An Exactly Solvable Quantum Four-Body Problem Associated with the Symmetries of an Octacube

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Introduction

In memory of Marvin Girardeau

Oct 3, 1930 - Jan 13, 2015

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 1, NUMBER 6 NOVEMBER–DECEMBER, 1960

Relationship between Systems of Impenetrable Bosons and Fermions in One Dimension*†

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(Received March 3, 1960)

A rigorous one-one correspondence is established between one-dimensional systems of bosons and of spinless fermions. This correspondence holds irrespective of the nature of the interparticle interactions, subject only to the restriction that the interaction have an impenetrable core. It is shown that the Bose and Fermi eigenfunctions are related by $\psi^B=\psi^F A$, where $A(x_1\cdots x_n)$ is $+1$ or -1 according as the order $pq\cdots r$, when the particle coordinates x_j are arranged in the order $x_p<x_q<\cdots<x_r$, is an even or an odd permutation of $1\cdots n$. The energy spectra of the two systems are identical, as are all configurational probability distributions, but the momentum distributions are quite different. The general theory is illustrated by application to the special case of impenetrable point particles; the one-one correspondence between bosons with this particular interaction and completely noninteracting fermions leads to a rigorous solution of this many-boson problem.

1. INTRODUCTION

In the following section a very simple and general relationship will be established between one-dimensional systems of impenetrable bosons and fermions. We shall find that the restrictions both to one dimension and to interactions with a completely impenetrable core are essential. Nevertheless, there are at least two motivations for studying such a relationship. First, one is enabled to obtain a rigorous solution of the many-boson problem for the case of impenetrable point particles in a one-dimensional periodic box, and this solution may serve as a useful testing ground for various approximation methods. Second, the relationship for the case of more general interactions may permit comparison of approximation methods designed for Fermi

where V includes all interactions except the hard cores' and is otherwise completely unrestricted. Consider first any Fermi wave function ψ^F satisfying (2); ψ^F is antisymmetric in the particle coordinates. We define a "unit antisymmetric function" A as follows:

$$A(x_1\cdots n_n)=\prod_{j>l}\operatorname{sgn}(x_j-x_l), \tag{3}$$

where $\operatorname{sgn}(x)$ is the algebraic sign of x; an equivalent definition is that A is $+1$ or -1 according as the order $pq\cdots r$, when the x_j are arranged in the order $x_p<x_q<\cdots<x_r$, is an even or an odd permutation of $1\cdots n$. Then the product

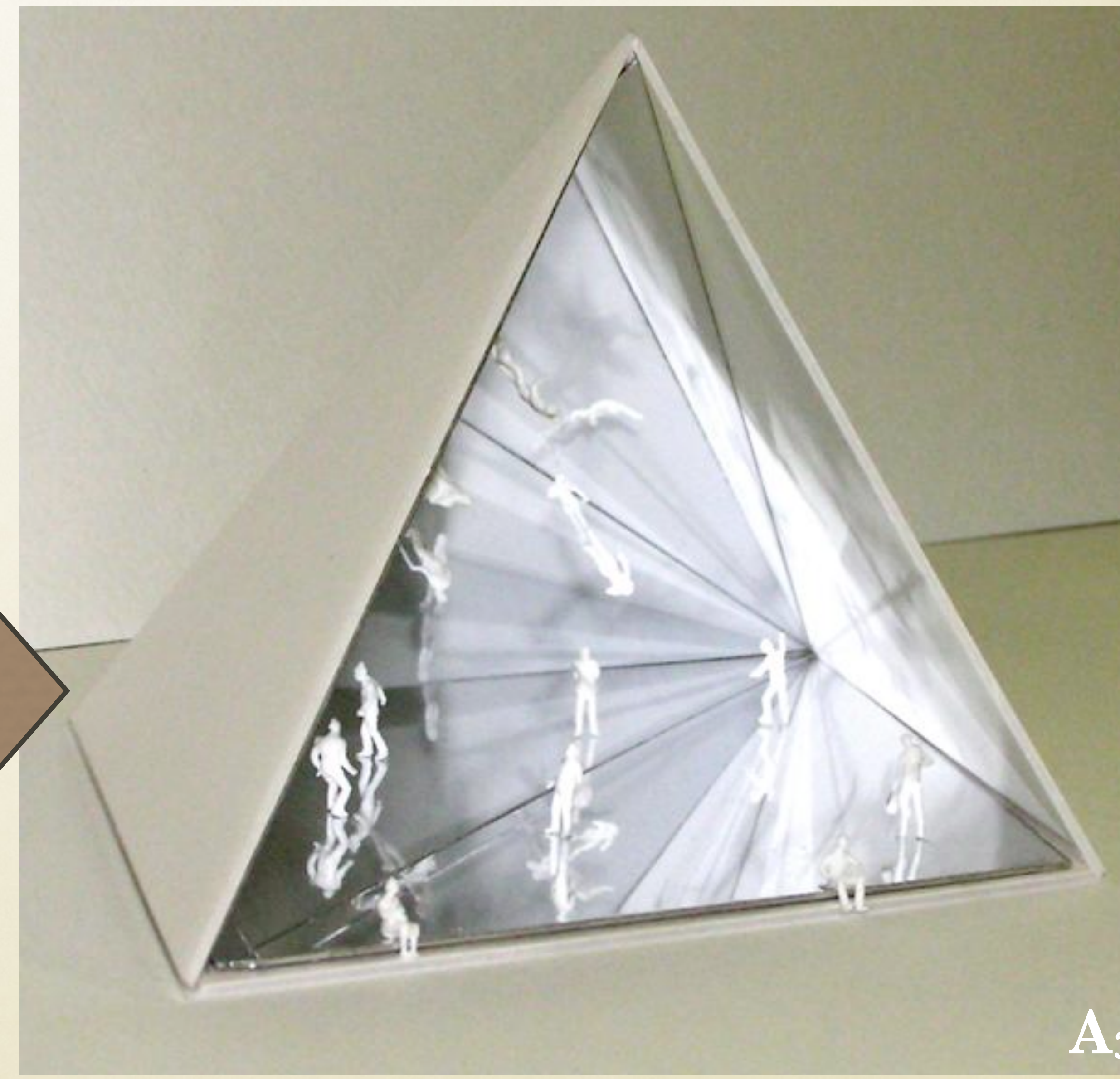
$$\psi^B=\psi^F A \tag{4}$$

is symmetric in the particle coordinates, and hence

Plan

Every instance of an integrable one-dimensional many-body system with zero-range two-body interactions can be traced to a multidimensional kaleidoscope

Example: 4 hard-core bosons on a line



H. Nishiyama

Kaleidoscopes are the systems of mirrors where the seams between the mirrors are do not seem to be there.

"Inside kaleidoscope"

exploratorium®
San Francisco

It is proven that the existing list of kaleidoscopes, or reflection groups,

$\widetilde{A}_N$, $\widetilde{B}_N$, $\widetilde{C}_N$, $\widetilde{D}_N$; $\widetilde{G}_2$, $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$; $I_2(n)$, H_3, H_4,

classical exceptional

crystallographic = closed mirror chamber

non-crystallographic = one mirror must be missing

is complete.

“Inside kaleidoscope”

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Plan

$\widetilde{C}_{3,\,4,\,\ldots}$

$\widetilde{B}_{2,\,3,\,\ldots}$

Affine

$\widetilde{D}_{4,\,5,\,\ldots}$

$\widetilde{A}_{2,3,\ldots}$ reflection groups

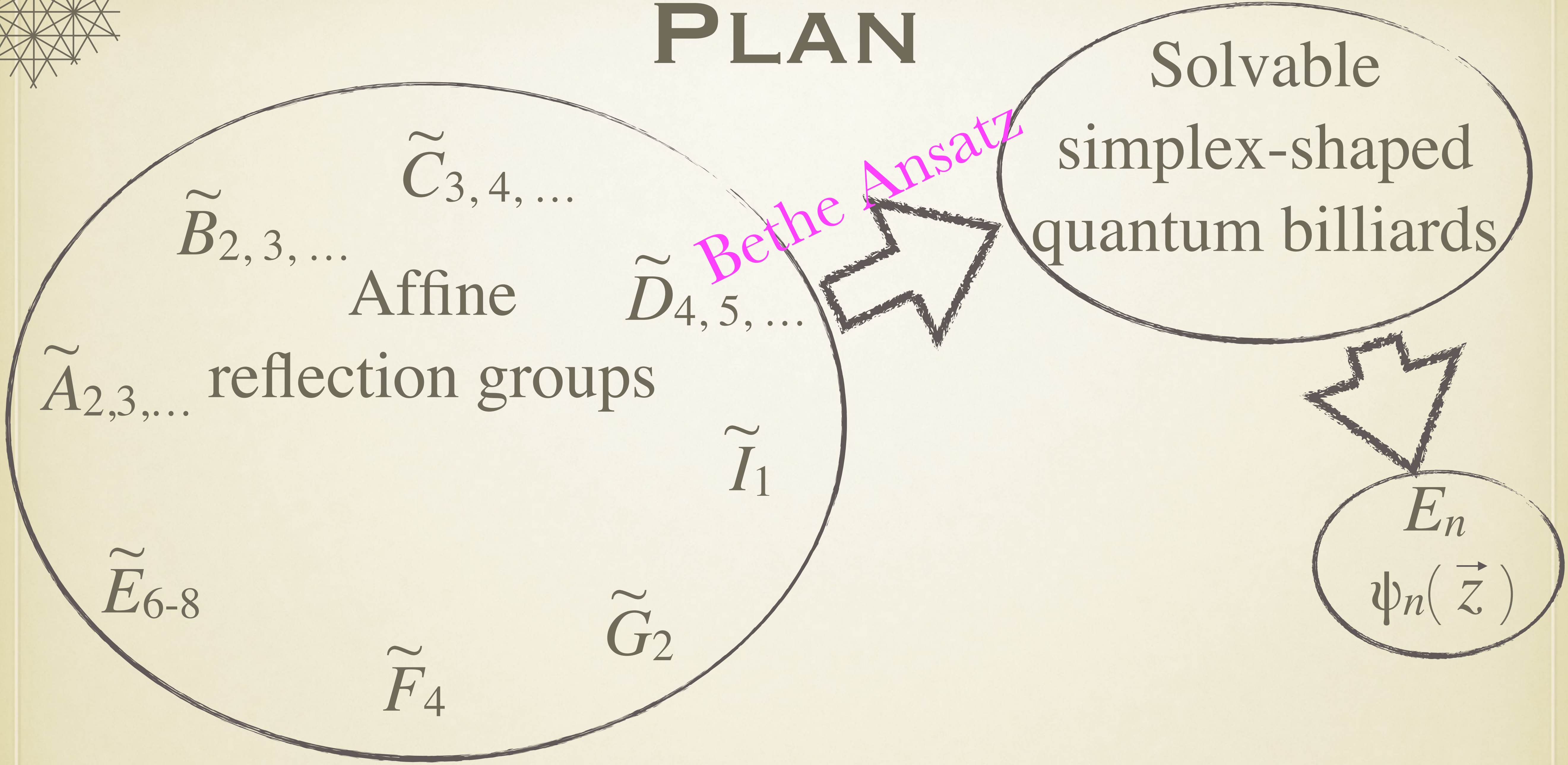
$\widetilde{I}_1$

$\widetilde{E}_{6\text{-}8}$

$\widetilde{G}_2$

$\widetilde{F}_4$

Plan



Gutkin-Sutherland, Emsiz-Opdam-Stokman

Plan

$\widetilde{C}_{3,\,4,\,\ldots}$

$\widetilde{B}$

$\widetilde{D}$

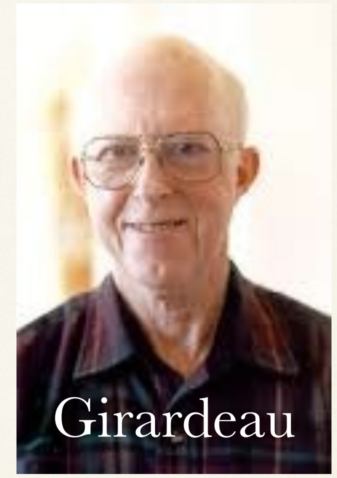
$\widetilde{A}_{2,3,\ldots}$

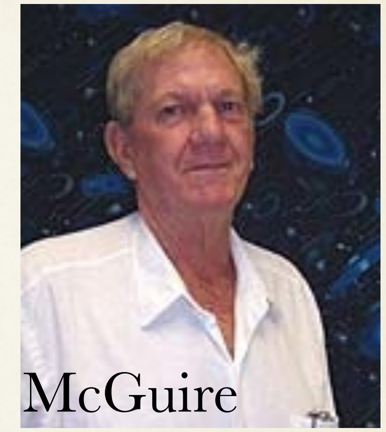
$\widetilde{I}_1$

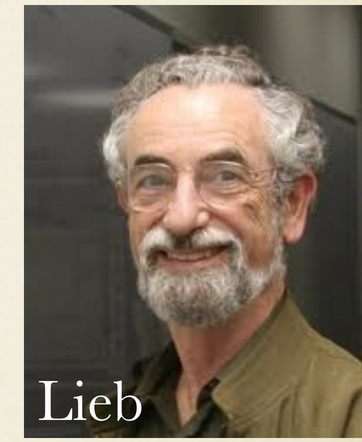
$\widetilde{E}$

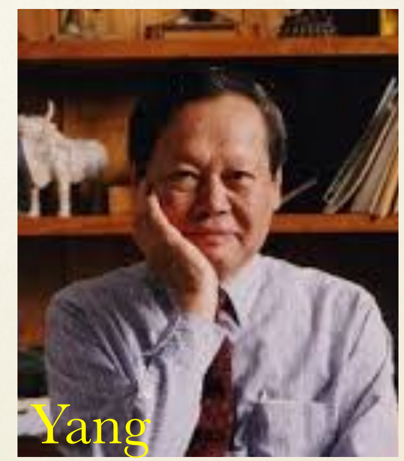
$\widetilde{G}$

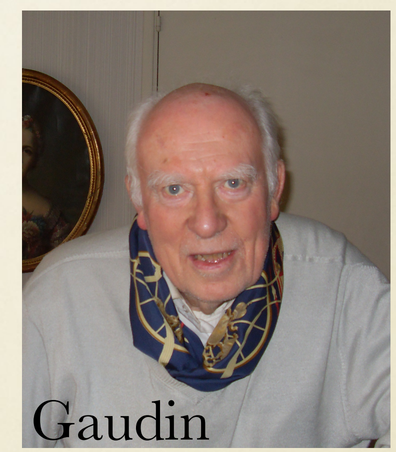
$\widetilde{F}$









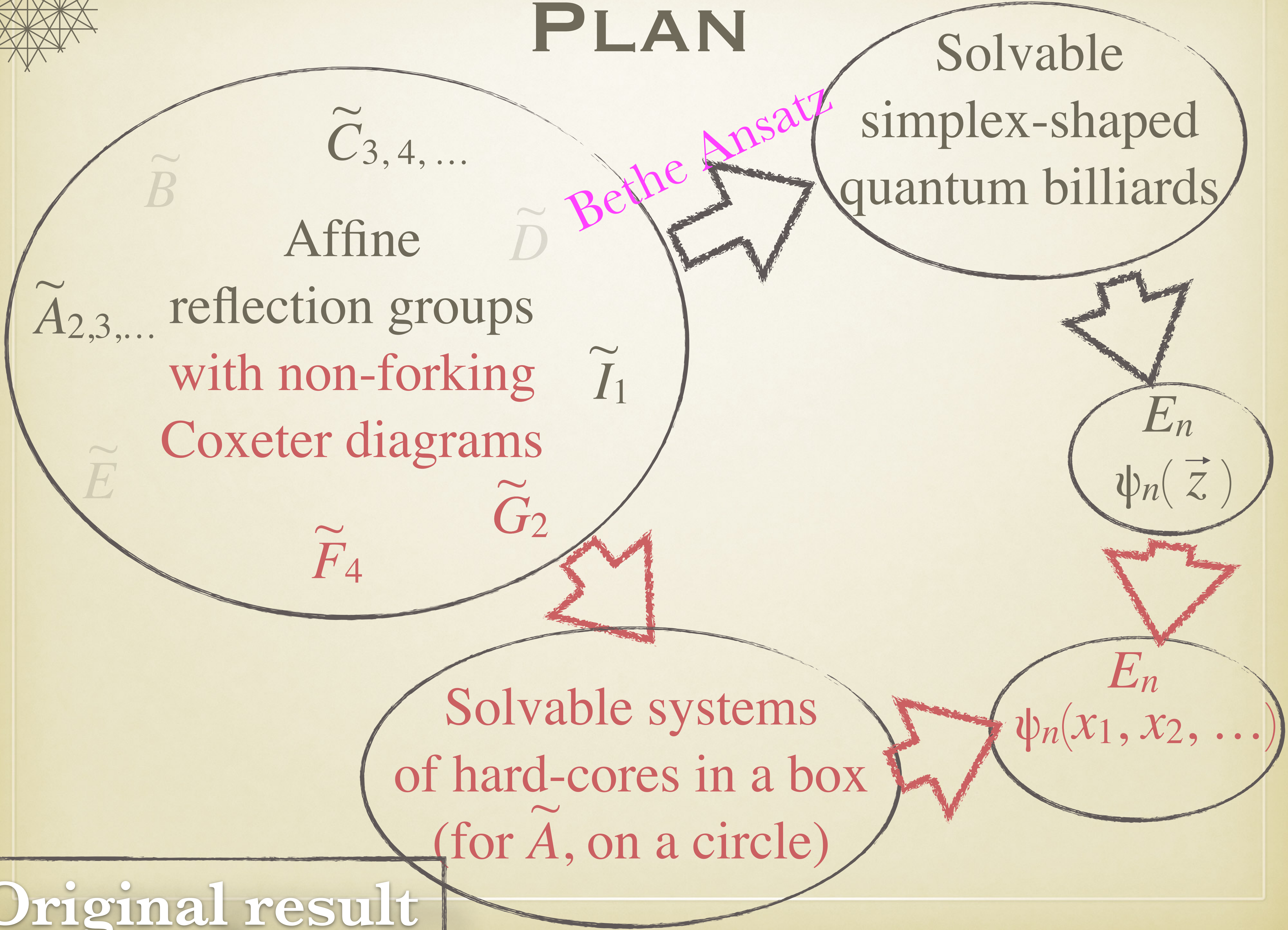


Solution for same-mass hard-cores on a circle and in a box

E_n

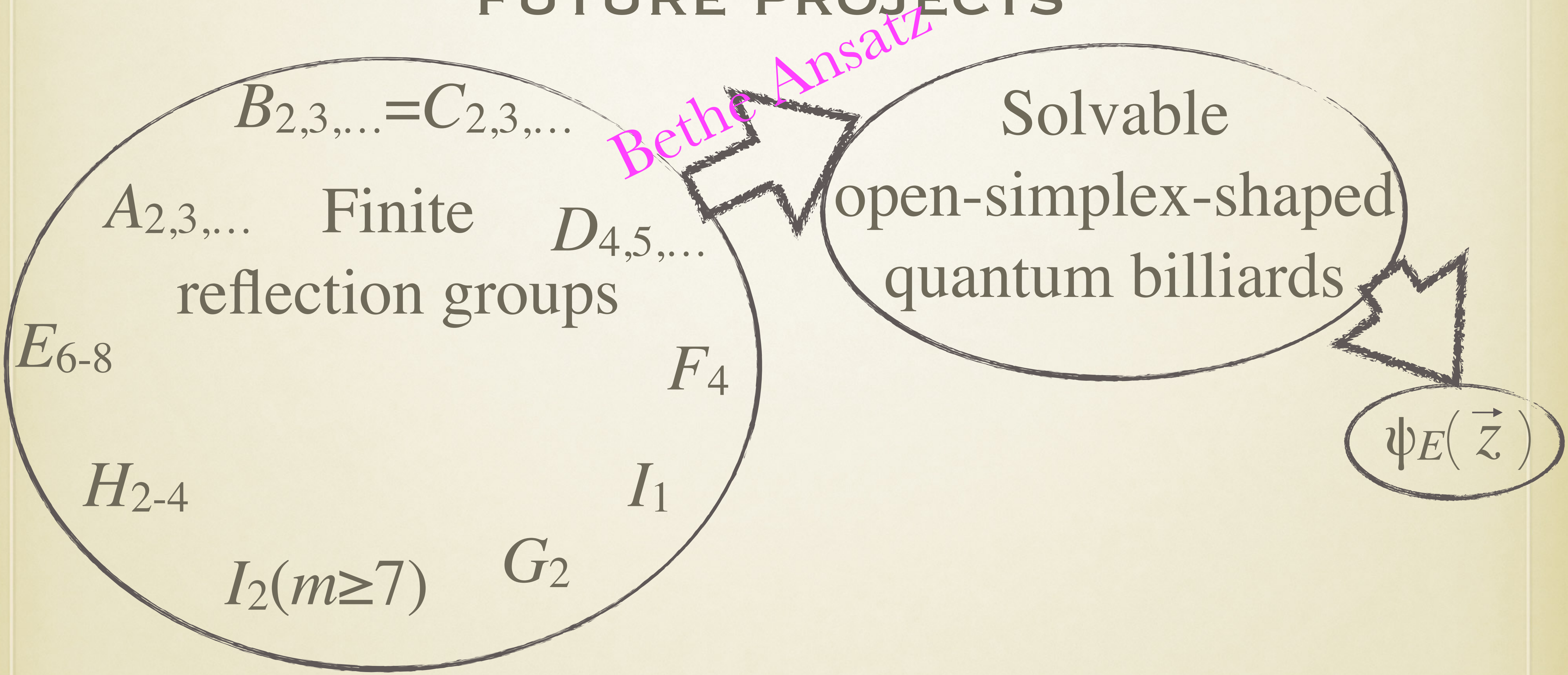
$\psi_n(x_1, x_2, \ldots)$

Plan

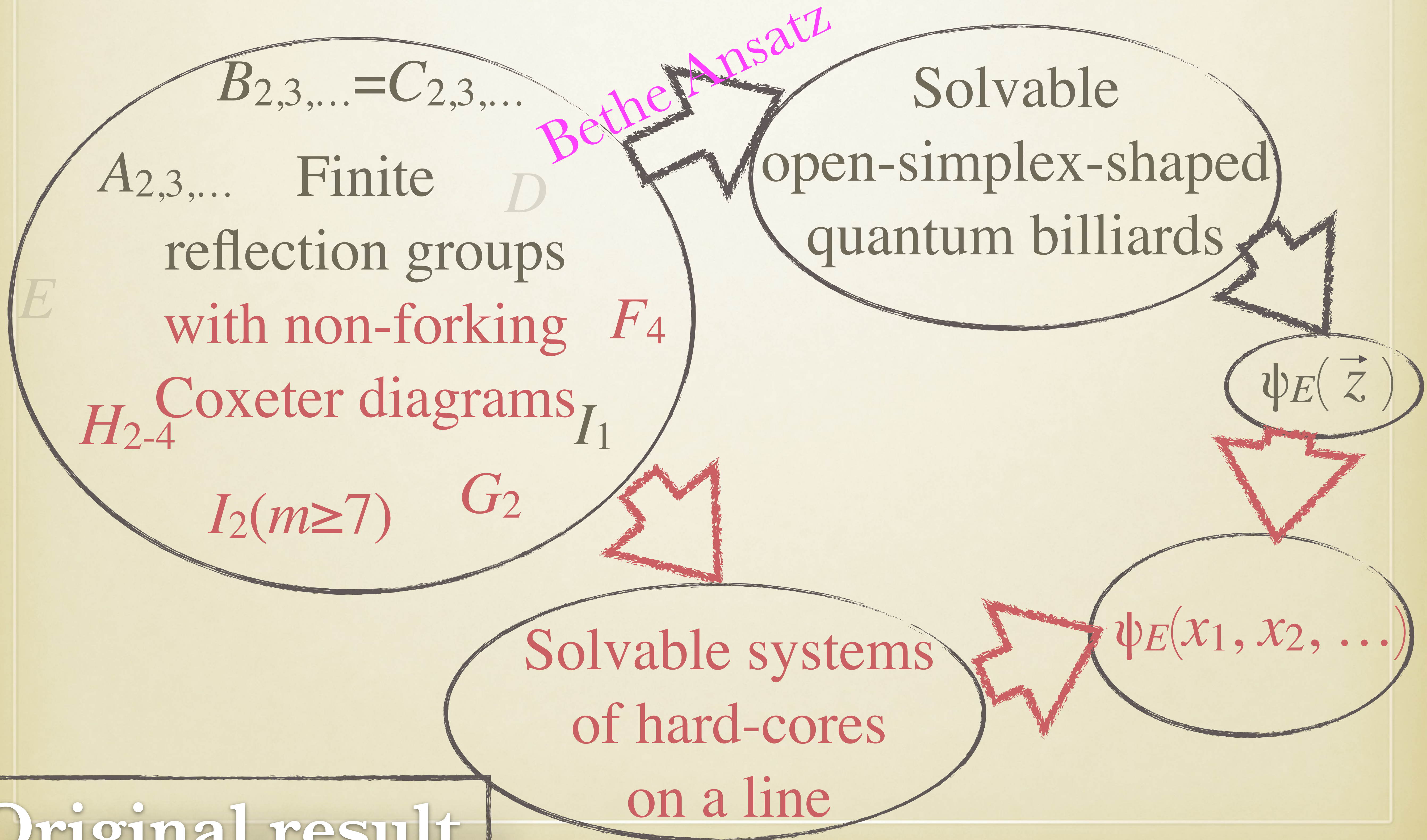


Affine reflection groups with non-forking Coxeter diagrams
$\widetilde{C}_{3,4,\ldots}$
$\widetilde{B}$
$\widetilde{D}$
$\widetilde{A}_{2,3,\ldots}$
$\widetilde{I}_1$
$\widetilde{E}$
$\widetilde{G}_2$
$\widetilde{F}_4$
Bethe Ansatz
Solvable simplex-shaped quantum billiards
E_n
$\psi_n(\vec{z})$
E_n
$\psi_n(x_1, x_2, \ldots)$
Solvable systems of hard-cores in a box (for $\widetilde{A}$, on a circle)
Original result

Also need finite reflection groups, both for technical reasons and for future projects



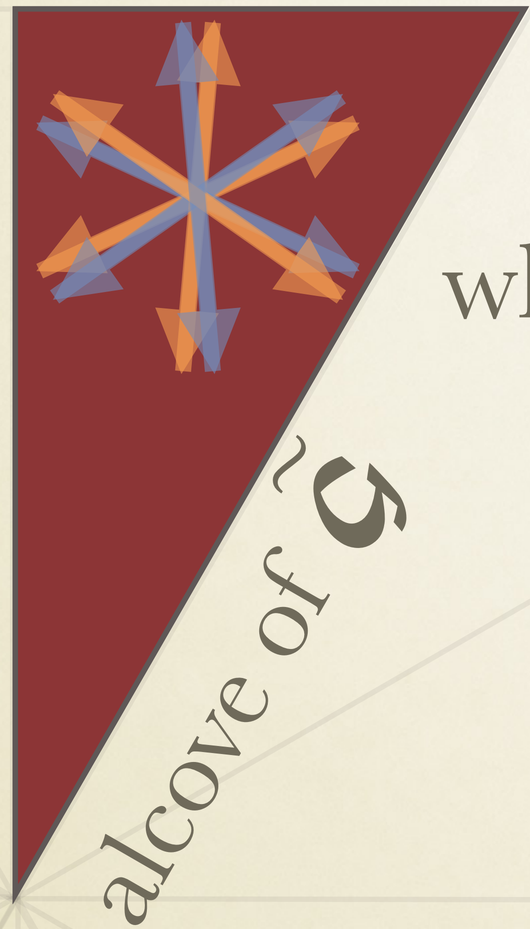
Also need finite reflection groups, both for technical reasons and for future projects



Original result

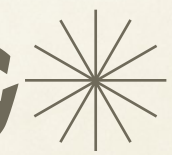
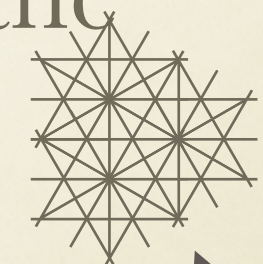
Affine reflection groups →
→ solvable billiards
(short summary
of known results
and new results)

Alcove of an affine reflection group as a solvable quantum hard-wall billiard



$$\psi(\vec{r}) = \sum_g (-1)^{\mathcal{P}[g]} \exp[(\hat{g}\vec{k})\vec{r}] \quad ,$$

where

g = an element of the finite nucleus $\mathcal{G}$  of the full affine group $\tilde{\mathcal{G}}$  ,

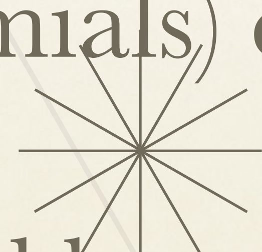
$\mathcal{P}[g]$ = parity of g,

$\vec{k} \in$ lattice reciprocal to the lattice $\tilde{\mathcal{G}}$.

After Gutkin-Sutherland, Emsiz-Opdam-Stokman
(covers Robin's boundary conditions, includes completeness)

Alcove of an affine reflection group as a solvable quantum hard-wall billiard

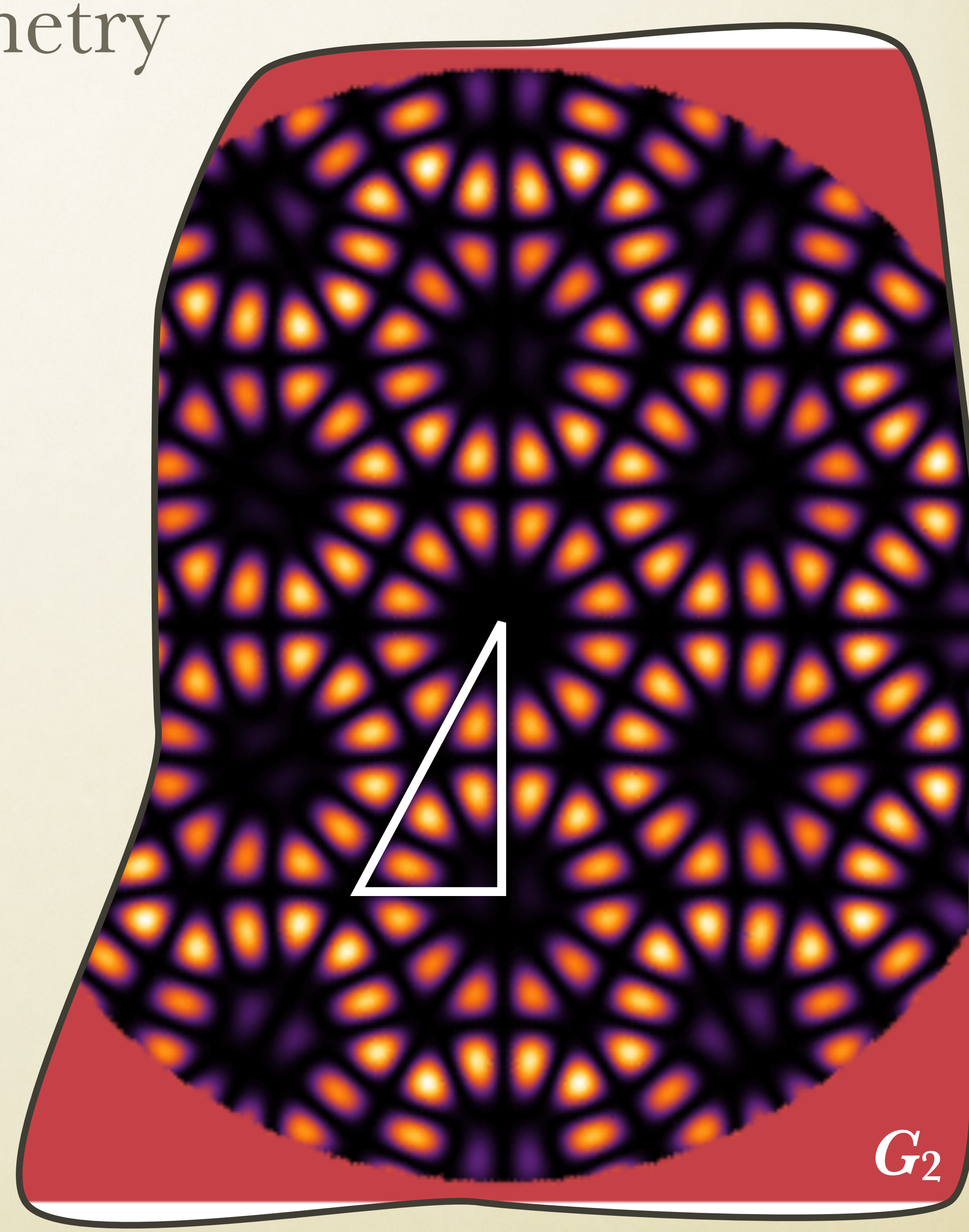
Original result

Integrals of motion in involution = invariant polynomials (Chevalley polynomials) of the non-affine nucleus  , with coordinates replaced by momenta (in the billiard coordinate system).

A hint to a Bethe Ansatz <=> Liouville's integrability connection

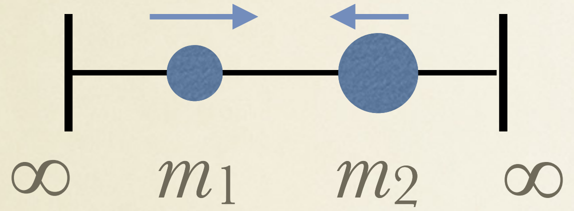
An example of a billiard solving

Above, we used G_2, the symmetry group of a hexagon, , as an example.

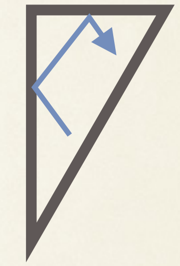


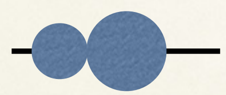
Non-forking affine reflection groups → solvable particle systems

d particles on a line in a box $\quad$ d-dimensional billiard



$$x_i = y_i/\sqrt{m_i}$$

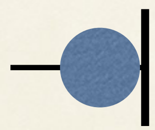


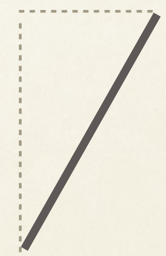


inter-particle contact

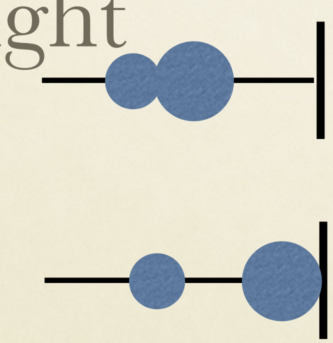
$(d\text{-}1)$-faces

particle-wall contact





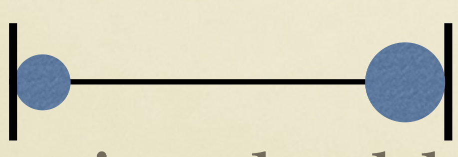
left-mid and mid-right contacts in a consecutive triplet



two $(d\text{-}1)$-faces at an angle

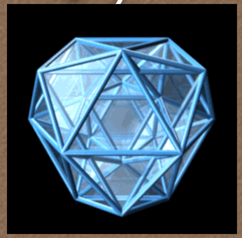
$$\alpha = \arctan\left[\frac{m_2(m_1+m_2+m_3)}{m_1 m_3}\right]$$

contacts in two unrelated consecutive doublets



two $(d\text{-}1)$-faces at 90°

A solvable particle system associated with the affine reflection group $\widetilde{F}_4$

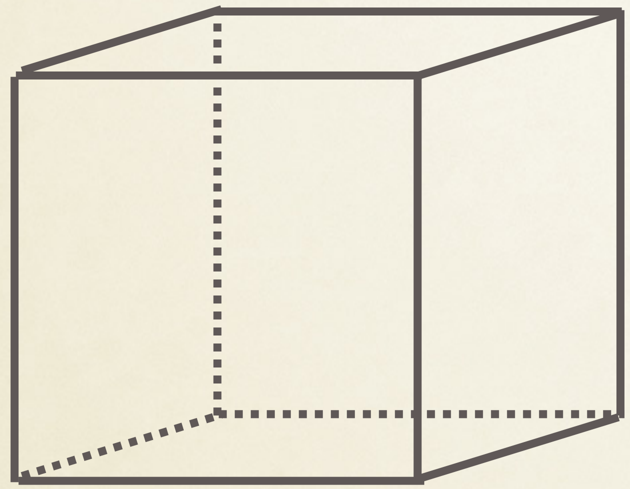
Our subject of is F_4, the symmetry group of an octacube,  , a <u>unique to 4D</u> Platonic solid, with no 3D analogue, <u>and its many-body realization</u>.

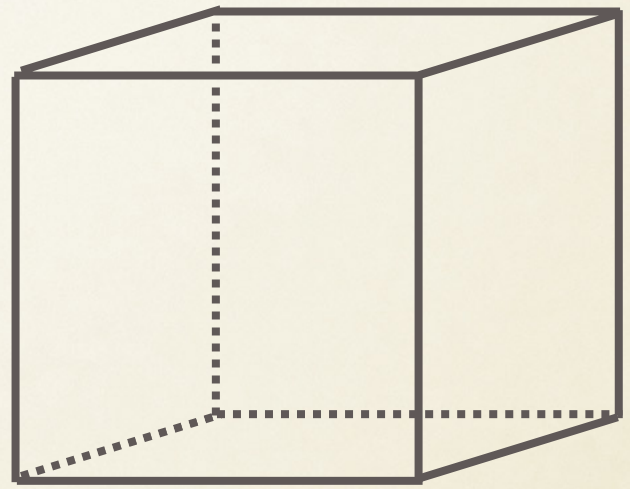
Our subject of is F_4, the symmetry group of an octacube,  , a unique to 4D Platonic solid, with no 3D analogue, and its many-body realization.

The “Octacube” and its designer,
Adrian Ocneanu, PennState

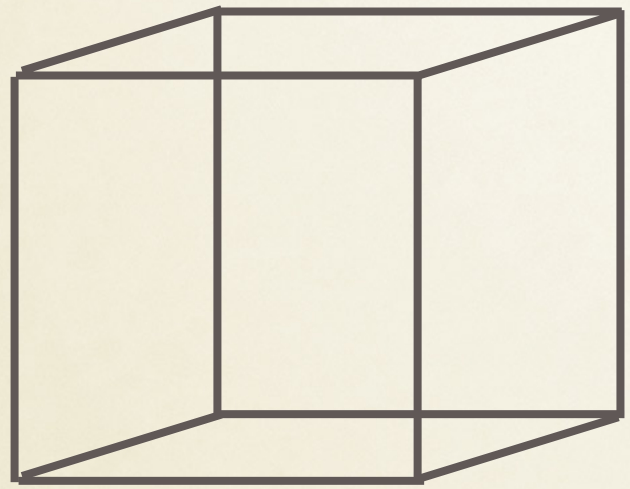
The "Octacube" and its designer, Adrian Ocneanu, PennState

Rhombic dodecahedron, the 3D cousin of the octacube

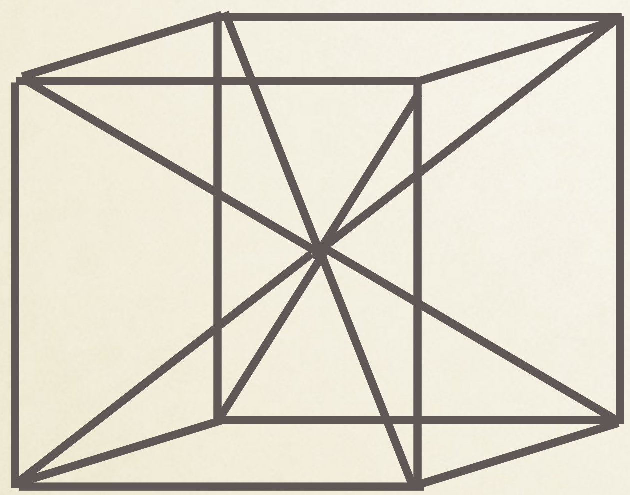


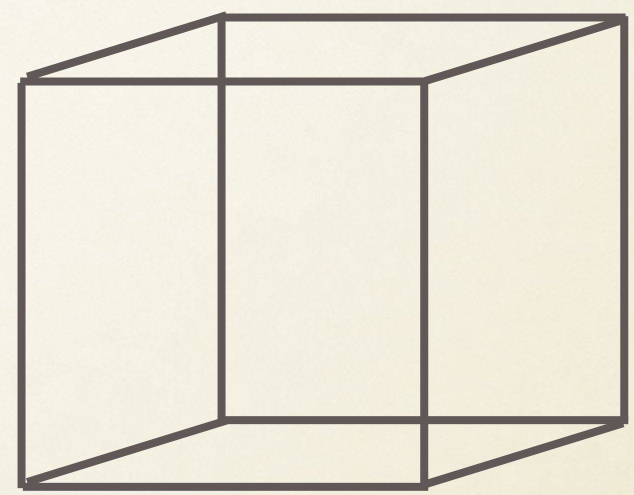


Rhombic dodecahedron, the 3D cousin of the octacube

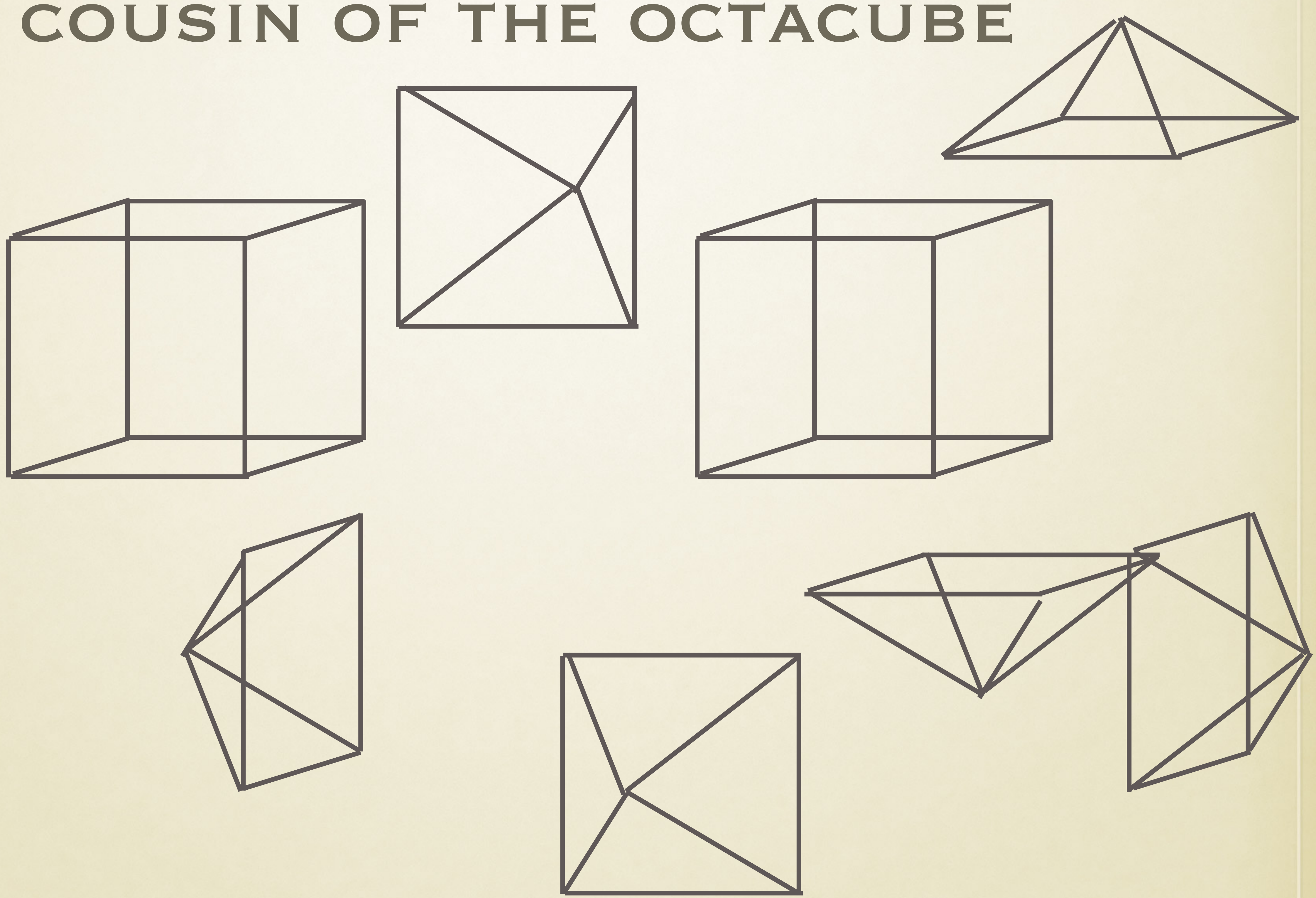


Rhombic dodecahedron, the 3D cousin of the octacube

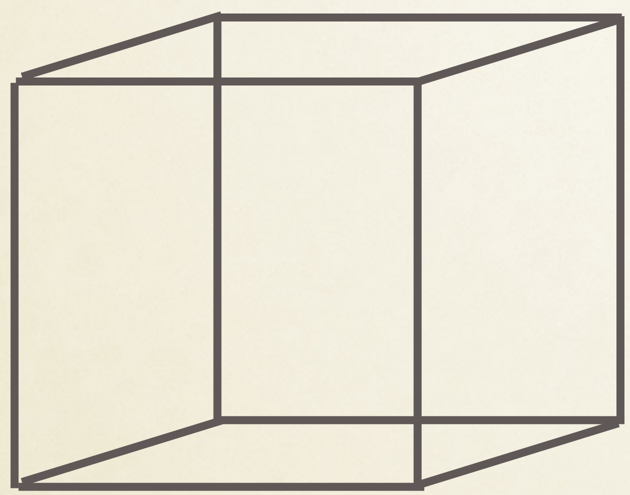


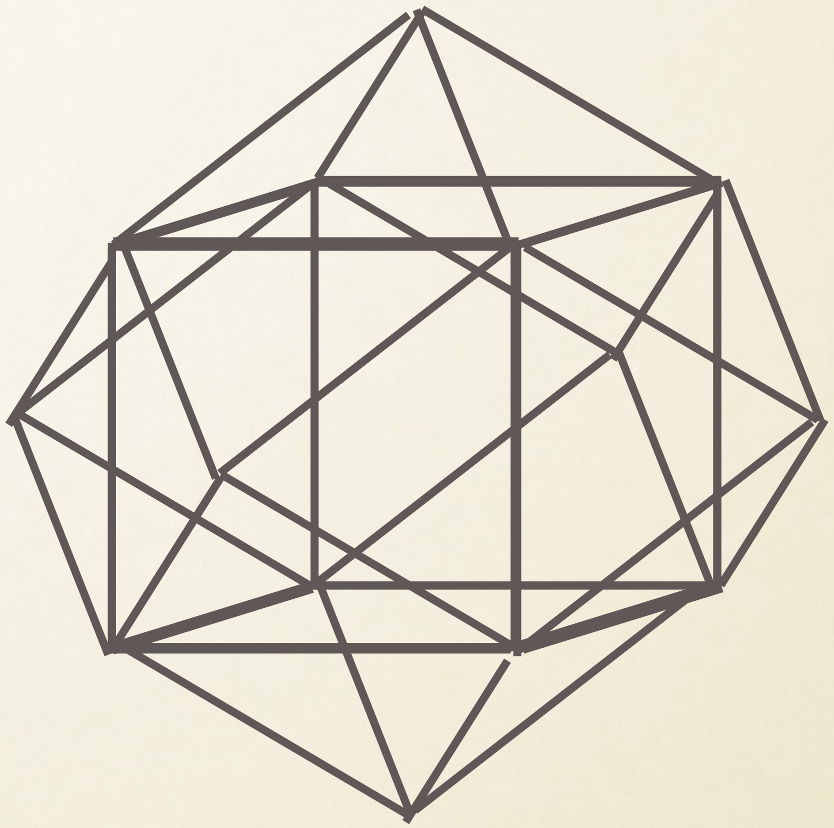


Rhombic dodecahedron, the 3D cousin of the octacube

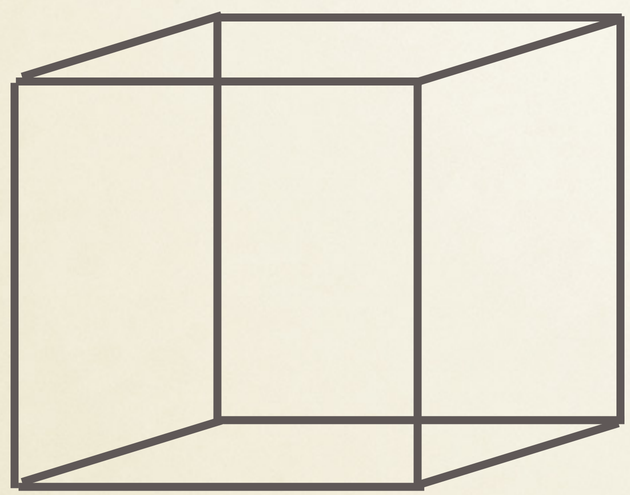


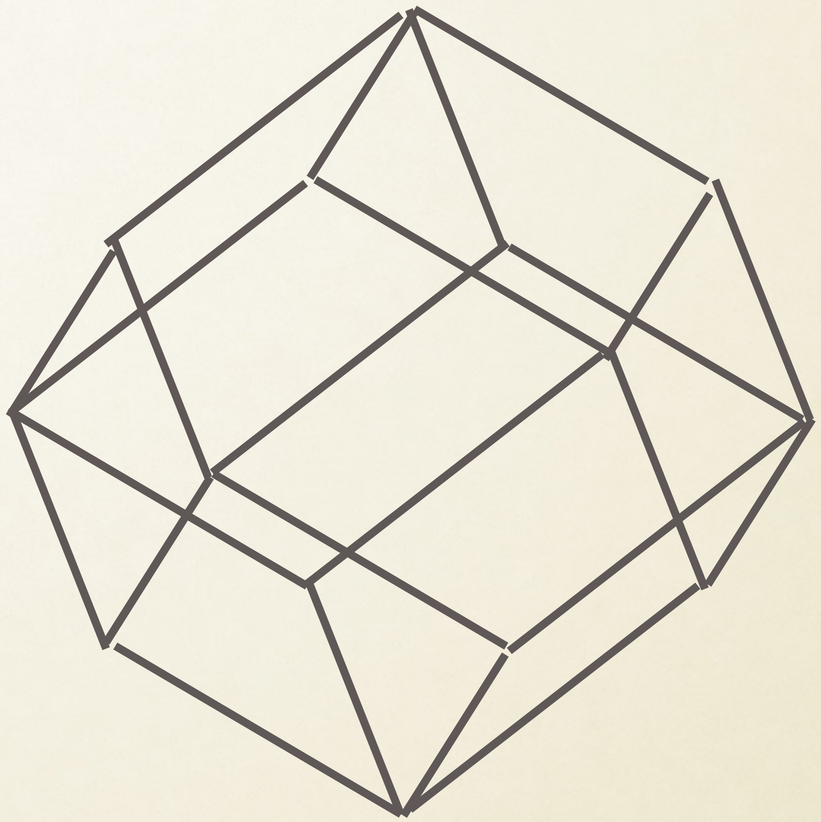
Rhombic dodecahedron, the 3D cousin of the octacube



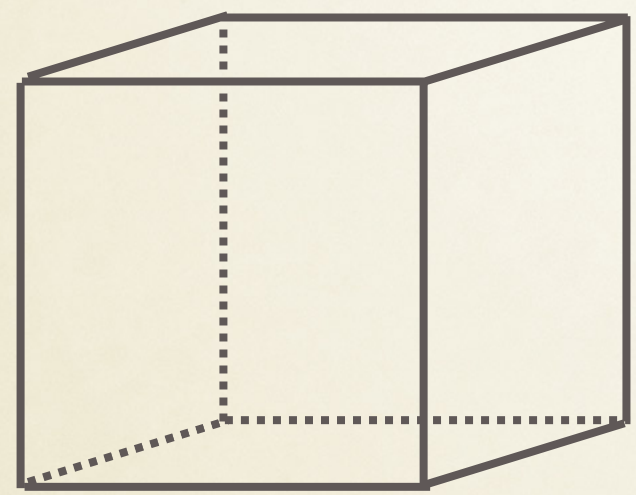


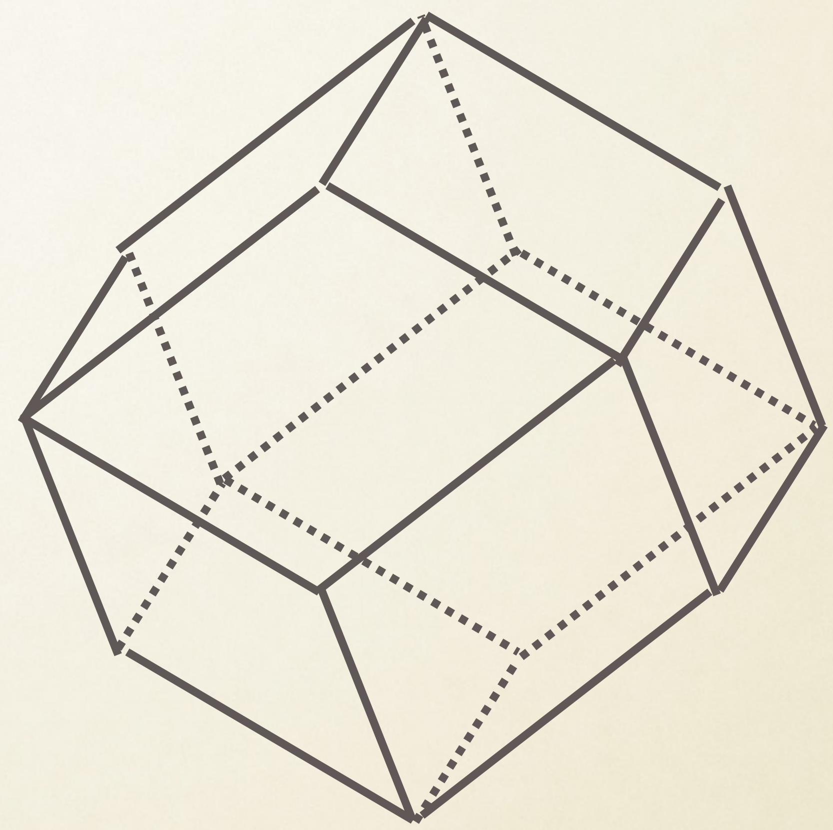
Rhombic dodecahedron, the 3D cousin of the octacube





Rhombic dodecahedron, the 3D cousin of the octacube





tintouen:

a **rhombic dodecahedron**, custom made for M.O.

Rhombic dodecahedron is the closest, albeit still distant cousin of the octacube, a unique four-dimensional Platonic solid with no three-dimensional analogues. The symmetry of a 4D space tiled by the octacubes is a key to the solution of a problem about four quantum particles with mass ratios 6:2:1:3 in a box.

7/10/15 — 11:45pm
FILED UNDER: #handmade #metal #geometry

TINTOUEN se trouve aux puces de Saint-Ouen,
venez par la 4 station porte de Clignancourt
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Le stand est à droite au pied des marches...
Broc. Sam et Dim 10h-18h. Atelier Mar à Ven

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1/27/15 — 10:58am

www.tintouen.fr

Repeat the steps above with two tesseracts and you will get an octacube. But unlike in 3D, in 4D you will get a Platonic solid.

The rhombic dodecahedron and the octacube are the 3D and 4D members of a family, that goes through all numbers of dimensions: in every dimension, the resulting polyhedron *tiles* the corresponding space

Building a particle system from the $\widetilde{F}_4$ Coxeter diagram

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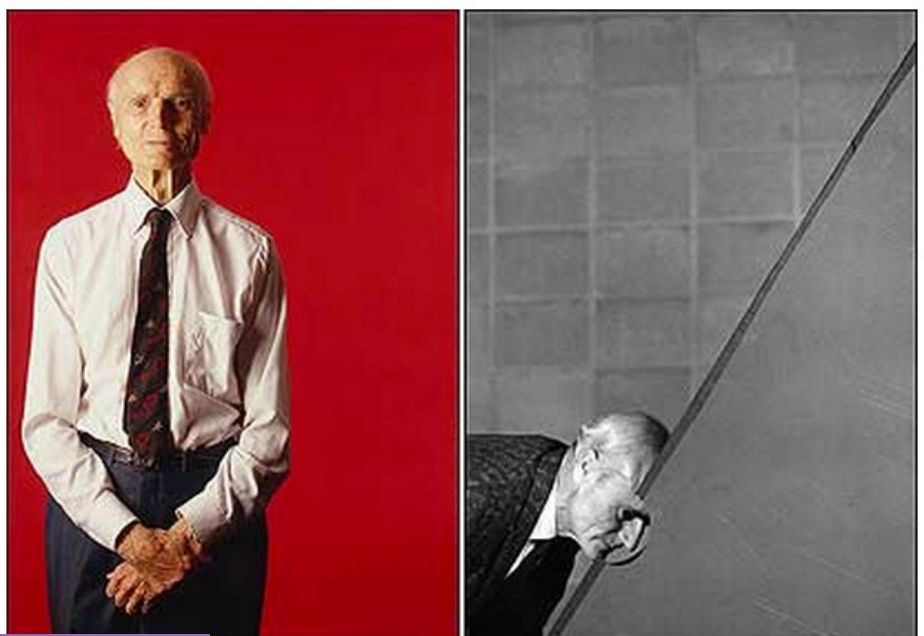
The man who saved geometry

Crying `Death to Triangles!' a generation of mathematicians tried to eliminate geometry in favor of algebra. Were it not for Donald Coxeter, they might have succeeded.



Donald Coxeter peers into a giant kaleidoscope (right). (Eden Robbins Photo at left) Eden Robbins Photo at left

By Siobhan Roberts
September 10, 2006

FOR A LOT OF PEOPLE, talk of geometry induces flashbacks to high school math class anxieties-fumbling with compasses and protractors and memorizing triangle theorems. So the idea that geometry was once on the brink of extinction as an academic subject does not elicit much regret or nostalgia. (Full article: 1299 words)

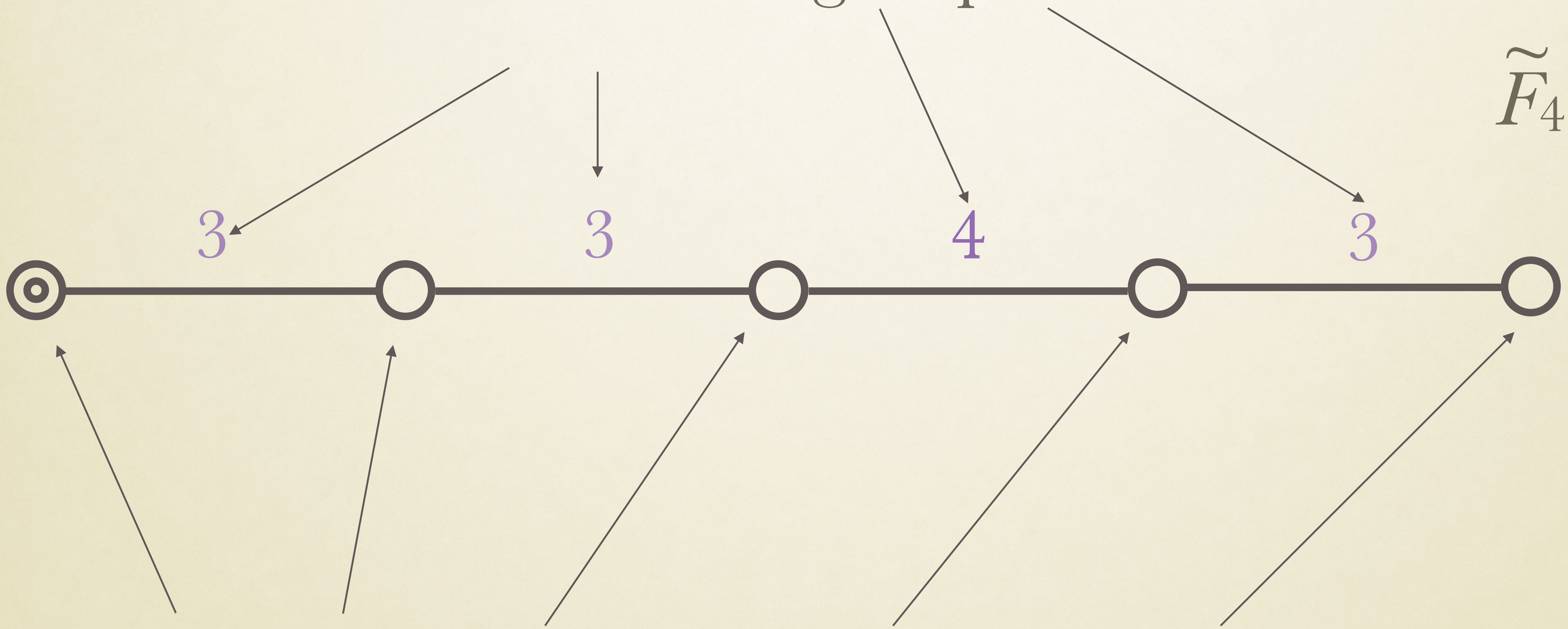
"[T]he angel of geometry and the devil of algebra share the stage, illustrating the difficulties of both."

Hermann Weyl

Alcoves of reflection groups (and many other geometric objects) are cataloged using Coxeter diagrams

Building a particle system from the $\widetilde{F}_4$ Coxeter diagram

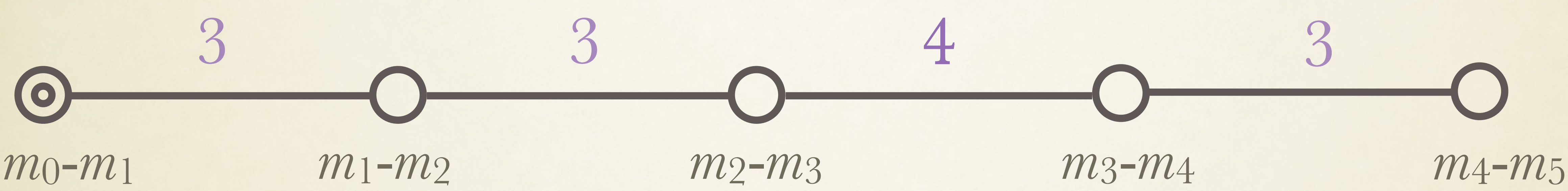
π / angles between the generating mirrors
of a reflection group



generating mirrors of the reflection group

Building a particle system from the $\widetilde{F}_4$ Coxeter diagram

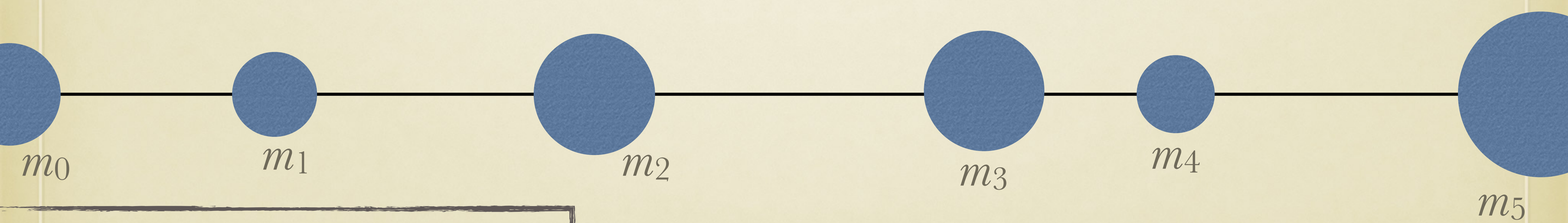
$\widetilde{F}_4$



$$\arctan\left[\frac{m_1(m_0+m_1+m_2)}{m_0 m_2}\right] = \pi/ \qquad \arctan\left[\frac{m_3(m_2+m_3+m_4)}{m_2 m_4}\right] = \pi/$$

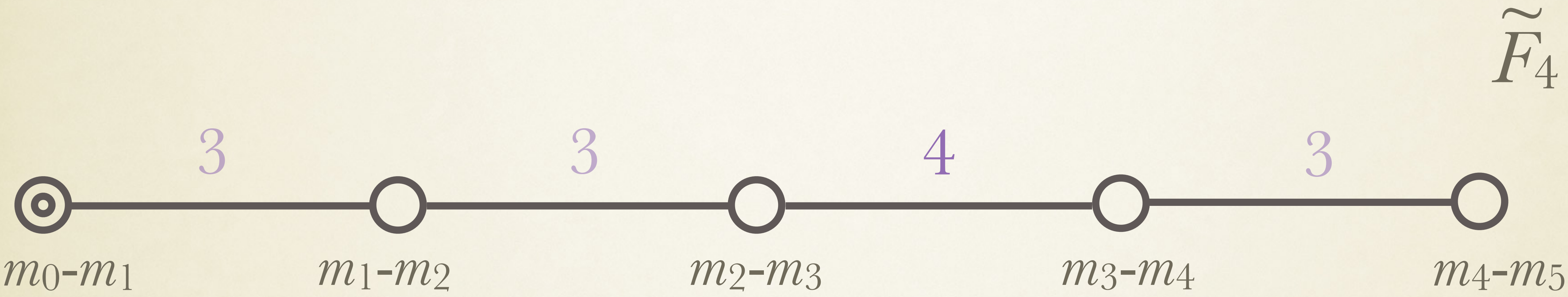
$$\arctan\left[\frac{m_2(m_1+m_2+m_3)}{m_1 m_3}\right] = \pi/ \qquad \arctan\left[\frac{m_4(m_3+m_4+m_5)}{m_3 m_5}\right] = \pi/$$

$m_0, m_1, m_2, m_3, m_4, m_5 > 0$



Original result

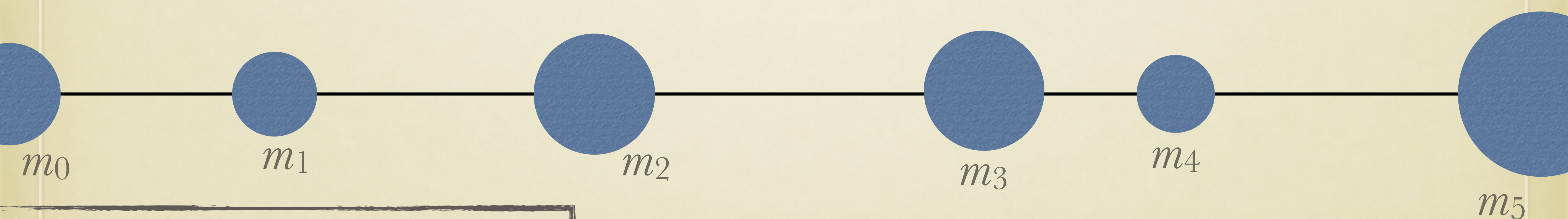
Building a particle system from the $\widetilde{F}_4$ Coxeter diagram



$$\arctan\left[\frac{m_1(m_0+m_1+m_2)}{m_0 m_2}\right] = \pi/3 \qquad \arctan\left[\frac{m_3(m_2+m_3+m_4)}{m_2 m_4}\right] = \pi/4$$

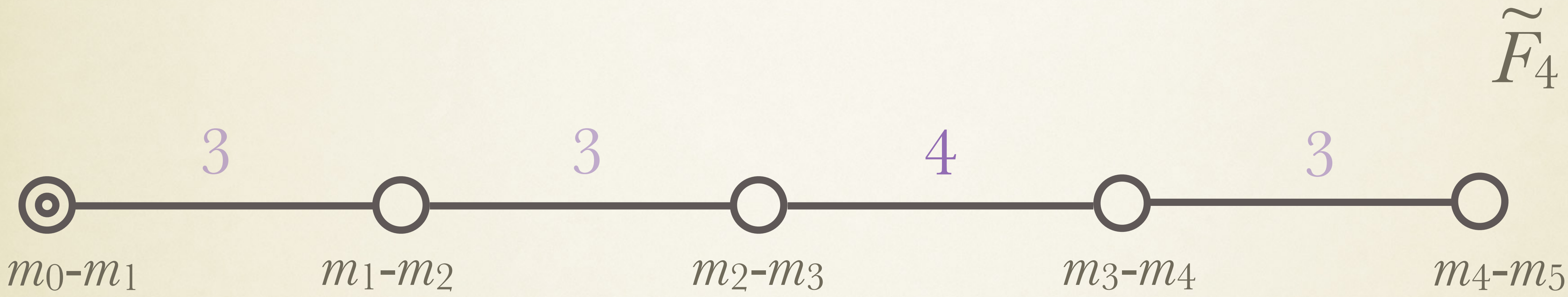
$$\arctan\left[\frac{m_2(m_1+m_2+m_3)}{m_1 m_3}\right] = \pi/3 \qquad \arctan\left[\frac{m_4(m_3+m_4+m_5)}{m_3 m_5}\right] = \pi/3$$

$m_0, m_1, m_2, m_3, m_4, m_5 > 0$



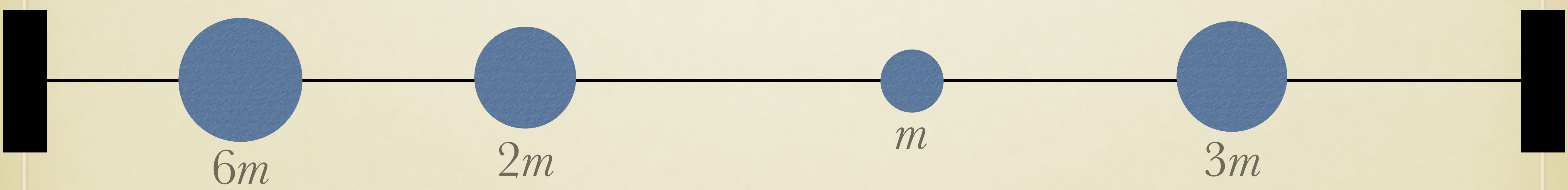
Original result

Building a particle system from the $\widetilde{F}_4$ Coxeter diagram

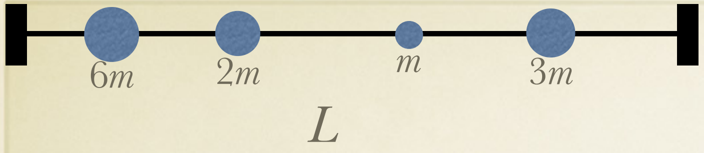


Single solution:

$m_0=\infty$, $m_1=6m$, $m_2=2m$, $m_3=m$, $m_4=3m$, $m_5=\infty$



Original result



Results

Periodicity cell:

octacube (24 octahedral 3-faces at all signs and permutations of $(\pm 1, \pm 1, 0, 0)$)

Energy spectrum:

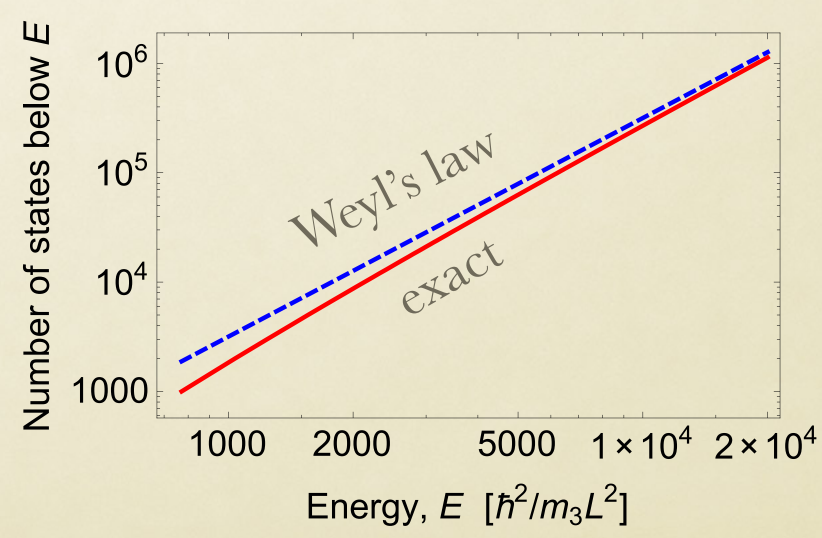
$$E_{n1,n2,n3,n4} = \frac{\pi^2 \hbar^2}{6mL^2}\left[2n_1(n_1+n_2+n_3+n_4) + n_2{}^2+n_3{}^2+n_4{}^2+n_2n_3+n_2n_4+n_3n_4\right]$$

$n_1 = 1, 2, 3, \ldots$

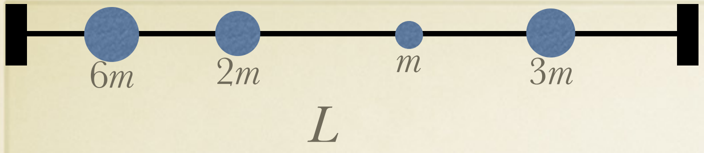
$n_2 = 1, 2, 3, \ldots$

$n_3 = n_2+1, n_2+2, n_2+3, \ldots$

$n_4 = n_3+1, n_3+2, n_3+3, \ldots$



Original result

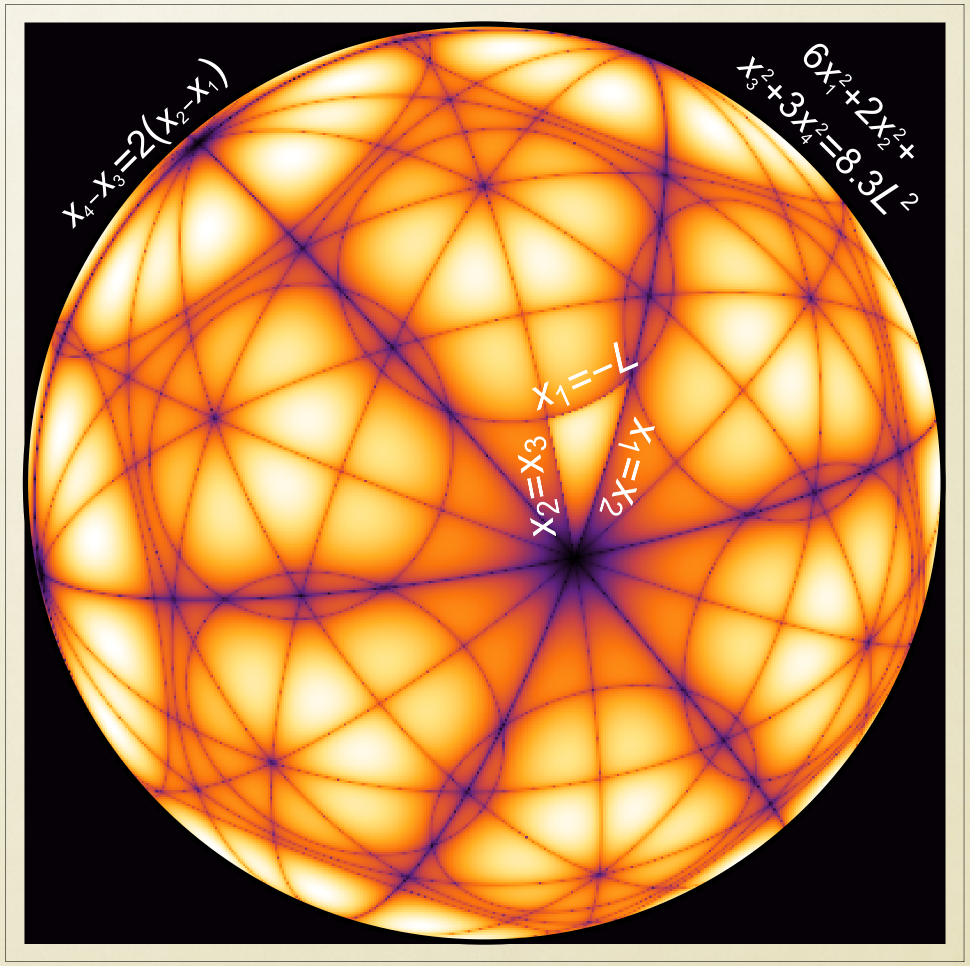


Results

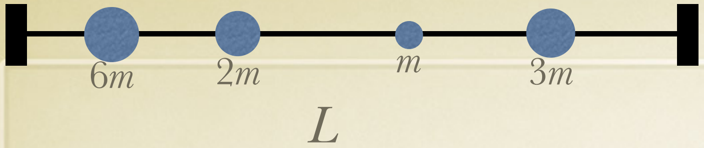
Ground state energy:

$$E_{1,1,2,3} = \frac{13\pi^2\hbar^2}{2mL^2}$$

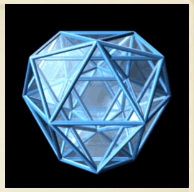
Ground state wavefunction:
consists of 1152 plane waves (the same for any other eigenstate)



Original result



Results



Four integrals of motion in involution:

$$I_l(\wp_1, \wp_2, \wp_3, \wp_4) \; = $$

$$(\wp_1+\wp_2)^l + (\wp_1-\wp_2)^l + (\wp_1+\wp_3)^l + (\wp_1-\wp_3)^l + (\wp_1+\wp_4)^l + (\wp_1-\wp_4)^l +$$
$$(\wp_2+\wp_3)^l + (\wp_2-\wp_3)^l + (\wp_2+\wp_4)^l + (\wp_2-\wp_4)^l + (\wp_3+\wp_4)^l + (\wp_3-\wp_4)^l \,,$$

$$l = 2,\, 6,\, 8,\, 12,$$

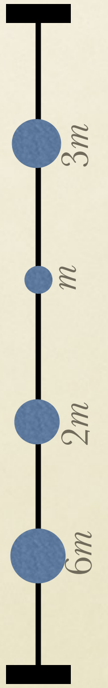
invariant polynomials of F_4

with

billiard momenta

$$\begin{bmatrix} \wp_1 \\ \wp_2 \\ \wp_3 \\ \wp_4 \end{bmatrix} \equiv \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

particle momenta



Remark: $I_2 \propto E$

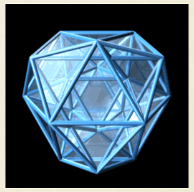
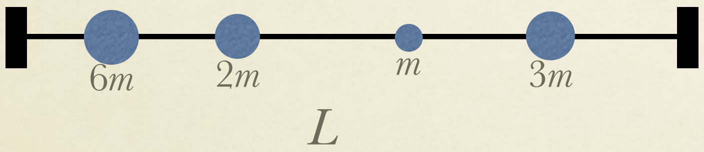
Remark: $A_{N\text{-}1}$->fermionic momentum moments

Original result

Summary

Summary

~ Extablished a map between affine reflection groups with non-forking Coxeter diagrams and exactly solvable quantum hard-core few-body problems on a line;

~ Worked the F_4 (symmetry of an octacube, ) to the end. The resulting integrable four-body system consists of four hard-cores with mass ratios 6:2:1:3, 

;

~ For F_4, found all four integrals of motion: Chevalley polynomials of square roots of particle kinetic energies.

Joint work with Maxim Olshanii
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Numerous discussions with:

Marvin Girardeau (U Arizona)
Vanja Dunjko (UMB)
Felix Werner (ENS)
Jean-Sébastien Caux (U Amsterdam)
Alfred G. Noël (UMB)
Dominik Schneble (Stony Brook)

Support by:

Thank you!