Composite String, S-Functions and Vertex Operators for Quantum Affine Algebras

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June 25, 2016

The intentions for this presentation:

- To present the Composite String Model.
- Associate the Infinite Dimensional Heisenberg Algebra to Symmetric Functions (S-Functions).
- Show the connection between replicated S-functions with Vertex Operator Traces.

Introduction

Standard theory of closed strings

Assumes the string to be homogeneous.

The composite string model

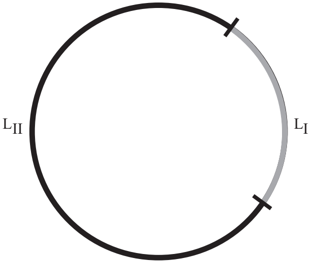
String consists of two or more separately different pieces.

- Important conditions are:
  - (i) $v_s = \sqrt{T/\rho} = c = 1$, tension T and the mass density ρ,
  - (ii) the transverse displacement $\psi = \psi(\sigma, \tau)$ is continuous,
  - (iii) the transverse force $T\partial\psi/\partial\sigma$ is continuous.
- This model was introduced in 1990, Brevik, I., Nielsen, H. B.

Simplest Case

The simplest model is when there are only two pieces:

- length L_I and L_{II}, $L = L_I + L_{II}$,
- tensions T_I and T_{II},
- mass density ρ_I and ρ_{II}.



Definitions

- Let $\psi = \psi(\sigma, \tau)$ be the transverse displacement of a point.
- The right- and left-moving waves in regions I and II:
  i) $\psi_I = \xi_I e^{i\omega(\sigma-\tau)} + \eta_I e^{-i\omega(\sigma+\tau)}$,
  ii) $\psi_{II} = \xi_{II} e^{i\omega(\sigma-\tau)} + \eta_{II} e^{-i\omega(\sigma+\tau)}$.
- These expressions satisfy the fundamental wave equation:

$$\left[\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right]\psi(\sigma, \tau) = 0.$$

Dispersion Equation

1. the transverse displacements must be continuous across the two junctions:

$$\psi_I(0) = \psi_{II}(L) \quad \longrightarrow \quad \xi_I + \eta_I = \xi_{II} e^{i\omega L} + \eta_{II} e^{-i\omega L},$$
$$\psi_I(L_I) = \psi_{II}(L_I) \quad \longrightarrow \quad \xi_I e^{i\omega L_I} + \eta_I e^{-i\omega L_I} = \xi_{II} e^{i\omega L_I} + \eta_{II} e^{-i\omega L_I},$$

2. the transverse elastic force on the string must be continuous across the junctions:

$$T_I \frac{\partial \psi_I}{\partial \sigma}\bigg|_{\sigma=0} = T_{II} \frac{\partial \psi_{II}}{\partial \sigma}\bigg|_{\sigma=L}, \quad T_I \frac{\partial \psi_I}{\partial \sigma}\bigg|_{\sigma=L_I} = T_{II} \frac{\partial \psi_{II}}{\partial \sigma}\bigg|_{\sigma=L_I}.$$

- The dispersion equation became

$$(1-x)^2 \cos(\omega L - 2\omega L_I) - (1+x)^2 \cos(\omega L) + 4x = 0,$$

  with $x = T_I/T_{II}$.
- Is invariant under the substitution $x \to 1/x$, let's consider x in the interval $0 < x \leqslant 1$.
- It allows the frequency spectrum to be calculated in terms of algebraic equations if the ratio between L_{II} and L_I is an integer.

Simple Cases

Dispersion

$$(1-x)^2\cos(\omega L - 2\omega L_I) - (1+x)^2\cos(\omega L) + 4x = 0$$

$x = 1$, $T_I = T_{II} \;\rightarrow\; \rho_I = \rho_{II}$

Uniform String:

$$\cos(wL) = 1.$$

$x \rightarrow 0$

$$\sin(\omega L_I)\sin(\omega L_{II}) = 0.$$

There are two sequences:

$$\omega_n(s) = \frac{n\pi}{L_I} = (1+s)n,\ \ \omega_n(s^{-1}) = \frac{n\pi}{L_{II}} = (1+s^{-1})n,$$

$$n \in \mathbb{Z}.$$

Definitions

- To formulate the division in 4 pieces let's introduce 2×2 transfer matrices:

$$\xi_{\mathbf{I}}^{(\mathbf{1})} = \begin{pmatrix} \xi_I^{(1)} \\ \eta_I^{(1)} \end{pmatrix}$$

- The boundary conditions became:

$$\xi_{\mathbf{I}}^{(\mathbf{1})} = \mathbf{M}^{(1)} \xi_{\mathbf{II}}^{(\mathbf{1})}; \qquad \xi_{\mathbf{II}}^{(\mathbf{1})} = \mathbf{M}^{(\mathbf{2})} \xi_{\mathbf{I}}^{(\mathbf{2})};$$
$$\xi_{\mathbf{I}}^{(\mathbf{2})} = \mathbf{M}^{(\mathbf{3})} \xi_{\mathbf{II}}^{(\mathbf{2})}; \qquad \xi_{\mathbf{II}}^{(\mathbf{2})} = \mathbf{M}^{(\mathbf{4})} \xi_{\mathbf{I}}^{(\mathbf{1})}.$$

- Where:

$$\begin{aligned}
\mathbf{M}^{(1)} &= \mathbb{I}\left(\frac{1+x^{-1}}{2}\right) + \sigma_x\left(\frac{1-x^{-1}}{2}\right)\cos(p) + \sigma_y\left(\frac{1-x^{-1}}{2}\right)\sin(p); \\
\mathbf{M}^{(2)} &= \mathbb{I}\left(\frac{1+x}{2}\right) + \sigma_x\left(\frac{1-x}{2}\right)\cos(2p) + \sigma_y\left(\frac{1-x}{2}\right)\sin(2p); \\
\mathbf{M}^{(3)} &= \mathbb{I}\left(\frac{1+x^{-1}}{2}\right) + \sigma_x\left(\frac{1-x^{-1}}{2}\right)\cos(3p) + \sigma_y\left(\frac{1-x^{-1}}{2}\right)\sin(3p); \\
\mathbf{M}^{(4)} &= \mathbb{I}\left(\frac{1+x}{2}\right)\cos(2p) + \sigma_x\frac{1-x}{2}\cos(2p) + \sigma_y\frac{1-x}{2}\sin(2p) - i\sigma_z\left(\frac{1+x}{2}\right)\sin(2p).
\end{aligned}$$

Eigenfrequencies w

- Under stationary conditions the eigenfrequencies ω of the string are all real quantities, determined from the equation

$$\det(\mathbf{M} - 1) = 0, \tag{1}$$

  with $\mathbf{M} = \mathbf{M}^{(1)}\mathbf{M}^{(2)}\mathbf{M}^{(3)}\mathbf{M}^{(4)}$.
- If $x = 1$, the system degenerates into that of a uniform string. Since the velocity of sound is required to be equal to c in everywhere, it is irrelevant whether the string is composed of type I or II material; the eigenvalue spectrum is determined from equation

$$\cos(\omega L) = 1,$$

  in either case. Thus $\omega_n L = 2\pi n$, $n \geq 0$.
- The eigenvalue spectrum of the system is in general invariant under the transformation $x \to 1/x$.

General Formalism

Recalling the notation:

- the tension ratio: $x = T_I / T_{II}$,
- symbols: $p_N = \omega L/N$, and $\alpha = \frac{(1-x)}{(1+x)}$,
- the eigenfrequencies ω of the string:

$$\det[\mathbf{M}_{2N}(x, p_N) - \mathbf{1}] = 0,$$

where

$$\mathbf{M}_{2N}(x, p_N) = \prod_{j=1}^{2N} \mathbf{M}^{(j)}(x, p_N), \tag{2}$$

with $j = 1, 2, \cdots, 2N$.

General Formalism

The component matrices can be expressed as

$$\mathbf{M}^{(j)}(x,p_N) = \begin{cases} \frac{1+x}{2x}\begin{pmatrix} 1 & -\alpha\, e^{-ijp_N} \\ -\alpha\, e^{ijp_N} & 1 \end{pmatrix}, & \text{if } j \text{ is odd} \\ \\ \frac{1+x}{2}\begin{pmatrix} 1 & \alpha\, e^{-ijp_N} \\ \alpha\, e^{ijp_N} & 1 \end{pmatrix}, & \text{if } j \text{ is even} \end{cases} \tag{3}$$

for $j = 1, 2, ..., (2N-1)$. At the last junction, for $j = 2N$, the matrix will be of a particular form

$$\mathbf{M}'^{(2N)}(x,p_N) = \frac{1+x}{2}\begin{pmatrix} e^{-iNp_N} & \alpha\, e^{-iNp_N} \\ \alpha\, e^{iNp_N} & e^{iNp_N} \end{pmatrix}. \tag{4}$$

General Formalism

The matrix $\mathbf{M}_{2N}$ will depend on x only through the variable $\alpha(x)$. It is possible to scale the matrices as

$$\mathbf{M}_{2N}(x, p_N) = [(1+x)^2/4x]^N \mathbf{m}_{2N}(\alpha, p_N).$$

The new matrices can be calculated as

$$\mathbf{m}_{2N}(\alpha, p_N) = \prod_{j=1}^{2N} \mathbf{m}^{(j)}(\alpha, p_N)\,, \tag{5}$$

where

$$\mathbf{m}^{(j)}(\alpha, p_N) = \begin{pmatrix} 1 & \mp\alpha\, e^{-ijp_N} \\ \mp\alpha\, e^{ijp_N} & 1 \end{pmatrix} \tag{6}$$

for $j = 1,\ 2, \ldots,\ (2N-1)$.

Exact Solution

- Establishing a recursion formula, for a string that is divided into $2(N+1)$ pieces:

$$\mathbf{m}_{2(N+1)} \quad = \quad \left[\mathbf{m}^{(1)} \cdot \dots \cdot \mathbf{m}'^{(2N)}\right] \cdot \left[(\mathbf{m}'^{(2N)})^{-1} \cdot \mathbf{m}^{(2N)} \cdot \mathbf{m}^{(2N+1)} \cdot \mathbf{m}'^{(2N+2)}\right] .$$

  All these matrices have p_{N+1} as their second argument.
- One can therefore write

$$\mathbf{m}_{2(N+1)}(\alpha, p_{N+1}) \quad = \quad \mathbf{m}_{2N}(\alpha, p_{N+1}) \cdot \mathbf{\Lambda}(\alpha, p_{N+1}). \tag{7}$$

  where the matrix $\mathbf{\Lambda}$ is a product of four matrices,

$$\mathbf{\Lambda} = (\mathbf{m}'^{(2N)})^{-1} \cdot \mathbf{m}^{(2N)} \cdot \mathbf{m}^{(2N+1)} \cdot \mathbf{m}'^{(2N+2)},$$

  evaluated at p_{N+1}.

Exact Solution

- Then it is possible to find that

$$\mathbf{\Lambda}(\alpha, p) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} , \qquad (8)$$

where

$$a = e^{-ip} - \alpha^2 , \qquad (9)$$
$$b = \alpha(e^{-ip} - 1) . \qquad (10)$$

The matrix $\mathbf{\Lambda}$ does not depend on N explicitly, but only through the variable $p = p_{N+1} = \omega L/(N+1)$.

- This fact will enable us to give an explicit solution, since then

$$\mathbf{m}_{2N}(\alpha, p_N) = \mathbf{\Lambda}^N(\alpha, p_N) . \qquad (11)$$

Diagonalization

- The eigenvalues $\lambda_\pm$ of $\mathbf{\Lambda}$ are roots of the polynomial

  $$P(\lambda) \;=\; \det(\mathbf{\Lambda} - \lambda\mathbf{1}) \;=\; \lambda^2 - 2(\cos p - \alpha^2)\lambda + (1-\alpha^2)^2 \;,$$

  giving

  $$\lambda_\pm \;=\; \cos p - \alpha^2 \pm \left[\left(\cos p - \alpha^2\right)^2 - (1-\alpha^2)^2\right]^{1/2} . \qquad (12)$$

  These eigenvalues are in general complex.
- Powers of the matrix $\mathbf{\Lambda}$ are

  $$\mathbf{\Lambda}^N \;=\; \mathbf{K}\begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix}\mathbf{K}^{-1} \;, \qquad (13)$$

  where $\mathbf{K}$ is a matrix whose columns consist of the eigenvectors of $\mathbf{\Lambda}$.

Diagonalization

- From (13) we get

$$\det(\mathbf{\Lambda}^N) = \lambda_+^N \lambda_-^N \,, \qquad \mathrm{tr}(\mathbf{\Lambda}^N) = \lambda_+^N + \lambda_-^N \,. \tag{14}$$

- And the relationships

$$\lambda_+ \lambda_- = (1-\alpha^2)^2, \quad \lambda_+ + \lambda_- = 2(\cos p - \alpha^2). \tag{15}$$

- Although the eigenvalues $\lambda_\pm$ are in general complex, the combinations $\lambda_+ \lambda_-$ and $(\lambda_+ + \lambda_-)$ are always real.

Oscillator coordinates

- Consider henceforth the motion of a two-piece classical string in flat D-dimensional space-time.
- Assume now that $L = \pi$, in conformity with usual practice. Thus $p_N = \pi\omega/N$.
- Let $X^\mu(\sigma, \tau)$ $(\mu = 0, 1, 2, \cdots (D-1))$ specify the coordinates on the world sheet.
- The general expression for X^μ in the form

$$X^\mu = x^\mu + \frac{p^\mu \tau}{\pi \bar{T}} + X_I^\mu, \quad \text{region I},$$

$$X^\mu = x^\mu + \frac{p^\mu \tau}{\pi \bar{T}} + X_{II}, \quad \text{region II},$$

where x^μ is the center of mass position, p^μ is the total momentum of the string and $\bar{T} = \frac{1}{2}(T_I + T_{II})$ denotes the mean tension.

The Case of Extreme Tensions

- Further, X_I^μ and X_{II}^μ are decomposed into oscillator coordinates,

$$X_I^\mu = \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{1}{n}\left[\alpha_{nI} e^{i\omega(\sigma-\tau)} + \tilde{\alpha}_{nI} e^{-i\omega(\sigma+\tau)}\right], \qquad (16)$$

$$X_{II}^\mu = \frac{i}{2}\ell_s \sum_{n\neq 0} \frac{1}{n}\left[\alpha_{nII} e^{i\omega(\sigma-\tau)} + \tilde{\alpha}_{nII} e^{-i\omega(\sigma+\tau)}\right]. \qquad (17)$$

The Case of Extreme Tensions

- Now assuming that T_{II} has a finite value, so that the limiting case $x \to 0$ corresponds to $T_I \to 0$.
- Since now $\alpha \to 1$, thus $\lambda_- = 0,\ \lambda_+ = \cos p_N - 1$.
- So we obtain the remarkable simplification that all the eigenfrequency branches degenerate into one single branch determined by $\cos p_N = 1$.
- That is, the eigenvalue spectrum becomes

$$\omega_n = 2Nn, \quad n = \pm 1, \pm 2, \pm 3, ... \tag{18}$$

The Case of Extreme Tensions

- Then, choosing the fundamental length equal to $\ell_s = (\pi T_I)^{-1/2}$, we can write the expansion in region I as

$$X_I^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^\mu e^{2iNn(\sigma-\tau)} + \tilde{\alpha}_n^\mu e^{-2iNn(\sigma+\tau)} \right]. \quad (19)$$

When $x \to 0$ the junction conditions reduce to the equations

$$\xi_I + \eta_I = 2\xi_{II} = 2\eta_{II}, \quad (20)$$

which show that the right– and left– moving amplitudes ξ_I and η_I in region I can be chosen freely and that the amplitudes ξ_{II}, η_{II} in region II are thereafter fixed.

The Case of Extreme Tensions

- The expansion in region II can be written as

$$X_{II}^{\mu} = \frac{i}{2\sqrt{\pi T_I}} \sum_{n\neq 0} \frac{1}{n} \gamma_n^{\mu} e^{-2iNn\tau} \cos(2Nn\sigma), \tag{21}$$

where we have defined γ_n^{μ} as

$$\gamma_n^{\mu} = \alpha_n^{\mu} + \tilde{\alpha}_n^{\mu}, \quad n \neq 0. \tag{22}$$

The oscillations in region II are thus standing waves.

The Hamiltonian

- The Hamiltonian is

$$H = \int_0^\pi \left[P_\mu(\sigma)\dot{X}^\mu - \mathcal{L}\right] d\sigma = \frac{1}{2}\int_0^\pi T(\sigma)(\dot{X}^2 + X'^2)d\sigma, \tag{23}$$

where $\mathcal{L}$ is the Lagrangian.

- With lightcone coordinates, region I,

$$\partial_- X^\mu = \frac{N}{\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \alpha_n^\mu e^{2iNn(\sigma-\tau)},$$

$$\partial_+ X^\mu = \frac{N}{\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \tilde{\alpha}_n e^{-2iNn(\sigma+\tau)}, \tag{24}$$

and in region II

$$\partial_\mp X^\mu = \frac{N}{2\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \gamma_n^\mu e^{\pm 2in(\sigma \mp \tau)}, \tag{25}$$

where we have defined

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{p^\mu}{NT_{II}}\sqrt{\frac{T_I}{\pi}}, \qquad \gamma_0^\mu = 2\alpha_0^\mu. \tag{26}$$

The Hamiltonian

- Inserting these expressions into the Hamiltonian

$$H = \int_0^{\pi} T(\sigma)\left(\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X\right) d\sigma$$

$$= NT_I \int_0^{\pi/(2N)} \left(\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X\right) d\sigma$$

$$+NT_{II} \int_{\pi/(2N)}^{\pi/N} \left(\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X\right) d\sigma \qquad (27)$$

the hamiltonian is given by

$$H = \frac{1}{2} N^2 \sum_{-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) + \frac{N^2}{4x} \sum_{-\infty}^{\infty} \gamma_{-n} \cdot \gamma_n. \quad (28)$$

Quantization

- The commutation rules in region I:

$$T_I[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}, \tag{29}$$

- in region II

$$T_{II}[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}, \tag{30}$$

  $\eta^{\mu\nu}$ being the D-dimensional flat metric.

Inserting the expansions for X^μ and $\dot{X}^\mu$ in regions I and II,

- in region I

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}\eta^{\mu\nu}, \tag{31}$$

  with a similar relation for $\tilde{\alpha}_n$.

- In region II,

$$[\gamma_n^\mu, \gamma_m^\nu] = 4nx\,\delta_{n+m,0}\,\eta^{\mu\nu}. \tag{32}$$

Quantization

- Introducing the annihilation and creation operators by

$$\alpha_n^\mu = \sqrt{n}\, a_n^\mu, \quad \alpha_{-n}^\mu = \sqrt{n}\, a_n^{\mu\dagger},$$

$$\gamma_n^\mu = \sqrt{4nx}\, c_n^\mu, \quad \gamma_{-n}^\mu = \sqrt{4nx}\, c_n^{\mu\dagger}, \tag{33}$$

and find for $n \geq 1$ the standard form

$$[a_n^\mu, a_m^{\nu\dagger}] = \delta_{nm}\eta^{\mu\nu},$$

$$[c_n^\mu, c_m^{\nu\dagger}] = \delta_{nm}\eta^{\mu\nu}. \tag{34}$$

Quantization

- From Eq.(28) we get, when separating out the $n = 0$ term,

$$H = -\frac{M^2}{\pi T_{II}} + \frac{1}{2} N \sum_{n=1}^{\infty} \omega_n \left(a_n^\dagger \cdot a_n + \tilde{a}_n^\dagger \cdot \tilde{a}_n + 2\, c_n^\dagger \cdot c_n \right). \tag{35}$$

Here $a_n^\dagger \cdot a_n \equiv a_n^{\mu\dagger} a_{n\mu}$, and $\omega_n = 2Nn$ as before.

Heisenberg Algebra and Symmetric Functions

This algebra is generated by operators $\{\alpha_i |\ i \in \mathbb{Z}\}$, with commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}.$$

These algebras can be realized on the space of symmetric functions by the association

$$\alpha_{-n} = p_n(x), \quad \alpha_n = n\frac{\partial}{\partial p_n(x)},$$

with $p_n(x) = \sum_{i=1}^{\infty} x_i^n$.

An alternative is a basis consisting of all Schur functions $s_\lambda(x)$.

The polynomial ring $\Lambda(X)$

- Let $\mathbb{Z}[x_1, \ldots, x_n]$ be the polynomial ring, or the ring of formal power series, in n commuting variables $x_1, \ldots, x_n$.

The symmetric group S_n acting on n letters acts on this ring by permuting the variables. For $\pi \in S_n$ and $f \in \mathbb{Z}[x_1, \ldots, x_n]$ we have $\pi f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

- The interest is in the subring of functions invariant under this action, $\pi f = f$, that is to say the ring of symmetric polynomials in n variables: $\Lambda(x_1, \ldots, x_n) = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$.

This ring may be graded by the degree of the polynomials, so that $\Lambda(X) = \oplus_n \Lambda^{(n)}(X)$, where $\Lambda^{(n)}(X)$ consists of homogenous symmetric polynomials in $x_1, \ldots, x_n$ of total degree n.

Symmetric Functions

For each partition λ, the Schur function is defined by

$$s_\lambda(X) \equiv s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) X^{\sigma(\lambda+\delta)}}{\prod_{i<j}(x_i - x_j)}\,, \qquad (36)$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$.
The study of projective representations of S_n led Schur to introduce the Q-functions, defined by:

$$Q_{(\lambda_1,\ldots,\lambda_p)}(x_1, \ldots, x_n) \stackrel{\text{def}}{=} 2^p \sum_{j_i,\ldots,j_p=1}^{n} \frac{x_{j_1}^{\lambda_1} \cdots x_{j_p}^{\lambda_p}}{u_{j_1} \cdots u_{j_p}} \mathcal{A}(x_{j_p}, \ldots, x_{j_2}, x_{j_1}), \qquad (37)$$

where

$$\mathcal{A}(y_1, \ldots, y_p) = \prod_{1\le i<j\le p} \frac{y_i - y_j}{y_i + y_j}, \quad u_j = \prod_{1\le i\le n, i\neq j} \frac{x_j - x_i}{x_j + x_i}\,. \quad (38)$$

Symmetric Functions

- Hall-Littlewood Functions

One generalization of the idea of symmetric functions is that of the Hall-Littlewood function [3, 4] in the variables $x_1, x_2, \ldots, x_n$ defined for a partition of length $\ell(\lambda) \le n$ by

$$Q_\lambda(x_1, \ldots, x_n; t) \stackrel{\text{def}}{=} (1-t)^{\ell(\lambda)} \sum_{\sigma \in S_n} \sigma \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{1 \le i < j \le n} \frac{x_i - t x_j}{x_i - x_j} \right) ,$$

where σ acts as $\sigma(x_1^{\lambda_1} \cdots x_n^{\lambda_n}) = x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n}$, and t is some parameter. When $t = 0$, Q_λ reduces to the S-function s_λ.

Symmetric Functions

- Generalized symmetric functions

One can define functions $P_\lambda(x;\xi)$ which span the ring Λ_F. Using the Gram-Schmidt orthogonalization procedure [2], one can derive a unique orthogonal basis for Λ_F. The interest is in the cases:

$$\xi_n = \alpha\left(\frac{q^{\kappa n} - q^{-\kappa n}}{q^{2n} - q^{-2n}}\right) = \alpha[\kappa/2]_q = \alpha\left(\frac{\sin(2\pi\kappa\vartheta n)}{\sin(4\pi\vartheta n)}\right), \qquad (39)$$

where $\alpha \in \mathbb{R}$ and $\kappa \in \mathbb{Z}$.

- Hall-Littlewood symmetric functions correspond to the case when $\xi_n = (1-t^n)^{-1}$.
- Maconald's functions correspond to the case $\xi_n = (1-q^n)/(1-t^n)$.

Replicated Functions

Introducing the symmetric functions with replicated variables. We want to be able to define the function $P_\lambda(X^{(\tau)}; q, \kappa, \alpha)$ so that when $\tau = m$, an integer, we have

$$P_\lambda(X^{(\tau)}; q, \kappa, \alpha) := P_\lambda(\overbrace{x_1, \ldots, x_1}^{m}, \overbrace{x_2, \ldots, x_2}^{m}, \ldots; q, \kappa, \alpha) \quad (40)$$

Using some orthogonality relations it is possible to define these functions $P_\lambda(X^{(\tau)}; q, \kappa, \alpha)$ of the replicated variable $X^{(\tau)}$ to be:

$$P_\lambda(X^{(\tau)}; q, \kappa, \alpha) = \sum_\mu B_{\lambda\mu}(\tau) P_\mu(X; q, \kappa, \alpha), \quad (41)$$

where $B_{\lambda\mu}(\tau)$ is a polynomial related with transition matrices between the power sums and the functions P_λ.

Vertex Operator Traces

Vertex operators

Have played a fruitful role in string theory, mathematical constructions of group representations as well as combinatorial constructions.

Consider here a general vertex operator which is able to connect with the symmetric and spectral functions. We specially note that realizations of (homogeneous) vertex operators are important in the high level representations theory of quantum affine algebras. Define a generalized vertex operator as

$$\begin{aligned} V(\vec{\tau} * Z; \vec{\eta} * W; \xi) &= \exp\left(\sum_{m>0} \frac{1}{m\xi_m} p_m(\tau_1 z_1^m + \cdots + \tau_n z_n^m)\right) \\ &\times \exp\left(\sum_{m>0} \frac{1}{m\xi_m} D(p_m)(\eta_1 w_1^m + \cdots + \eta_n w_n^m)\right), \quad (42) \end{aligned}$$

where $D(p_m) = m\xi_m \frac{\partial}{\partial p_m}$.

Vertex Operator Traces

The matrix elements of the above vertex operator in a basis of (Kerov's) symmetric functions take the form:

$$\langle P_\mu(x;\xi), VQ_\nu(x;\xi)\rangle = \sum_\sigma P_{\mu/\sigma}(\overrightarrow{\tau} * Z;\xi)Q_{\nu/\sigma}(\overrightarrow{\eta} * W;\xi) \tag{43}$$

Calculating the regularized trace of the vertex operator V is the same as calculate [2]

$$S_{p/1} = \sum_{\mu\nu} p^{|\mu|} P_{\mu/\nu}(\overrightarrow{\tau} * Z;\xi)Q_{\mu/\nu}(\overrightarrow{\eta} * W;\, \xi). \tag{44}$$

Suppose that the Kerov functions with replicated arguments obey a very general Cauchy identity

$$\sum_\lambda r^{|\lambda|} P_\lambda(X^{(\tau)};\xi)\; Q_\lambda(Y^{(\eta)};\xi) = J_r^{\tau\eta}(X,Y;\xi)\,, \tag{45}$$

so that for the functions $P_\lambda(X;\xi)$ with ξ_λ defined by (39) for example, the expression on the right has the form

$$\begin{aligned} J_r^{\tau\eta}(X,Y;\xi) &= \prod_{i,j}\left(\frac{(x_i y_j q^{\kappa+2} r; q^{2\kappa})_\infty}{(x_i y_j q^{\kappa-2} r; q^{2\kappa})_\infty}\right)^{\tau\eta/\alpha} \\ &= \prod_{i,j}\left(\frac{\mathcal{R}(s=(\Omega(x_i y_j r;\vartheta)+2)(1-i\varrho(\vartheta))-2)}{\mathcal{R}(s=(\Omega(x_i y_j r;\vartheta)-2)(1-i\varrho(\vartheta))-2)}\right)^{\tau\eta/\alpha}. \end{aligned}$$

Vertex Operator Traces

We then form the generating function $\mathcal{J} = \sum_{\lambda\mu} A_{\lambda\mu} \, P_\lambda(\mathfrak{A}) \, Q_\mu(\mathfrak{B})$, which allow us to finaly arrive at

$$S_{p/1} = \sum_{j=1}^{\infty} \frac{1}{(1-p^j)} \prod_{i,j=1}^{n} J_{p^j}^{\tau_i \eta_j}(z_i, w_j; \xi).$$

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