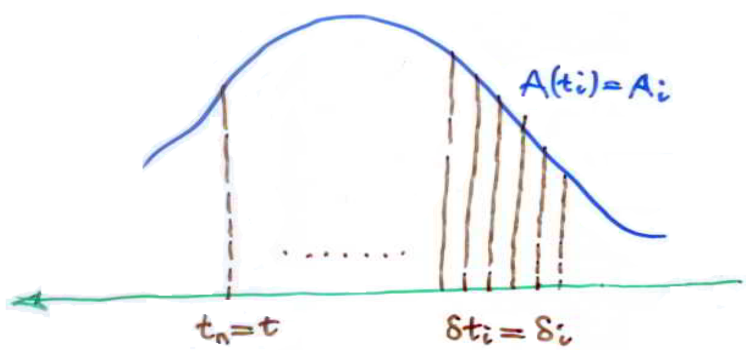
$$U = e^{-i\tau_n H_n} \dots e^{-i\tau_2 H_2} e^{-i\tau_1 H_1} =$$

$$= e^{a_n} \dots e^{a_2} e^{a_1} = e^{F(a_n,\dots,a_1)}$$ Dynkin (1947)

2 THE CONTINUOUS CASE

2.1 The limiting process

Some interesting consequences of the traditional Baker–Campbell–Hausdorff formula arise for increasing number of the exponential operators with infinitesimally small exponents



$$U(t) \cong e^{A_n \delta_n} \cdots e^{A_i \delta_i} \cdots e^{A_2 \delta_2} e^{A_1 \delta_1}$$

$$\Vert$$

$$e^{\Omega(t)}$$

la tarea de escribir $U(t)$
como un operador exponencial
$\equiv$ el problema de BCH continuo

$$\Omega(t) \neq \lim_{n \to \infty} \sum_{k=1}^{n} A_k \delta_k$$

But good news! If $\delta_k \longrightarrow 0$ the results suggest the differential equation for $U(t)$:

$$\frac{dU}{dt} = A(t)U(t)\ ; \quad U(t_0) = 1$$

$$U(t) = 1 + \int_{t_0}^{t} A(t_1)U(t_1)\,dt_1$$

$$U(t) = 1 + \int_{t_0}^{t} A(t_1)\,dt_1 + \int_{t_0}^{t}\int_{t_0}^{t_1} A(t_1)\,A(t_2)\,U(t_2)\,dt_2\,dt_1$$

$$U(t) = 1 + \int_{t_0}^{t} A(t_1)\,dt_1 + \int_{t_0}^{t}\int_{t_0}^{t_1} A(t_1)A(t_2)\,dt_2 dt_1 + \\ + \int_{t_0}^{t}\int_{t_0}^{t_1}\int_{t_0}^{t_2} A(t_1)A(t_2)A(t_3)\,dt_3 dt_2 dt_1 + \cdots$$

VERBAL SOLUTION:

$$U(t) = T\left\{1 + \int_{t_0}^{t} A(t_1)\,dt_1 + \frac{1}{2!}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)A(t_2)\,dt_2 dt_1 + \right. \\ \left. + \frac{1}{3!}\iiint_{t_0}^{t} A(t_1)\,A(t_2)\,A(t_3)\,dt_3 dt_2 dt_1 + \cdots\right\}$$

$$= T\left\{ e^{\int_{t_0}^{t} A(t')\,dt'} \right\}$$

"cronological" ordering operation

A bad news: it is O.K. to represent U(t) but can suggest wrong inspirations about its exponent!

2.2 Approximations of Magnus and Wilcox

To obtain more information about the exponent Ω, Magnus [7] and Wilcox [9] introduced the parameter λ into the continuous Baker–Campbell–Hausdorff problem

$$\frac{dU}{dt}(\lambda, t) = \lambda A(t)U(\lambda, t), \quad U(\lambda, 0) = \mathbf{1}.$$

They assume that $U(\lambda, t) = e^{\Omega(\lambda,t)}$. Then they are looking for Ω in form of a symbolic series

$$\Omega(\lambda, t) = \sum_{n=1}^{\infty} \lambda^n \Delta_n(t)$$

and tried to find it by using the integral equation inspired by Hausdorff [3]

$$\int_0^1 e^{\mu\Omega} \frac{d\Omega}{dt} e^{-\mu\Omega} d\mu = \lambda A(t).$$

Magnus obtain Δ_1, Δ_2 and Δ_3 and conclude the rest is "combinatorial mess". Wilcox obtained still Δ_4. Among them only Δ_1 and Δ_2 are easy to guess:

$$\Delta_1(t) = \int_0^t A(t_1)dt_1,$$
$$\Delta_2(t) = \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} [A(t_1), A(t_2)]dt_2.$$

The remaining Δ_3 and Δ_4 are indeed involved even in multi commutator terms [5, 9, 11]. Yet, much simpler results follow directly from $U(\lambda, t)$.

2.3 The search for explicit expressions

An authentic breakthrough came from the formal series expression for the $U(t)$.

$$U(t) =$$

$$= 1 + \int_{t_0}^{t} A(t_1)\,dt_1 + \int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\theta_{12}\,A(t_2)\,dt_2\,dt_1 +$$
$$+ \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\theta_{12}\,A(t_2)\,\theta_{23}\,A(t_3)\,dt_3\,dt_2\,dt_1 + \cdots$$
$$= 1 + Z$$

where

$$Z =$$

$$\int_{t_0}^{t} A(t_1)\,dt_1 + \int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\theta_{12}\,A(t_2)\,dt_2\,dt_1 +$$
$$+ \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\theta_{12}\,A(t_2)\,\theta_{23}\,A(t_3)\,dt_3\,dt_2\,dt_1 + \cdots \tag{2.1}$$

and

$$\theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}, \quad \theta_{ij} = \theta(t_i - t_j).$$

$$\text{If } U(t) = e^{\Omega} \Rightarrow$$

$$\Rightarrow \Omega = \ln(1+Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 - \cdots$$

where

$$Z^2 = \int_{t_0}^{t}\int_{t_0}^{t} A(t_1)A(t_2)\,dt_2\,dt_1 + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\theta_{12}\,A(t_2)A(t_3)\,dt_3\,dt_2\,dt_1 +$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)A(t_2)\,\theta_{23}\,A(t_3)\,dt_3\,dt_2\,dt_1 + \ldots$$

It is important to notice that the formula for Z^2 can be obtained from the expression for Z by an operation X of dropping some θ'as:

$$Z^2 = \int_{t_0}^{t} \cancel{A(t_1)}\,dt_1 + \int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\cancel{\theta_{12}}\,A(t_2)\,dt_2\,dt_1 +$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\cancel{\theta_{12}}\,A(t_2)\,\theta_{23}\,A(t_3)\,dt_3\,dt_2\,dt_1 +$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} A(t_1)\,\theta_{12}\,A(t_2)\,\cancel{\theta_{23}}\,A(t_3)\,dt_3\,dt_2\,dt_1 + \ldots$$

It turns out that X is the differentiation which acts on θ:

$$X = \frac{d}{d\theta} \quad \Big\} \quad \frac{d}{d\theta} A(\,) = 0 \,; \quad \frac{d}{d\theta}\theta_{ij} = 1$$

$$\boxed{Z^2 = \frac{d}{d\theta} Z} \qquad \text{¿ linearización de } Z^2 \text{?}$$

$$Z^3 = ?? \qquad Z^3 = \frac{1}{2}\left[Z^2 Z + Z Z^2\right] =$$

$$= \frac{1}{2}\left[\left(\frac{d}{d\theta}Z\right)Z + Z\frac{d}{d\theta}Z\right] =$$

$$= \frac{1}{2}\frac{d}{d\theta}Z^2 = \frac{1}{2}\frac{d}{d\theta}\left(\frac{d}{d\theta}Z\right) = \frac{1}{2}\frac{d^2}{d\theta^2}Z$$

Inducción :

$$Z^n = \frac{1}{(n-1)!}\frac{d^{n-1}}{d\theta^{n-1}} Z$$

Henceforth, the operation X permits to write down the continuous analog of Baker–Campbell–Hausdorff formula with all terms linear in Z:

$$\Omega(t) = \ln\left[1 + Z\right] =$$

$$= Z - \frac{1}{2} Z^2 + \frac{1}{3} Z^3 - \cdots$$

$$= Z - \frac{1}{2}\frac{d}{d\theta} Z + \frac{1}{3}\frac{1}{2}\frac{d^2}{d\theta^2} Z - \cdots =$$

$$= \left[1 - \frac{1}{2!}\frac{d}{d\theta} + \frac{1}{3!}\frac{d^2}{d\theta^2} - \frac{1}{4!}\frac{d^3}{d\theta^3} + \cdots\right] Z =$$

$$= \left[\frac{e^{-\frac{d}{d\theta}} - 1}{-\frac{d}{d\theta}}\right] Z =$$

$$= \frac{e^{-\frac{d}{d\theta}} - 1}{-\frac{d}{d\theta}} \sum_{n=1}^{+\infty} \underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{t}}_{n} \theta_{21} \cdots \theta_{n,n-1} A(t_1) \cdots A(t_n) \, dt_n \cdots dt_1$$

$$= \sum_{n=1}^{\infty} \int_{t_0}^{t} \cdots \int_{t_0}^{t} L_n(t_1, \ldots, t_n) \, A(t_1) \cdots A(t_n) \, dt_n \cdots dt_1$$

where $L_n(t_1, \ldots, t_n)$ are numerical integration kernels

$$L_n(t_1,\dots,t_n) = \frac{e^{-\frac{d}{d\theta}}-1}{-\frac{d}{d\theta}}\,\theta_{21}\cdots\theta_{n,n-1}$$

Since every product $\theta_{2,1}\theta_{3,2}\ldots\theta_{k,k-1}$ contains only the finite number of θ'as, then the all higher order derivatives $\frac{d^n}{d\theta^n}$, with $n > k$, vanish and every $L_n(t_1,\ldots,t_n)$ reduces itself into an explicitly known, finite combination of θ-products. For instance:

$$\begin{aligned}
L_1 =& 1,\\
L_2 =& \theta_{2,1} - \frac{1}{2},\\
L_3 =& \theta_{3,2}\theta_{2,1} - \frac{1}{2}\theta_{2,1} - \frac{1}{2}\theta_{3,2} + \frac{1}{3},\\
L_4 =& \theta_{4,3}\theta_{3,2}\theta_{2,1} - \frac{1}{2}\theta_{4,3}\theta_{3,2} - \frac{1}{2}\theta_{4,3}\theta_{2,1} - \frac{1}{2}\theta_{3,2}\theta_{2,1} +\\
& + \frac{1}{3}\theta_{2,1} + \frac{1}{3}\theta_{3,2} + \frac{1}{3}\theta_{4,3} - \frac{1}{4},\\
& etc.
\end{aligned}$$

It is interesting to apply also the integral expression

$$\frac{e^{-\frac{d}{d\theta}}-1}{-\frac{d}{d\theta}} = \int_0^1 e^{-\mu\frac{d}{d\theta}}\, d\mu.$$

By using the permutation rule

$$e^{-\mu\frac{d}{d\theta}}\,\theta_{ij} = (\theta_{ij}-\mu)\, e^{-\mu\frac{d}{d\theta}}$$

one obtains:

$$L_n(t_1,\ldots,t_n) = \left(\frac{e^{-\frac{d}{d\theta}}-1}{-\frac{d}{d\theta}}\right)\theta_{21}\ldots\theta_{n,n-1} =$$

$$= \int_0^1 e^{-\mu\frac{d}{d\theta}}\left(\theta_{21}\ldots\theta_{n,n-1}\right)d\mu =$$

$$= \int_0^1 (\theta_{21}-\mu)\ldots(\theta_{n,n-1}-\mu)\,d\mu.$$

An interesting consequence was derived in [11]:

$$\Theta_n = \theta_{21} + \theta_{32} + \ldots + \theta_{n,n-1}$$

$$L_n(t_n,\ldots,t_1) = \frac{(-1)^{n-1-\Theta_n}}{n}\binom{n-1}{\Theta_n}^{-1}$$

$$= (-1)^{n-1-\Theta_n}\,\frac{\Theta_n!\,(n-1-\Theta_n)!}{n!}$$

Ample discussions of the results in [10, 11] were offered by J. Czyż [12] and I.M. Gelfand [13].

Due to the theorem of Friedrichs, all terms of the explicit formula (2.1) are the Lie elements of the free algebra containing the operators $A(t)$. However, if applied to A(t) of a finite dimensional Lie group, the increasing multicommutator terms will show repetitions leading to finite dimensional matrix algebras. The simplest case of this mechanism are the quantum theories of oscillators with time dependent elastic forces.

3 VARIABLE OSCILLATORS

3.1 Classical–Quantum equivalence

Between the quantum systems with Hamiltonians quadratic in canonical variables, the simplest solutions are obtained for $1D$ oscillators (1 position + 1 momentum) permitting to simplify the theory due to a notable phenomenon: the 2×2 matrices of the evolution of classical and quantum (q, p) variables are exactly the same!

CLASSICAL $\equiv$ QUANTUM

$$H(t) = \frac{p^2}{2} + \beta(t)\frac{q^2}{2} \qquad [q, p] = i$$

t. clasica — t. cuantica

$$\frac{dq}{dt} = p \qquad \equiv \qquad \frac{d}{dt} q = [iH, q] = p$$

$$\frac{dp}{dt} = -\beta(t) q \qquad \equiv \qquad \frac{d}{dt} p = [iH, p] = -\beta(t) q$$

$$\begin{Vmatrix} q(t) \\ p(t) \end{Vmatrix} = \underbrace{\begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}}_{b(t, t_0)} \begin{Vmatrix} q(t_0) \\ p(t_0) \end{Vmatrix}$$

$$\frac{d}{dt} b(t,t_0) = \Lambda(t)\, b(t,t_0)$$

$$\frac{d}{dt_0} b(t,t_0) = -b(t,t_0)\,\Lambda(t_0)$$

$$\Lambda(t) = \begin{Vmatrix} 0 & 1 \\ -\beta(t) & 0 \end{Vmatrix}$$

Note that the two unitary operators U_1, U_2 which generate the same transformation of the q, p pairs can differ only by a $\mathbb{C}$-number phase. Indeed:

Lema $\quad U_1^\dagger q U_1 = U_2^\dagger q U_2$ and

$$U_1^\dagger p U_1 = U_2^\dagger p U_2$$

$$\Rightarrow [U_1 U_2^\dagger, q] = [U_1 U_2^\dagger, p] = 0$$

$$\Rightarrow [U_1 U_2^\dagger, f(q,p)] = 0$$

$\boxed{U_1 \equiv U_2}$

$$\Rightarrow U_1 U_2^\dagger = e^{i\varphi} \Rightarrow U_1 = e^{i\varphi} U_2$$

One can consider the operators U_1 and U_2 *equivalent* in any physical experiment. Atte! To underline the relation between the evolution operator $U = U(t,t_0)$ the corresponding *evolution matrix* will be denoted $u = u(t,t_0)$.

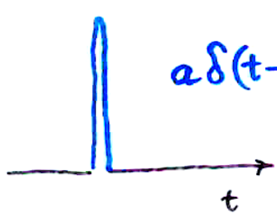
Consequences: The classical canonical transformations determine the unitary evolution operators. $\Longrightarrow$ The classical motion permits reconstruct the quantum evolution for general time dependent elastic force $-\beta(t)q$.

3.2 Kick operations

The general solutions of both classical and quantum problem requires the computer solutions of the common 2×2 evolution matrix $u(t)$. However, the problem admits some extremely simple exact solutions.

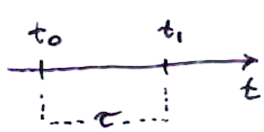
Soluciones Exactas (singulares)

Pulsos δ-ta:



$$a\delta(t-t_0)\frac{q^2}{2} \Rightarrow U(a) = e^{-ia\frac{q^2}{2}}$$

Evolución libre:



$$\Rightarrow U(\tau) = e^{-i\tau\frac{p^2}{2}}$$

Both are easily obtained from the general law of Baker [1]:

FORMULA DE BAKER:

$$e^{\lambda a} B e^{-\lambda a} = B + \lambda [a, B] + \frac{\lambda^2}{2!} [a,[a,B]] + \cdots + \frac{\lambda^n}{n!} [a,[a,\ldots[a,B]\ldots] + \ldots$$

Hence, the sequence of 3 operations (a pair of free evolution steps separated by one oscillator kick):

$$\underbrace{e^{-i\tau\frac{p^2}{2}} e^{-i\frac{1}{\tau}\frac{q^2}{2}} e^{-i\tau\frac{p^2}{2}}} \equiv F$$

generates:

$$u_F = \begin{pmatrix} 0 & \tau \\ -\frac{1}{\tau} & 0 \end{pmatrix}$$

which might be called a "squeezed Fourier operation". Curiously, the same is generated by two kicks separated by the free evolution interval:

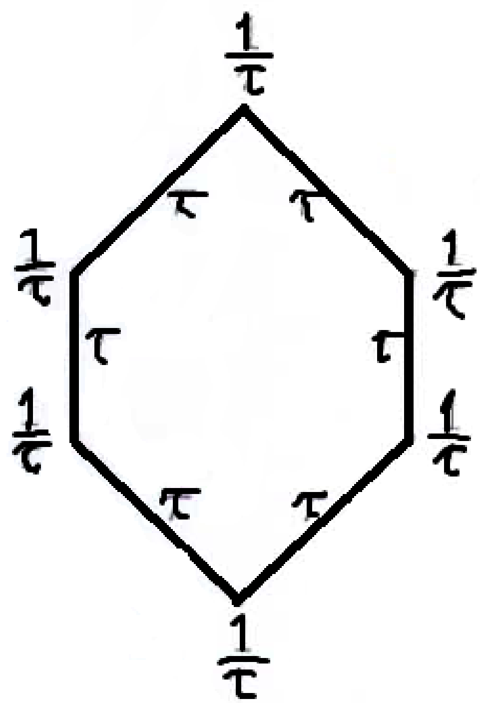
$$\underbrace{e^{-i\frac{1}{\tau}\frac{q^2}{2}}\, e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}}} \equiv F.$$

So, the following 6 operations yield $q \to -q$, $p \to -p$ (the parity operator)

$$\underbrace{e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}} \cdots e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}}} \equiv P,$$

while the sequence of 12 unitary terms produces an evolution loop:

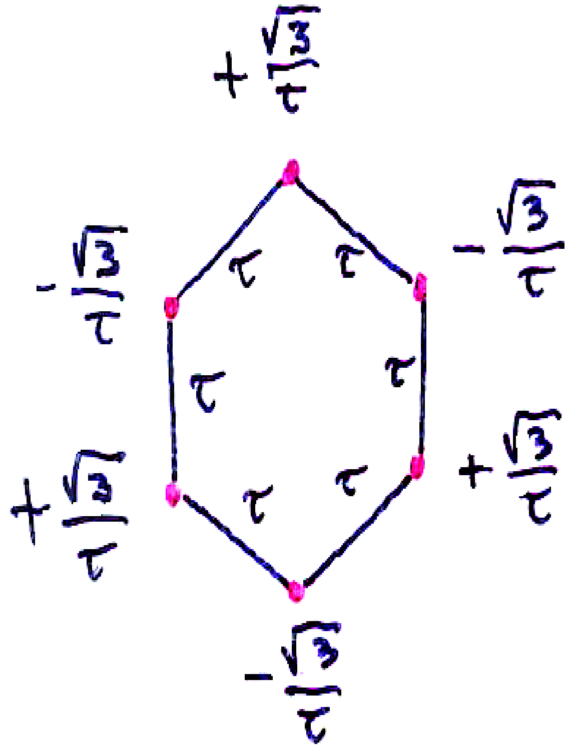
$$\underbrace{e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}} \cdots\cdots e^{-i\tau\frac{p^2}{2}}\, e^{-i\frac{1}{\tau}\frac{q^2}{2}}}_{\text{12 terms}} \equiv 1$$

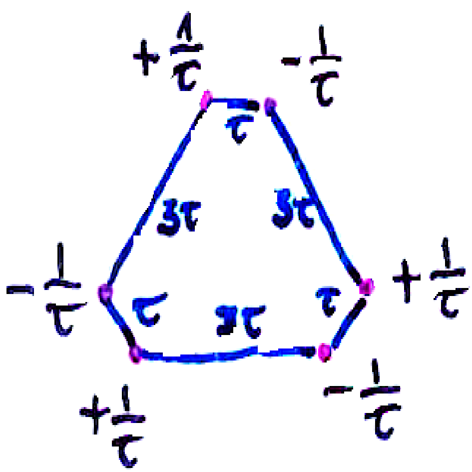


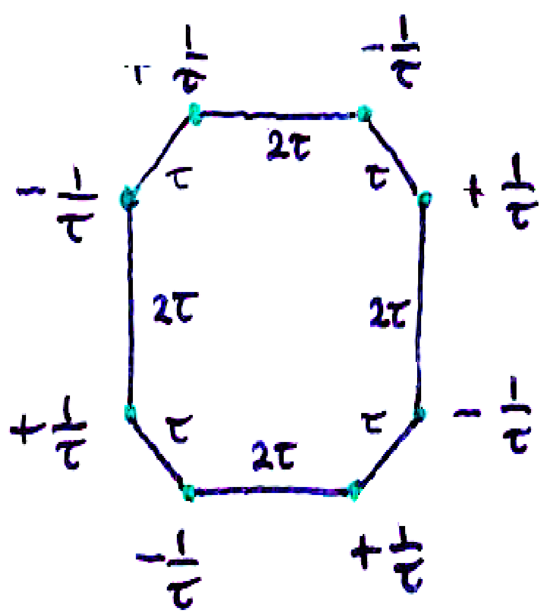
Here, the sides of the hexagon represent the 6 identical free evolution intervals for the time τ and all vertices stand for the oscillator potential kicks with $\frac{1}{\tau}$ forces. By the same the sequence of 11 operators only must invert the free evolution during the time τ:

$$e^{-i\frac{1}{\tau}\frac{q^2}{2}} \cdots\cdots e^{-i\tau\frac{p^2}{2}} e^{-i\frac{1}{\tau}\frac{q^2}{2}} \equiv e^{i\tau\frac{p^2}{2}}$$

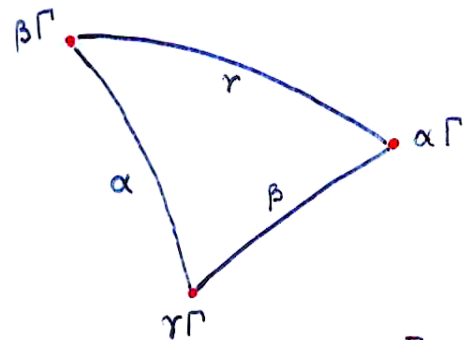
OTHER CURIOUS CASES OF BCH







TRIANGULO DE INVERSIÓN:



$$\Gamma = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$$

$$e^{+i\alpha\frac{p^2}{2}} \equiv e^{-i\beta\Gamma\frac{q^2}{2}} e^{-i\gamma\frac{p^2}{2}} e^{-i\alpha\Gamma\frac{q^2}{2}} \cdot$$

$$\cdot\, e^{-i\beta\frac{p^2}{2}} e^{-i\gamma\Gamma\frac{q^2}{2}}$$

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3.3 The general operations

The description of the soft manipulations in general requires the computer solutions for the matrix equations:

$$\frac{du}{dt}(t,\tau) = \Lambda(t)u(t,\tau),$$
$$\frac{du}{d\tau}(t,\tau) = -u(t,\tau)\Lambda(\tau).$$

An interesting observation was, however, that if $\beta(t)$ and therefore $\Lambda(t)$ is symmetric with respect to some given moment $t = t_0$, then some results of the evolution can be exactly predicted. Assume for simplicity $t_0 = 0$ and consider the evolution matrix $u = u(t,-t)$ in a symmetric interval $[-t,t]$. Then the equation for u has the anticommutator form:

$$\frac{du}{dt} = \Lambda(t)u + u\Lambda(t)$$

or explicitly

$$\frac{du}{dt} = \begin{pmatrix} u_{21} - \beta u_{12} & \mathrm{Tr}\ u \\ -\beta \mathrm{Tr}\ u & u_{21} - \beta u_{12} \end{pmatrix} =$$
$$= (u_{21} - \beta u_{12})\,\mathbf{1} + \mathrm{Tr}\ u \begin{pmatrix} 0 & 1 \\ -\beta & 0 \end{pmatrix}.$$

Therefore,

$$\frac{d}{dt}\,(u_{12}u_{21}) = \mathrm{Tr}\ u\,(u_{21} - \beta u_{12}) =$$

$$= \text{Tr } u \frac{1}{2} \frac{d}{dt} \text{Tr } u = \frac{1}{4} \frac{d}{dt} (\text{Tr } u)^2$$

and integrating

$$\frac{d}{dt} \left[u_{12} u_{21} - \frac{1}{4} (\text{Tr } u)^2 \right] = 0$$

$$\Downarrow$$

$$u_{12} u_{21} - \frac{1}{4} (\text{Tr } u)^2 = C = const.$$

The initial condition $u(0,0) = 1$ imply $C = -1$ and so

$$u_{12} u_{21} = -\frac{1}{4} (\text{Tr } u)^2 - 1. \tag{3.1}$$

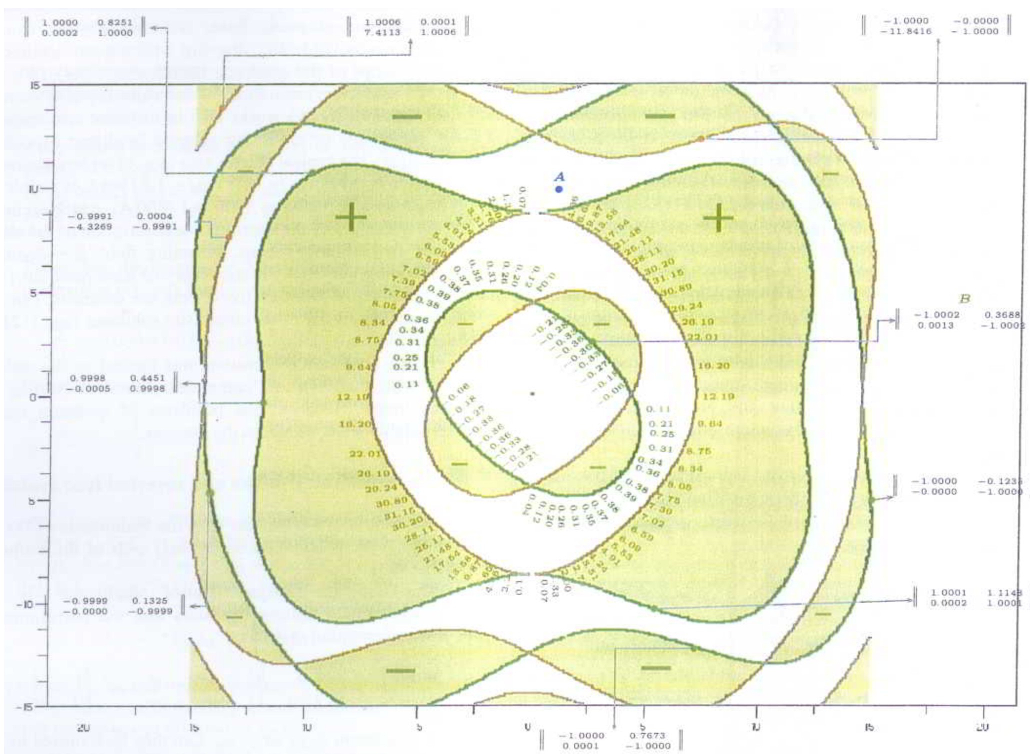
Hence, one arrives at the following lemma.

Lemma 3.1. *Whenever the evolution matrix* $u(t) = u(t, -t)$ *for symmetric* $\beta(t)$ *reaches the threshold values* $Tr\ u = \pm 2$, *(3.1) implies that either* u_{12} *or* u_{21} *(or both) must vanish and simultaneously* $u_{11} = u_{22} = \pm 1$, *leading to the soft evolution cases imitating the oscillator kicks, incidents of distorted free evolution, or just one of the evolution loops, all of them with or without simultaneous parity transformation (see [16, 17]).*

3.4 The manipulation by time dependent magnetic fields

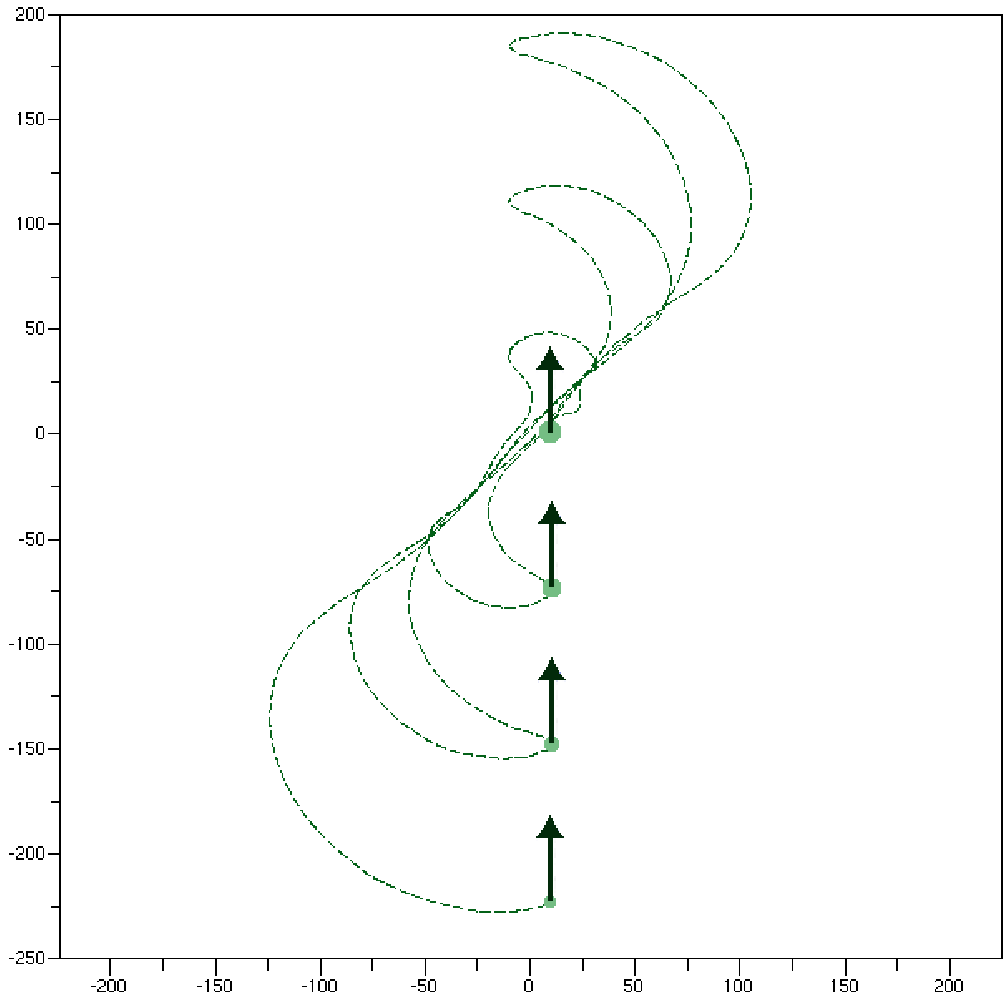
The above results were applied to the evolution operators induced by homogeneous variable magnetic fields

$B(t)$. If the time dependence of $B(t)$ is not too violent, the non-relativistic approximation still holds up to $\frac{1}{c^2}$-terms, (see [14, 15, 16, 17, 18]) the motion of charged particles obeys the 2 dimensional oscillators with time-dependence radial force. The stability and instability areas for fields of two frequencies ω and 2ω were determined in [16], leading to the following map of A. Ramirez:



The Ramirez map in [16]. The point on the blue and red stability borders represent the field parameters permitting to approximate the free evolution with modified (faster, slower or inverted) evolution time, or the softly achieved radial oscillator pulses.

Below, the motion of the center of a Gaussian wave packet of a charged particle, in a pulsating magnetic field [16]. The centers perform displacements opposite to the initial velocity.



The search for variable oscillator pulses, permitting to achieve some physically interesting result, in spite of its narrow subject, is still an open area. In particular, you might be interested to consult [19] (non-hermitian problems), also [20] (non-linear equations for higher dimensional models) [21] (the exponential formula for higher dimensional matrices, though the physical applications are still an open problem).

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