# Soliton hierarchies and matrix loop algebras

Wen-Xiu Ma

Department of Mathematics and Statistics
University of South Florida, USA

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# Outline

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# Soliton equations

$$\phi_x = U(u,\lambda)\phi \text{ or } E\phi = U(u,\lambda)\phi \quad \Leftrightarrow \quad u_t = \Phi^n K_0[u]$$

$$\Updownarrow \qquad \boxed{\begin{aligned} u_t = K_0[u] \Leftrightarrow U_t - V_x + [U,V] = 0 \\ \text{or} \quad U_t + UV - (EV)U = 0 \end{aligned}} \qquad \Updownarrow$$

spectral matrix $U$ $\quad\Leftrightarrow\quad$ recursion operator $\Phi$

# KdV equation

The KdV equation:

$$u_t - \frac{3}{2} b_1 u u_x - \frac{1}{4} b_1 u_{xxx} = 0.$$

Lax Pair:

$$U = \begin{bmatrix} 0 & 1 \\ \lambda - u & 0 \end{bmatrix},$$

$$V = b_1 \begin{bmatrix} -\frac{1}{4} u_x & \lambda + \frac{1}{2} u \\ -(\lambda + \frac{1}{2} u)(u - \lambda) - \frac{1}{4} u_{xx} & \frac{1}{4} u_x \end{bmatrix}.$$

# NLS equations

The nonlinear Schrödinger equations:

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q, \\ q_t = \frac{1}{2}q_{xx} - pq^2. \end{cases}$$

Lax Pair:

$$U = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},$$

$$V = \begin{bmatrix} -\lambda^2 + \frac{1}{2}pq & \lambda p - \frac{1}{2}p_x \\ \lambda q + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq \end{bmatrix}.$$

# Symmetry and conservation law

**Symmetry:**

$S$ is called a symmetry of $u_t = K(u)$, if

$$[K, S] = K'[S] - S'[K] = 0, \; P'[S] = \frac{\partial}{\partial \varepsilon} P(u + \varepsilon S)\Big|_{\varepsilon=0}.$$

This defines a commuting flow with $u_t = K(u)$.

**Conservation law:**

A conservation law is

$$\partial_t T + \partial_x X = 0 \;\; \text{when} \;\; u_t = K(u).$$

This gives a conserved density:

$$\frac{d}{dt} \int T \, dx = 0 \;\; \text{when} \;\; u_t = K(u).$$

# The fundamental question

**Question:**

How to generate soliton equations with infinitely many symmetries and/or conservation laws?

**Starting point:**

Spectral problems on matrix loop algebras

# Spectral problems

Let $g$ be a semisimple Lie algebra and its loop algebra

$$\tilde{g} = g \otimes C[\lambda, \lambda^{-1}].$$

Choose a peudoregular element $e_0(\lambda)$:

$$\text{(a) } \mathrm{Ker}(\mathrm{ad} e_0) \oplus \mathrm{Im}(\mathrm{ad} e_0) = \tilde{g},$$

$$\text{(b) } \mathrm{Ker}(\mathrm{ad} e_0) \text{ is commutative.}$$

Spectral problem with linearly independent $e_i \in \tilde{g},\ 0 \leq i \leq q$:

$$\phi_x = U\phi,\ \ U = U(\lambda, u) = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_q e_q(\lambda).$$

# Zero curvature equations

Solve the stationary zero curvature equation

$$V_x = [U, V], \;\; V = \sum_{i \geq 0} V_i \lambda^{-i}.$$

Select $\Delta_n$ so that

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n$$

where $+$ means to take the polynomial part, satisfies

$$V_x^{(n)} - [U, V^{(n)}] \in \mathrm{span}(e_1, \cdots, e_q).$$

# Zero curvature equations

- P.D. Lax, *Comm. Pure Appl. Math.*, **21**(1968), 467-490.

Lax pairs:

$$U, V^{(n)}$$

or

$$\phi_x = U\phi, \ \phi_{t_n} = V^{(n)}\phi.$$

Zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$$

present a soliton hierarchy

$$u_{t_n} = K_n(u), \ n \geq 0.$$

# Algebraic structure of Lax operators

- W.X. Ma, *J. Phys. A*, **25**(1992), 5329-5343; **26**(1993), 2573-2582.

An evolution equation

$$u_t = K(u)$$

$$\Updownarrow$$

$$U'[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0$$

$$\Updownarrow$$

$$\phi_x = U(u, \lambda)\phi, \ \ \phi_t = V(u, \lambda)\phi$$

under $\lambda_t = f(\lambda)$.

# Commutators

Introduce

$$[K,S]=K'[S]-S'[K],$$
$$[\![V,W]\!]=V'[S]-W'[K]+[V,W]+gV_\lambda-fW_\lambda,$$
$$[\![f,g]\!](\lambda)=f'(\lambda)g(\lambda)-f(\lambda)g'(\lambda),$$

where

$$P'[S]=\frac{\partial}{\partial\epsilon}P(u+\epsilon S)\Big|_{\epsilon=0}.$$

# Algebraic structure

- W.X. Ma, *J. Phys. A*, **26**(1993), 2573-2582.

If

$$U'[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0,$$
$$U'[S] + g(\lambda)U_\lambda - W_x + [U, W] = 0,$$

then

$$U'[\,[K, S]\,] + [\![f, g]\!](\lambda)U_\lambda - [\![V, W]\!]_x + [U, [\![V, W]\!]] = 0.$$

# Lie algebraic structure

- W.X. Ma, *British J. Appl. Sci. Tech.*, **3**(2013), 1336-1344.

All $(K, V, f)$ form a Lie algebra under the binary operation:

$$[\![(K, V, f), (S, W, g)]\!] = ([K, S], [\![V, W]\!], [\![f, g]\!]).$$

That is, the above operation satisfies

- Bilinearity:

$$\begin{aligned}&[\![\alpha(K_1, V_1, f_1) + \beta(K_2, V_2, f_2), (K_3, V_3, f_3)]\!] \\ &= \alpha[\![(K_1, V_1, f_1), (K_3, V_3, f_3)]\!] + \beta[\![(K_2, V_2, f_2), (K_3, V_3, f_3)]\!].\end{aligned}$$

- Anticommutativity:

$$[\![(K_1, V_1, f_1), (K_2, V_2, f_2)]\!] = -[\![(K_2, V_2, f_2), (K_1, V_1, f_1)]\!].$$

- The Jacobi Identity:

$$[\![(K_1, V_1, f_1), [\![(K_2, V_2, f_2), (K_3, V_3, f_3)]\!]]\!] + \mathrm{cycle}(1, 2, 3) = 0.$$

# Symmetry algebras

**Symmetry algebras in (1+1)-dimensions:**

$$[K_m, K_n] = 0,$$

$$[K_n, \tau_{s,m}] = (m+\gamma+1)K_{m+n},$$

$$[\tau_{s,n}, \tau_{s,m}] = (m-n)\tau_{s,m+n},$$

where $\tau_{s,m} = \sigma_{m+1} + t[K_s, \sigma_{m+1}]$.

**Graded symmetry algebras in higher-dimensions:**

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$$

for KP, MKP, etc.

# Trace and variational identities

**Semisimple Lie algebras:**

$$\frac{\delta}{\delta u}\int \mathrm{tr}(V\frac{\partial U}{\partial \lambda})\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}(V\frac{\partial U}{\partial u}),\ \ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(V^2)|.$$

**Non-semisimple Lie algebras:**

$$\frac{\delta}{\delta u}\int \langle V,\frac{\partial U}{\partial \lambda}\rangle\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle V,\frac{\partial U}{\partial u}\rangle,\ \ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln\langle V,V\rangle,$$

where $\langle\cdot,\cdot\rangle$ is an ad-invariant, symmetric and non-degenerate bilinear form.

# Lie algebra sl(2, $\mathbb{R}$)

sl(2, $\mathbb{R}$):

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,$$

where

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

# The KdV equations

The KdV spectral problem:

$$\phi_x = U\phi,\ \ U = U(u,\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda - u & 0 \end{bmatrix} \in \tilde{g},$$

where

$$\text{(a) Lie algebra: } \ \tilde{g} = \mathrm{sl}(2) \otimes C[\lambda, \lambda^{-1}],$$

$$\text{(b) Pseudoregular element: } \ e_0 = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}.$$

# Stationary zero curvature equation

A solution to $V_x = [U, V]$:

$$V = \begin{bmatrix} -\frac{1}{2} b_x & b \\ -\frac{1}{2} b_{xx} + (\lambda - u) b & \frac{1}{2} b_x \end{bmatrix},$$

where

$$b = \sum_{i \geq 0} b_i \lambda^{-i},$$

with

$$b_0 = 0, \;\; b_1 = 1, \;\; b_{i+1} = \Psi b_i, \;\; \Psi = \frac{1}{4}\partial^2 + u - \frac{1}{2}\partial^{-1} u_x, \;\; i \geq 1.$$

# Soliton hierarchy

Lax pairs:

$$\phi_x = U\phi,\ \phi_t = V^{(n)}\phi,\ V^{(n)} = (\lambda^{n+1}V)_+ + \begin{bmatrix} 0 & 0 \\ -b_{n+2} & 0 \end{bmatrix},\ n \geq 0.$$

The KdV soliton hierarchy:

$$u_{t_n} = \Phi^n u_x,\ \ n \geq 0,$$

with the recursion operator $\Phi$:

$$\Phi = \Psi^\dagger = \frac{1}{4}\partial^2 + u + \frac{1}{2}u_x\partial^{-1}.$$

# Symmetry algebra

Non-isospectral flows ($\lambda_t = \lambda^{n+1}$):

$$u_{s_n} = \sigma_n = \Phi^n \sigma_0, \ \ \sigma_0 = u + \frac{1}{2} x u_x, \ \ n \geq 0.$$

Symmetry algebra:

$$[K_m, K_n] = 0,$$

$$[K_n, \tau_{s,m}] = (m + \frac{1}{2}) K_{m+n},$$

$$[\tau_{s,n}, \tau_{s,m}] = (m - n)\tau_{s,m+n},$$

where

$$\tau_{s,m} = \sigma_m + t[K_s, \sigma_m].$$

# The AKNS equations

- M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, *Stud. Appl. Math.*, **53**(1974), 249-315.

The AKNS spectral problem:

$$\phi_x = U\phi, \ \ U = U(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix} \in \tilde{g}, \ \ u = \begin{bmatrix} p \\ q \end{bmatrix},$$

where

(a) Lie algebra: $\tilde{g} = \mathrm{sl}(2) \otimes C[\lambda, \lambda^{-1}]$,

(b) Pseudoregular element: $e_0 = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

# Stationary zero curvature equation

A solution to $V_x = [U, V]$:

$$V = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0} V_i \lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}$$

with the initial data

$$a_0 = -1,\ b_0 = c_0 = 0$$

and

$$\begin{cases} a_{ix} = qc_i - rb_i, \\ b_{ix} = -2b_{i+1} - 2qa_i, \qquad i \geq 0. \\ c_{ix} = 2c_{i+1} + 2ra_i, \end{cases}$$

# Soliton hierarchy

Lax pairs:

$$\phi_x = U\phi,\ \ \phi_t = V^{(n)}\phi,\ \ V^{(n)} = (\lambda^n V)_+,\ \ n \geq 0.$$

The AKNS soliton hierarchy:

$$u_{t_n} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_n} = K_n = \begin{bmatrix} -2b_{n+1} \\ 2c_{n+1} \end{bmatrix} = \Phi^n \begin{bmatrix} -2p \\ 2q \end{bmatrix},\ \ n \geq 0,$$

with the recursion operator $\Phi$:

$$\Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}.$$

# Symmetry algebra

Non-isospectral flows ($\lambda_t = \lambda^n$):

$$u_{s_n} = \sigma_n = \Phi^n \sigma_0, \;\; \sigma_0 = \begin{bmatrix} -2xp \\ 2xq \end{bmatrix}, \;\; n \geq 0.$$

Symmetry algebra:

$$[K_m, K_n] = 0,$$

$$[K_n, \tau_{s,m}] = mK_{m+n},$$

$$[\tau_{s,n}, \tau_{s,m}] = (m-n)\tau_{s,m+n},$$

where

$$\tau_{s,m} = \sigma_{m+1} + t[K_s, \sigma_{m+1}].$$

# Lie algebra $(3, \mathbb{R})$

so$(3, \mathbb{R})$:

$$[e_1, e_2] = e_3, \ [e_2, e_3] = e_1, \ [e_3, e_1] = e_2,$$

where

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

# Spectral problems associated with so(3, $\mathbb{R}$)

An AKNS type spectral problem

$$\phi_x = U\phi, \ \ U = U(u, \lambda) = \lambda e_1 + pe_2 + qe_3.$$

A Kaup-Newell type spectral problem

$$\phi_x = U\phi, \ \ U = U(u, \lambda) = \lambda^2 e_1 + \lambda pe_2 + \lambda qe_3.$$

A WKI type spectral problem

$$\phi_x = U\phi, \ \ U = U(u, \lambda) = \lambda e_1 + \lambda pe_2 + \lambda qe_3.$$

# Recursion operators

- W.X. Ma, Appl. Math. Comput., 220(2013), 117
- W.X. Ma, J. Math. Phys., 54(2013), 103509

AKNS type recursion operator

$$\Phi = \begin{bmatrix} q\partial^{-1}p & \partial + q\partial^{-1}q \\ -\partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix}.$$

Kaup-Newell type recursion operator

$$\Phi = \begin{bmatrix} -\partial p\partial^{-1}p & \partial - \partial p\partial^{-1}q \\ -\partial - \partial q\partial^{-1}p & -\partial q\partial^{-1}q \end{bmatrix}.$$

# Lax pairs

## Selection of Lax operators:

How to determine modifications $\Delta_n$ so that

$$V_x^{(n)} - [U, V^{(n)}] \in \mathrm{span}(e_1, \cdots, e_q),$$

where $V^{(n)} = (\lambda^n V)_+ + \Delta_n$?

## Existence of solutions:

Determine when there exist solutions to

$$V_x = [U, V],\ U \in \tilde{g} = g \otimes C[\lambda, \lambda^{-1}],$$

when $g$ is non-semisimple.

# Criterion for existence of Hamiltonian structures

**Open question:**

Is there any concrete criterion which tells when there exist Hamiltonian structures for integrable couplings, even bi- and tri-integrable couplings?

More specially, how to generalize the variational identity on matrix loop algebras?

A concrete example is the bi-integrable coupling

$$u_t = K(u), \; v_t = K'(u)[v], \; w_t = K'(u)[w].$$

Is there any Hamiltonian structure for this integrable coupling?

# Conjecture on integrability of commuting soliton equations

- For a soliton hierarchy $u_t = K_m(u), \ m \geq 0$,

$$K_n \ \Rightarrow \ \text{Lie group of solutions } S_n(\varepsilon_n), \varepsilon_n \in I_n \subseteq \mathbb{R}.$$

- Let $S$ be the set of solutions to a system $u_t = K_m$, and make a metric space $(S_{\mathcal{D}}, d)$ with a bounded domain $\mathcal{D}$:

$$S_{\mathcal{D}} = \{f|_{\mathcal{D}} | f \in S\}, \ d(f,g) = sup_{(t,x)\in\mathcal{D}} |f(t,x) - g(t,x)|.$$

**Open question:**

Is the union $\cup_{n=0}^{\infty} S_n$ dense in $S_{\mathcal{D}}$ with any bounded domain $\mathcal{D}$ for each system $u_t = K_m$?

If yes, the solution to any Cauchy problem can be approximated by solutions generated from those Lie symmetries.

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# NMMP Workshop 2017

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Thank you!