Deformation quantization with separation of variables on a split supermanifold

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Introduction

Star products with separation of variables on Kähler manifolds arise in the context of Berezin's quantization. The construction of these star products can be naturally generalized to super-Kähler manifolds.

In this talk I will describe star products with separation of variables on a split super-Kähler manifold ΠE obtained from a holomorphic vector bundle E over a Kähler manifold M. Such a star product $*$ depends on a star product with separation of variables $\star$ on M and some function Y on ΠE (a "nilpotent potential"). I will describe the canonical supertrace for this star product.

This construction is used in the heat kernel proof of the index theorem for a star product with separation of variables (work in progress).

Star products with separation of variables

Let M be a complex manifold and $C^\infty(M)[\nu^{-1},\nu]]$ be the space of formal Laurent series with a finite principal part,

$$f = \nu^r f_r + \nu^{r+1} f_{r+1} + \ldots,$$

where ν is a formal parameter, $r \in \mathbb{Z}$, and $f_k \in C^\infty(M)$ for $k \geq r$. A star product with separation of variables of the anti-Wick type on M is an associative product $\star$ on $C^\infty(M)[\nu^{-1},\nu]]$,

$$f \star g = fg + \sum_{r=1}^{\infty} \nu^r C_r(f,g),$$

where $C_r, r \geq 1$, is a bidifferential operator which differentiates the first argument in antiholomorphic directions and the second argument in holomorphic ones.

The star product $\star$ induces a Kähler-Poisson bracket on M of type (1,1) with respect to the complex structure,

$$\{f,g\} := -i(C_1(f,g) - C_1(g,f)).$$

It is called a star product on the Kähler-Poisson manifold $(M, \{\cdot,\cdot\})$.
Normalization: We assume that the unit constant 1 is the unity of a star product.
Localization: A star product on M can be restricted (localized) to any open subset $U \subset M$.
Nondegeneracy: We call a star product with separation of variables on M *nondegenerate* if the corresponding Kähler-Poisson structure is nondegenerate and therefore is induced by a pseudo-Kähler structure on M.

Left and right star multiplication operators

For formal functions f, g on M we denote by $L^\star_f$ and $R^\star_g$ the left and the right $\star$-multiplication operators by f and g, respectively, so that

$$L^\star_f g = f \star g = R^\star_g f.$$

The operators $L^\star_f$ and $R^\star_g$ commute.
For a local holomorphic function a and antiholomorphic function b the $\star$-multiplication operators $L^\star_a$ and $R^\star_b$ are the pointwise multiplication operators by a and b, respectively:

$$L^\star_a = a \text{ and } R^\star_b = b,$$

so that

$$a \star f = af \text{ and } f \star b = fb.$$

Formal Berezin transform

Given a star product with separation of variables $\star$ on a complex manifold M, there exists a unique formal differential operator B globally defined on M such that for a local holomorphic function a and antiholomorphic function b,

$$B(ba) = b \star a.$$

It is called the formal Berezin transform of the star product $\star$. A star product with separation of variables can be recovered from its formal Berezin transform.

The dual star product

Given a star product with separation of variables $\star$ of the anti-Wick type on $(M, \{\cdot, \cdot\})$ with the formal Berezin transform B, the star product $\star'$ on $(M, \{\cdot, \cdot\})$ given by the formula

$$f \star' g := B^{-1}(Bf \star Bg)$$

is a star product with separation of variables *of the Wick type*, where the rôles of the holomorphic and antiholomorphic coordinates are swapped.

The opposite product

$$f \tilde{\star} g := g \star' f$$

is a star product with separation of variables of the anti-Wick type on $(M, -\{\cdot, \cdot\})$.

The star product $\tilde{\star}$ is called *the dual star product to* $\star$. Its formal Berezin transform is B^{-1}.

The Kähler-Poisson tensor

Let M be a complex manifold equipped with a Kähler-Poisson structure, $(U, \{z^k, \bar{z}^l\})$ be a holomorphic coordinate chart on M, and g^{lk} be the Kähler-Poisson tensor on U. The Kähler-Poisson bracket on M is

$$\{u, v\} = ig^{lk} \left(\frac{\partial u}{\partial z^k} \frac{\partial v}{\partial \bar{z}^l} - \frac{\partial u}{\partial \bar{z}^l} \frac{\partial v}{\partial z^k} \right).$$

If g^{lk} is nondegenerate, its inverse g_{kl} is a pseudo-Kähler metric tensor and

$$\omega_{-1} := ig_{kl} dz^k \wedge d\bar{z}^l$$

is a pseudo-Kähler form globally defined on M.

The operator C_1

Given a star product with separation of variables $\star$ on a complex manifold M, the local expression for the operator C_1 of $\star$ is

$$C_1(u, v) = g^{lk} \frac{\partial u}{\partial \bar{z}^l} \frac{\partial v}{\partial z^k},$$

so that

$$u \star v = uv + \nu g^{lk} \frac{\partial u}{\partial \bar{z}^l} \frac{\partial v}{\partial z^k} + \dots$$

The space $\Omega(M)$

Given a complex manifold M which admits a pseudo-Kähler structure, denote by $\Omega(M)$ the space of all formal forms

$$\omega = \nu^{-1}\omega_{-1} + \omega_0 + \nu\omega_1 + \dots$$

on M such that all $\omega_r, r \geq -1$, are closed forms of type (1,1) with respect to the complex structure on M and ω_{-1} is some pseudo-Kähler form on M.

Theorem. There is a natural bijection $\omega \mapsto \star_\omega$ from Ω onto the space of all nondegenerate deformation quantizations with separation of variables of the anti-Wick type on M.

The form ω is called the classifying form of the star product $\star_\omega$.

We say that $\star_\omega$ is a star product on the pseudo-Kähler manifold (M, ω_{-1}).

The star product $\star_\omega$ is completely determined by the following property. Let $U \subset M$ be an arbitrary contractible coordinate chart and Φ be a local potential of ω on U, so that

$$\Phi = \nu^{-1}\Phi_{-1} + \Phi_0 + \ldots \text{ and } \omega = i\partial\bar{\partial}\Phi.$$

Then

$$L^\star_{\frac{\partial \Phi}{\partial z^k}} = \frac{\partial \Phi}{\partial z^k} + \frac{\partial}{\partial z^k} \text{ and } R^\star_{\frac{\partial \Phi}{\partial \bar{z}^l}} = \frac{\partial \Phi}{\partial \bar{z}^l} + \frac{\partial}{\partial \bar{z}^l}$$

on U.

The dual form

Given a form $\omega \in \Omega(M)$, the dual star product of $\star_\omega$ corresponds

to a form $\tilde{\omega} \in \Omega(M)$,

$$\widetilde{(\star_\omega)} = \star_{\tilde{\omega}}.$$

If $\omega = \nu^{-1}\omega_{-1} + \omega_0 + \dots$, then $\tilde{\omega} = -\nu^{-1}\omega_{-1} + \tilde{\omega}_0 + \dots$.
The mapping $\omega \mapsto \tilde{\omega}$ is an involution on $\Omega(M)$. We call $\tilde{\omega}$ the dual form of ω.

A local trace density

Given a nondegenerate star product with separation of variables $\star_\omega$ on M, for any contractible coordinate chart $U \subset M$ and any potential $\Phi = \nu^{-1}\Phi_{-1} + \dots$ of ω on U there exists a potential $\Psi = -\nu^{-1}\Phi_{-1} + \dots$ of the dual form $\tilde{\omega}$ on U satisfying the equations

$$\frac{\partial \Phi}{\partial z^k} = -B\left(\frac{\partial \Psi}{\partial z^k}\right) \text{ and } \frac{\partial \Phi}{\partial \bar{z}^l} = -B\left(\frac{\partial \Psi}{\partial \bar{z}^l}\right),$$

where B is the formal Berezin transform for the product $\star_\omega$. The potential Ψ is determined up to a formal additive constant.
Then

$$e^{\Phi+\Psi} dz d\bar{z},$$

where $dzd\bar{z}$ is a Lebesgue measure on U, is a local trace density for the product $\star_\omega$.

A local ν-derivation

Given a star product with separation of variables $\star_\omega$ on a pseudo-Kähler manifold (M, ω_{-1}), on any contractible coordinate chart $U \subset M$ there exists a derivation of the product $\star$ of the form

$$\delta = \frac{d}{d\nu} + A,$$

where A is a formal differential operator on U.

A canonical trace density

There exists a global trace density μ for the product $\star$ on M uniquely determined by the conditions that (i)

$$\mu = \frac{1}{m!\nu^m}\,(\omega_{-1})^m + \ldots,$$

where m is the complex dimension of M,
and that (ii) on any contractible chart U the equality

$$\frac{d}{d\nu}\int_U f\,\mu = \int_U \delta(f)\,\mu$$

holds for any function f with compact support on U.
It is called the canonical trace density for the product $\star$.

Functions on a split supermanifold

Let E be a holomorphic vector bundle of rank d over a complex manifold M and ΠE be the corresponding split supermanifold. If E is trivializable over an open subset $U \subset M$, consider a holomorphic trivialization $\Pi E|_U \cong U \times \mathbb{C}^{0|d}$.

Let $\theta^\alpha, \bar{\theta}^\beta$ be the odd fiber coordinates on $U \times \mathbb{C}^{0|d}$. We consider the ordered subsets of $[d] := \{1, 2, \ldots, d\}$ as tensor indices and set

$$\theta^I = \theta^{\alpha_1} \ldots \theta^{\alpha_k} \text{ and } |I| = k$$

for $I = \{\alpha_1, \ldots, \alpha_k\} \subset [d]$.

A function f on $\Pi E|_U$ can be written in coordinates as

$$f = f_{IJ} \theta^I \bar{\theta}^J,$$

where $f_{IJ} \in C^\infty(U)$ and summation over repeated indices is assumed.

A product on a split supermanifold

Given a star product with separation of variables $\star$ on M, an open subset $U \subset M$ such that E is trivializable over U, and a formal function u on $\Pi E|_U$ such that $u - 1$ is nilpotent. Fix a holomorphic trivialization $\Pi E|_U \cong U \times \mathbb{C}^{0|d}$ and set $u = u_{IJ}\theta^I\bar{\theta}^J$. We call the function u admissible with respect to the product $\star$ if there exists a matrix (v^{JK}) on U such that

$$u_{IJ} \star v^{JK} = \delta_I^K \text{ and } v^{JK} \star u_{KL} = \delta_L^J.$$

We define a product $*$ on formal functions on $\Pi E|_U$ by the formula

$$f * g = u^{-1}((uf)_{KQ} \star v^{QP} \star (ug)_{PL})\theta^K\bar{\theta}^L.$$

The admissibility of u and the product $*$ do not depend on the choice of trivialization of $\Pi E|_U$.

Properties of the product $*$

The product $*$ is not necessarily a star product (i.e., it is not necessarily a deformation of the supercommutative product on $\Pi E|_U$).

The product $*$ has the property of separation of variables. If $a = a_I\theta^I$ and $b = \beta_J\bar{\theta}^J$, where a_I is holomorphic and b_J is antiholomorphic on U, then

$$a * f = af \text{ and } f * b = fb.$$

If the function u is globally defined on ΠE, then the product $*$ is also globally defined on ΠE.

The product $*$ is $\mathbb{Z}_2$-graded with respect to the parity of the functions on ΠE.

Denote by δ_K the operator on the functions on $U \times \mathbb{C}^{0|d}$ which maps $f = f_{IJ}\theta^I\bar{\theta}^J$ to $f = f_{KJ}\bar{\theta}^J$. For any function f on $U \times \mathbb{C}^{0|d}$ there exist uniquely defined functions f_I^K on U such that

$$L_f^* = u^{-1}\left(L^\star_{f_I^K}\theta^I\delta_K\right)u.$$

The mapping

$$(f_I^K) \mapsto f = \left(u^{-1}\left(L^\star_{f_I^K}\theta^I\delta_K\right)u\right)1$$

is an isomorphism of the algebra of matrices (f_I^K) over the algebra $(C^\infty(U)[\nu^{-1},\nu]], \star)$ onto the algebra $(C^\infty(U \times \mathbb{C}^{0|d})[\nu^{-1},\nu]], *)$.

Canonical supertrace

We define a canonical supertrace σ for the product $*$ on U using the isomorphism

$$(f_I^K) \mapsto f = \left(u^{-1}\left(L^{\star}_{f_I^K}\theta^I\delta_K\right)u\right)1$$

Assume that the functions f_I^K have compact supports on U. Then

$$\sigma(f) = \int_U \sum_I (-1)^{|I|} f_I^I \, \mu,$$

where μ is the canonical trace density for the product $\star$.
The supertrace σ does not depend on the choice of trivialization of $\Pi E|_U$.
It can be given by an integral of f with respect to a Berezin density.

Nilpotent potential

We call an even nilpotent function $Y = \nu^{-1} Y_{-1} + Y_0 + \nu Y_1 + \dots$ on $U \times \mathbb{C}^{0|d}$ a nondegenerate nilpotent potential if the matrix

$$\left(\overrightarrow{\frac{\partial}{\partial \theta^\alpha}} Y_{-1} \overleftarrow{\frac{\partial}{\partial \bar{\theta}^\beta}} \right)$$

is nondegenerate at every point of U.

Theorem

Given any star product with separation of variables $\star$ on U and any nondegenerate nilpotent potential Y on $U \times \mathbb{C}^{0|d}$, the function $u = \exp Y$ is admissible with respect to the product $\star$. The corresponding product $*$ is a star product with separation of variables on $U \times \mathbb{C}^{0|d}$.

Example

Let E be a holomorphic Hermitian vector bundle over a complex manifold M. If $h_{\alpha\beta}$ is the fiber metric on E, then there is a global function h on ΠE given locally by the formula

$$h = h_{\alpha\beta}\theta^{\alpha}\bar{\theta}^{\beta}.$$

The function $Y = \nu^{-1}h$ is a global nondegenerate nilpotent potential on ΠE.
If $\star$ is a star product with separation of variables on M, then $u = \exp(\nu^{-1}h)$ is a global admissible function on ΠE and the corresponding product $*$ is a global star product with separation of variables on ΠE.

A nondegenerate superpotential

Let $\star_\omega$ be a nondegenerate star product with separation of variables on a contractible open set $U \subset \mathbb{C}^m$ and Y be a nondegenerate nilpotent potential on $U \times \mathbb{C}^{0|d}$. Denote by $*$ the corresponding star product with separation of variables on $U \times \mathbb{C}^{0|d}$.
Let Φ be a potential of ω on U. We call

$$X = \Phi + Y$$

a nondegenerate superpotential on $U \times \mathbb{C}^{0|d}$. We have

$$L^*_{\frac{\partial X}{\partial z^k}} = \frac{\partial X}{\partial z^k} + \frac{\partial}{\partial z^k} \text{ and } L^*_{\frac{\partial X}{\partial \theta^\alpha}} = \frac{\partial X}{\partial \theta^\alpha} + \frac{\partial}{\partial \theta^\alpha}.$$

Graded right $*$-multiplication operators

Denote by R^*_f the graded right $*$-multiplication operator by f such that if f, g are homogeneous, then

$$R^*_f g = (-1)^{|f||g|} g * f.$$

We have

$$L^*_f g - R^*_f g = [f, g]_*,$$

where $[f, g]_*$ is the supercommutator of f and g with respect to the product $*$.

The operators L^*_f and R^*_g supercommute.

If X is a nondegenerate superpotential for the product $*$ on $U \times \mathbb{C}^{0|d}$, then

$$R^*_{\frac{\partial X}{\partial \bar{z}^l}} = \frac{\partial X}{\partial \bar{z}^l} + \frac{\partial}{\partial \bar{z}^l} \text{ and } R^*_{\frac{\partial X}{\partial \bar{\theta}^\beta}} = \frac{\partial X}{\partial \bar{\theta}^\beta} + \frac{\partial}{\partial \bar{\theta}^\beta}.$$

The super Kähler-Poisson tensor for the product $*$

For a star product with separation of variables $*$ of the anti-Wick type on $U \times \mathbb{C}^{0|d}$ the operator C_1 is of the form

$$C_1(f,g) = \frac{\partial f}{\partial \bar{z}^l} A^{lk} \frac{\partial g}{\partial z^k} + \frac{\partial f}{\partial \bar{z}^l} B^{l\alpha} \overrightarrow{\frac{\partial}{\partial \theta^\alpha}} g + \\ f \overleftarrow{\frac{\partial}{\partial \bar{\theta}^\beta}} C^{\beta k} \frac{\partial g}{\partial z^k} + f \overleftarrow{\frac{\partial}{\partial \bar{\theta}^\beta}} D^{\beta\alpha} \overrightarrow{\frac{\partial}{\partial \theta^\alpha}} g,$$

where the matrix

$$\begin{pmatrix} A^{lk} & B^{l\alpha} \\ C^{\beta k} & D^{\beta\alpha} \end{pmatrix}$$

is an even Poisson tensor of type (1,1) on $U \times \mathbb{C}^{0|d}$ (so that $A^{lk}, D^{\beta\alpha}$ are even and $B^{l\alpha}, C^{\beta k}$ are odd).

Classification of nondegenerate star products on ΠE

A star product with separation of variables $*$ of the anti-Wick type on $U \times \mathbb{C}^{0|d}$ is called nondegenerate if the matrix

$$\begin{pmatrix} A^{lk} & B^{l\alpha} \\ C^{\beta k} & D^{\beta\alpha} \end{pmatrix}$$

is nondegenerate at every point of U (i.e., when the matrices A^{lk} and $D^{\beta\alpha}$ are nondegenerate).

A star product with separation of variables $*$ on $U \times \mathbb{C}^{0|d}$ is nondegenerate if and only if there exists a nondegenerate superpotential X such that

$$L^*_{\frac{\partial X}{\partial z^k}} = \frac{\partial X}{\partial z^k} + \frac{\partial}{\partial z^k}, L^*_{\frac{\partial X}{\partial \theta^\alpha}} = \frac{\partial X}{\partial \theta^\alpha} + \frac{\partial}{\partial \theta^\alpha},$$
$$R^*_{\frac{\partial X}{\partial \bar{z}^l}} = \frac{\partial X}{\partial \bar{z}^l} + \frac{\partial}{\partial \bar{z}^l} \text{ and } R^*_{\frac{\partial X}{\partial \bar{\theta}^\beta}} = \frac{\partial X}{\partial \bar{\theta}^\beta} + \frac{\partial}{\partial \bar{\theta}^\beta}.$$

The superpotential X is unique up to a summand $a + b$, where a is holomorphic and b is antiholomorphic.

Equivalently, a nondegenerate star product $*$ is determined by a formal pseudo-Kähler form

$$\Omega = \nu^{-1}\Omega_{-1} + \Omega_0 + \dots$$

on the supermanifold $U \times \mathbb{C}^{0|d}$ such that

$$\Omega = i\partial\bar{\partial}X,$$

where the operators ∂ and $\bar{\partial}$ are extended to $U \times \mathbb{C}^{0|d}$.