Higher order Koszul brackets

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The talk is based on the work with Ted Voronov

Contents

Papers that talk is based on are

[1] H.M.Khudaverdian, Th. Voronov *Higher Poisson brackets and differential forms*, 2008a In: Geometric Methods in Physics. AIP Conference Proceedings 1079, American Institute of Physics, Melville, New York, 2008, 203-215., arXiv: 0808.3406

[2] Th. Voronov, *Nonlinear pullback on functions and a formal category extending the category of supermanifolds]*, arXiv: 1409.6475

[3] Th. Voronov, *Microformal geometry*, arXiv: 1411.6720

Abstract...

For an arbitrary manifold M, we consider supermanifolds ΠTM and ΠT^*M, where Π is the parity reversion functor. The space ΠT^*M possesses canonical odd Schouten bracket and space ΠTM posseses canonical de Rham differential d. An arbitrary even function P on ΠT^*M such that $[P, P] = 0$ induces a homotopy Poisson bracket on M, a differential, d_P on ΠT^*M, and higher Koszul brackets on ΠTM. (If P is fiberwise quadratic, then we arrive at standard structures of Poisson geometry.) Using the language of Q-manifolds and in particular of Lie algebroids, we study the interplay between canonical structures and structures depending on P. Then using just recently invented theory of thick morphisms we construct a non-linear map between the L_∞ algebra of functions on ΠTM with higher Koszul brackets and the Lie algebra of functions on ΠT^*M with the canonical odd Schouten bracket.

Poisson manifold

Let M be Poisson manifold with Poisson tensor $P = P^{ab}\partial_b \wedge \partial_a$

$$\{f,g\} = \{f,g\}_P = \frac{\partial f}{\partial x^a} P^{ab} \frac{\partial g}{\partial x^b}\,.$$

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0\,,$$

$$\Updownarrow$$

$$P^{ar}\partial_r P^{bc} + P^{br}\partial_r P^{ca} + P^{cr}\partial_r P^{ab} = 0\,.$$

If P is non-degenerate, then $\omega = (P^{-1})_{ab} dx^a \wedge dx^b$ is closed non-degenerate form defining symplectic structure on M.

Differentials

d—de Rham differential, $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$,

$$d^2 = 0\,, df = \frac{\partial f}{\partial x^a} dx^a\,, \qquad d(\omega \wedge \rho) = d\omega \wedge \rho + (-1)^{p(\omega)} \omega \wedge d\rho$$

d_P—Lichnerowicz- Poisson differential, $d_P\colon \mathfrak{A}^k(M) \to \mathfrak{A}^{k+1}(M)$,

$$d_P^2 = 0, d_P f = \frac{\partial f}{\partial x^b} P^{ba} \frac{\partial}{\partial x^a}$$

$$d_P P = 0 \leftrightarrow \text{Jacobi identity for Poisson bracket } \{\,,\,\}$$

Differential forms and multivector fields

$\mathfrak{A}^*$ space multivector fields on M,
Ω^* space of differential forms on M,

$$\begin{array}{ccc} \mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) \\ \uparrow & & \uparrow \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \end{array}$$

Differential forms and multivector fields

$\mathfrak{A}^*$— multivector fields on M= functions on $\Pi T^* M$
Ω^*— differential forms on M= functions on ΠTM,

$$
\begin{array}{ccccccc}
\mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) & \qquad & C(\Pi T^* M) & \xrightarrow{d_P} & C(\Pi T^* M) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & & C(\Pi TM) & \xrightarrow{d} & C(\Pi TM)
\end{array}
$$

$$
d\omega(x,\xi) = \xi^a \frac{\partial}{\partial x^a}\omega(x,\xi)\,, d_P F(x,\theta) = [P,F]\,,
$$

$[P,F]$-canonical odd Poisson bracket on $\Pi T^* M$.

$x^a = (x^1, \ldots, x^n)$— coordinates on M
$(x^a, \xi^b) = (x^1, \ldots, x^n; \xi^1, \ldots, \xi^n)$, —coordinates on ΠTM

$$p(\xi^a) = p(x^a) + 1, x^{a'} = x^{a'}(x^a) \to \xi^{a'} = \xi^a \frac{\partial x^{a'}}{\partial x^a}. \qquad (dx^a \leftrightarrow \xi^a).$$

' Respectively
$(x^a, \theta_b) = (x^1, \ldots, x^n; \theta_1, \ldots, \theta_n)$, —coordinates on ΠT^*M

$$p(\theta_a) = p(x^a) + 1, x^{a'} = x^{a'}(x^a) \to \theta_{a'} = \theta_a \frac{\partial x^a}{\partial x^{a'}}. \qquad (\partial_a \leftrightarrow \theta_a).$$

'

Example

$$\Omega^* \ni \omega = l_a dx^a + r_{ab} dx^a \wedge dx^b \leftrightarrow \omega(x, \xi) = l_a \xi^a + r_{ab} \xi^a \xi^b \in C(\Pi TM)$$

$$\mathfrak{A}^* \ni F = X^a \partial_a + M^{ab} \partial_a \wedge \partial_b \leftrightarrow F(x, \theta) = X^a \theta_a + M^{ab} \theta_a \theta_b \in C(\Pi T^*M).$$

Canonical odd Poisson bracket

F, G multivector fields $[F, G]$ Schouten commutator'	F, G functions on $\Pi T^* M$ $[F, G]$ odd Poisson bracket'
$\mathbf{X} = X^a \partial_a, [\mathbf{X}, F] = \mathfrak{L}_{\mathbf{X}} F$ $P = P^{ab} \partial_a \wedge \partial_b\,, [P, F] = d_P F$'	$[\mathbf{X}, F] = [X^a \theta_a, F(x, \theta)]$ $d_P F = (P, F) = [P^{ab} \theta_a \theta_b, F(x, \theta)]$

$$[F(x,\theta), G(x,\theta)] = \frac{\partial F(x,\theta)}{\partial x^a} \frac{\partial G(x,\theta)}{\partial \theta_a} + (-1)^{p(F)} \frac{\partial F(x,\theta)}{\partial \theta_a} \frac{\partial G(x,\theta)}{\partial x^a}\,.$$

Names are

odd Poisson bracket
Schouten bracket
Buttin bracket
anti-bracket

Koszul bracket on differential forms

$$\varphi_P^*: \quad \begin{array}{c} C(\Pi T^*M) \\ \uparrow \\ \mathbf{C}(\Pi TM) \end{array} \qquad \xi^a = P^{ab}\theta_b \,\text{or}\, dx^a = P^b\partial_b$$

From $\{\,,\}$ on functions to Koszul bracket on differential forms

$$[\omega,\sigma]_P = (\varphi_P^*)^{-1}\left([\varphi_P^*(\omega),\varphi_P^*(\sigma)]\right) .$$

$[f,g]_P = 0\,,\ [f,dg]_P = (-1)^{p(f)}\{f,g\}_P\,,\ [df,dg]_P = (-1)^{p(f)} d\left(\{f,g\}_P\right)$

This formula survives the limit if P is degenerate.

Lie algebroid

$E \to M$—vector bundle,
$[[\,,\,]]$—commutator on sections, $\rho\colon E \to TM$—-anchor

$$[[\mathbf{s}_1(x), f(x)\mathbf{s}_2(x)]] = f(x)[[\mathbf{s}_1(x), \mathbf{s}_2(x)]] + \left(\rho\widehat{(\mathbf{s}_1(x)})f(x)\right)\mathbf{s}_2(x)\,,$$

Jacobi identity:

$$[[[[\mathbf{s}_1\,,\mathbf{s}_2]],\mathbf{s}_3]] + \text{cyclic permutations} = 0\,.$$

$$\mathbf{s}(x) = s^i(x)\mathbf{e}_i(x)\,,\quad [[\mathbf{e}_i(x), \mathbf{e}_k(x)]] = c^m_{ik}(x)\mathbf{e}_m(x)\,, \rho(\mathbf{e}_i) = \rho^\mu_i\partial_\mu\,,$$

$$[[\mathbf{s}_1(x), \mathbf{s}_2(x)]] = \left(s_1^i s_2^k c^m_{ik} + s_1^i\rho^\mu_i\partial_\mu s_2^m(x) - s_2^i\rho^\mu_i\partial_\mu s_1^m(x)\right)\mathbf{e}_m$$

Trivial examples of Lie algebroid

$$\mathscr{G} -- \text{Lie algebra}\,, \quad \begin{array}{c} \mathscr{G} \\ \downarrow \\ * \end{array}, \text{ where } [[\,,\,]] \text{— usual commutator}\,,$$

$$\text{tangent bundle} \quad \begin{array}{c} TM \\ \downarrow \\ M \end{array}, \quad \text{where } [[\,,\,]] \text{— commutator of vector fields}$$

For *TM* anchor is identity map

Poisson algebroid

(M,P) Poisson manifold, $(P = P^{ab}\partial_b \wedge \partial_a$, $\{f,g\} = \partial_a f P^{ab}\partial_b g)$

$$\begin{array}{c} T^*M \\ \downarrow \\ M \end{array}, \quad [[df,dg]] = d\{f,g\}\,, \ \text{anchor}\,\rho: \ \rho(\omega_a dx^a) = D_\omega = P^{ab}\omega_b\frac{\partial}{\partial x^a}\,,$$

$$[[\omega_a dx^a, \sigma_b dx^b]] = \left(\frac{1}{2}\omega_a\sigma_b\partial_c P^{ab} + P^{ab}\omega_b\partial_a\sigma_c - (\omega \leftrightarrow \sigma)\right) dx^x$$

(This is Koszul bracket $[\,,\,]_P$ on 1-forms).

Anchor—morphism of algebroids

$$\text{Anchor } \rho : \begin{pmatrix} T^*M \\ \downarrow \\ M \end{pmatrix} \to \begin{pmatrix} TM \\ \downarrow \\ M \end{pmatrix},$$

morphism of algebroid T^*M to tangent algebroid.

$$\rho[[\omega,\sigma]] = [\rho(\omega),\rho(\sigma)].$$

One very useful object—Q manifold

Definition

A pair (M, Q) where M is (super)manifold, and Q is odd vector field on it such that

$$Q^2 = \frac{1}{2}[Q, Q] = 0$$

is called Q-manifold.
Q is called homological vector field.

Lie algebroid and its neighbours

Algebroid has diffferent manifestations

$$\begin{array}{c} \Pi E \\ \downarrow \\ M \\ \Pi E \text{ is } Q \text{ manifold with} \\ Q = \xi^k \xi^i c_{ik}^m \frac{\partial}{\partial \xi^m} + \xi^i \rho_i^\mu \frac{\partial}{\partial x^\mu} \end{array} , \qquad \begin{array}{c} E \\ \downarrow \\ M \\ E \to M \text{ is Lie algebroid with} \\ [[\mathbf{e}_i, \mathbf{e}_k]] = c_{ik}^m , \rho(\mathbf{e}_i) = \rho_i^\mu \frac{\partial}{\partial x^\mu} \end{array}$$

$$\begin{array}{c} E^* \\ \downarrow \\ M \end{array} , \quad \begin{array}{c} \Pi E^* \\ \downarrow \\ M \end{array} \quad \text{—(even, odd)Poisson manifolds}$$

Lie–Poisson bracket:

$$\{u_i, u_k\} = c_{ik}^m u_m , \{x^\mu, u_i\} = \rho_i^\mu , \{x^\mu, x^\nu\} = 0 .$$

Neighbours of $\mathscr{G} \to *$

$$
\begin{array}{c}\Pi\mathscr{G}\\ \downarrow\\ *\end{array}\qquad\qquad
\begin{array}{c}\mathscr{G}\\ \downarrow\\ *\end{array}\qquad\qquad
\begin{array}{c}\mathscr{G}^*\\ \downarrow\\ *\end{array}
$$

$$
\underbrace{Q = \xi^i \xi^k c_{ik}^m \frac{\partial}{\partial \xi^m}}_{\text{homological vector field}}\ , \quad \underbrace{[\mathbf{e}_i, \mathbf{e}_k] = c_{ik}^m \mathbf{e}_m}_{\text{structure constants}}\ , \quad \underbrace{\{u_i, u_k\} = c_{ik}^m u_m}_{\text{Lie-Poisson bracket}}
$$

Neighbours of tangent algebroid $TM \to M$

$$\underbrace{\begin{array}{c} \Pi TM \\ \downarrow \\ M \\ Q = \xi^m \dfrac{\partial}{\partial x^m} \end{array}}_{}\qquad ,$$

homological vector field—de Rham differential d
(functions on ΠTM)—differential forms on M)

$$\begin{array}{c} T^*M \\ \downarrow \\ M \end{array} \qquad , \qquad \begin{array}{c} \Pi T^*M \\ \downarrow \\ M \end{array}$$

canonical symplectic structure — canonical odd sympletic structure

Neighbours of Poisson algebroid $T^*M \to M$

(M,P)—Poisson manifold, $\{x^a, x^b\} = P^{ab}$

$$\begin{array}{c}\Pi T^*M\\ \downarrow\\ M\end{array}$$

$$\underbrace{Q = \theta_a\theta_b \frac{\partial P^{ba}}{\partial x^c}\frac{\partial}{\partial \theta_c} + \theta_a P^{ab}\frac{\partial}{\partial x^b}}_{\text{homological vector field}},$$

$$\begin{array}{c}T^*M\\ \downarrow\\ M\end{array}$$

Poisson algebroid

$$[[dx^a, dx^b]] = dP^{ab}\,,\ \rho(dx^a) = P^{ab}\partial_b$$

$$\begin{array}{c}\Pi TM\\ \downarrow\\ M\end{array}$$

$\{\,,\,\} = [\,,\,]_P$ is Koszul bracket on ΠTM.

$$\begin{array}{ccc} \Pi T^*M & & TM \\ \downarrow & \text{is in the neighbourhood of tangent algebroid} & \downarrow \\ M & & M \end{array}$$

$$\begin{array}{ccc} \Pi TM & & T^*M \\ \downarrow & \text{is in the neighbourhood of Poisson algebroid} & \downarrow \\ M & & M \end{array}$$

$$\underbrace{\Pi T^*M}_{\text{Odd canonical Poisson bracket}} \quad \to \quad \underbrace{\Pi TM}_{\text{Odd Koszul bracket}}$$

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Linear map $\xi^a = \frac{1}{2}\frac{\partial P(x,\theta)}{\partial \theta_a} = P^{ab}\theta_b$, $\quad (dx^a = P^{ab}\partial_b \qquad)$

Question

What happens if even function $P = P^{ab}(x,\theta)\theta_a\theta_b$ is replaced by an arbitrary even function $P = P(x,\theta)$ which obeys the master-equation

$$[P,P] = 2\frac{P(x,\theta)}{\partial x^a}\frac{P(x,\theta)}{\partial \theta^a} = 0\,.$$

(In the case $P = P^{ab}(x,\theta)\theta_a\theta_b$ master-equation is just Jacobi identity for Poisson bracket $\{\,,\,\}_P$ on M.)

Higher Poisson brackets on M

$P\colon [P,P]=0$ defines higher brackets (homotopy Poisson brackets)

$$\{f_1,f_2,\ldots,f_n\}_P=[\ldots[P,f_1],\ldots,f_p]\big|_M\,,\qquad \big|_M=\big|_{\theta=0}\,.$$

$$P=P^a\theta_a+P^{ab}\theta_b\theta_a+P^{abc}\theta_c\theta_b\theta_a+\ldots$$

$$\{x^a\}_P=P^a\,,\{x^a,x^b\}=P^{ab}\,,\{x^a,x^b,x^c\}=P^{abc}\,\ldots$$

From ΠT^*M to ΠTM

$$C(\Pi T^*M) \to \mathfrak{X}(\Pi T^*M) \to C(T^*(\Pi T^*M)) \to C(T^*(\Pi TM))$$

Function $P(x,\theta) \to$ Hamiltonian vector field $D_F \to$
$\to$ Hamiltonian in $T^*(\Pi T^*M) \to T^*(\Pi T^*M)$
The last map is Mackenzie Xu symplectomorphism

$$C(\Pi TM) \ni P = P(x,\theta) \to K = K_P(x,\xi) \in T^*(\Pi T^*M)$$

$$K_P(x,\xi,p,\pi) = \left(p_a \frac{\partial}{\partial \theta_a} P(x,\theta) + \xi^a \frac{\partial}{\partial x^a} P(x,\theta)\right)\Big|_{\theta\to\pi}$$

$(x^a, \xi^b | p_a, \pi_b)$ coordinates on $T^*(\Pi TM)$.

Higher Koszul brackets on M

$P \in \Pi T^* M$ defines homotopy Poisson bracket (higher Poisson brackets) on M,
$K_P \in T^*(\Pi TM)$ defines homotopy odd Poisson bracket (higher Koszul bracket) on ΠM,

$$\{F_1, F_2, \ldots, F_n\}_{K_P} = [\ldots[K_P, F_1], \ldots, F_p]\big|_{\Pi M}, \qquad \big|_{\Pi M} = \big|_{p=\pi=0}).$$

$$F = F(x,\xi) = f(x) + \xi^a f_a(x) + \ldots, (df = \xi^a \partial_a f),$$

$$[f]_P = 0, [f_1, f_2, \ldots, f_k]_P = 0$$

$$[f_1, df_2, \ldots, df_n] = \{f_1, f_2, \ldots, f_n\},$$

$$[df_1, df_2, \ldots, df_n] = d\{f_1, f_2, \ldots, f_n\},$$

$$C(\Pi T^*M) \quad \overset{\text{morphism of } Q\text{-manif.}}{\longleftarrow} \quad C(\Pi TM)$$

ΠT^*M		ΠTM
Lichnerowicz Poisson differential d_P	$\rightarrow$	de Rham differential
Odd Poisson canonical bracket		Odd Koszul bracket

$$d = \xi^a \partial_a \,,$$

$$d_P\colon \;\; d_P f = [P, f] \,, \qquad d_P = \frac{\partial P}{\partial x^a} \frac{\partial}{\partial \theta_a}, + \frac{\partial P}{\partial \theta_a} \frac{\partial}{\partial x^a}$$

If $P = P^{ab}\theta_b\theta_a$ then the map

$$\Pi T^*M \to \Pi TM: \quad \xi^a = \frac{\partial P}{\partial \theta^a} = P^{ab}(x)\theta_b\,,$$

is linear in fibres. Morphism of Q-manifolds

$$C(\Pi T^*M) \leftarrow C(\Pi TM)$$

is its pull-back.
These linear maps are interwining maps for differentials d and d_P, their Hamiltonians, and their homological vector fields on infinite-dimensional spaces of functions.

Let $P(x,\theta)$ be an arbitrary even function, solution of master-equation $[P,P]=0$. The map

$$\Pi T^*M \to \Pi TM\colon \qquad \xi^a = \frac{\partial P}{\partial \theta^a}\,,$$

is in general non-linear map.

$$\Pi T^*M \quad \overset{\text{non-linear}}{\rightarrow} \quad \Pi TM$$
$$\Pi TM \quad \overset{\text{thick}}{\leftarrow} \quad \Pi T^*M$$

i.e.

$$C(\Pi TM) \,--\text{non-linear map to}\, C(\Pi T^*M)$$

This non-linear map defines morphism of Q-manifolds.

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