Short-Time Behavior for the Exciton-Polariton Equations

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XXXV Workshop on Geometric Methods in Physics
Białowieza, Poland - July 2, 2016

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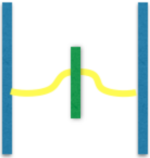
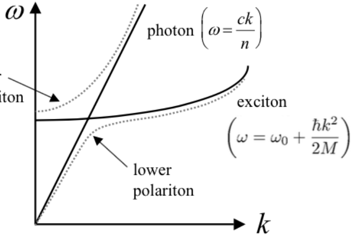
Exciton-Polariton

Semiconductor cavities

Half-light & Half-matter



Strong coupling: Excitons - Photons



Exciton-Polariton: 1d micro-cavity

$\phi(x,t)$: photons $\qquad$ $\psi(x,t)$: excitons

$$\begin{cases} i\partial_t \phi = (\omega_c - i\kappa_c - \frac{\hbar}{2m_C}\Delta)\phi + \gamma\psi \\ i\partial_t \psi = (\omega_x - i\kappa_x + g|\psi|^2)\psi + \gamma\phi, \end{cases}$$

Parameters:

Space: $x \in \mathbb{R}^n$

Time: $t \in \mathbb{R}$

Frequencies: ω_x, ω_c

Half of Rabi frequency: γ

Attractive: $g > 0$

Repulsive $g < 0$

Attenuation rates: κ_x, κ_c

$$i\partial_t u \quad = \quad -\Delta u + g|u|^2 u \qquad \mathrm{NLS}_3(\mathbb{R}^n)$$

Parameters:

Space: $x \in \mathbb{R}^n$ Time: $t \in \mathbb{R}$

Attractive: $g > 0$ Repulsive $g < 0$

Invariance

- Spatial translation

$$u(x,t) \quad \mapsto \quad u(x+x_0,t)$$

- Time translation

$$u(x,t) \quad \mapsto \quad u(x,t+t_0)$$

- Galilean transformation

$$u(x,t) \quad \mapsto \quad u(x-\xi t,t)e^{i(k\cdot\mathbf{x}-\omega t)}$$

- Scaling

$$u(x,t) \quad \mapsto \quad \lambda u(\lambda^2 x,\lambda t)$$

Conserved Quantities

- Number of particles (Mass)

$$M[u](t) = \int_{\mathbb{R}^n} |u(x,t)|^2 dx$$

- Hamiltonian (Energy)

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx + \frac{g}{4}\int_{\mathbb{R}^n} |u|^4 dx.$$

- Momentum

$$P[u](t) = \mathrm{Im}\int_{\mathbb{R}^n} \bar{u}\nabla u$$

Norms

- Lebesgue norm:

$$p_c = n$$

$$\|u\|_{L^{p_c}(\mathbb{R}^n)} = \|u_\lambda\|_{L^{p_c}(\mathbb{R}^n)}$$

- Sobolev norm:

$$s_c = \frac{n}{2} - 1$$

$$\|u\|_{\dot{H}^{s_c}(\mathbb{R}^n)} = \|u_\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^n)}$$

Others conserved quatities

$$\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^{s} \qquad \text{and} \qquad M[u]^{1-s}E[u]^s$$

- $s_c < 0 \implies n = 1$ mass-subcritical
- $s_c = 0 \implies n = 2$ mass-critical
- $0 < 1 s_c < 1 \implies n = 3$ mass-supercritical energy subcritical
- $s_c = 1 \implies n = 5$ energy critical
- $s_c > 1 \implies n = 5$ energy supercritical

Ground State Soliton Solution

$u_Q(x,t) = e^{i\beta t} Q(\alpha x)$. - solution if $-\beta\, Q + \alpha^2\, \Delta Q + Q^3 = 0$,

$$\alpha := \sqrt{n}, \text{ and } \beta := 1 - \tfrac{n-2}{2}.$$

- Elliptic PDE theory: infinitely many H^1 solutions
  choose the minimal L^2 norm solution $\Rightarrow$ ground state Q
  Properties:
  1. Q - positive, radial, vanishing at ∞
  2. Minimizer for Gagliardo-Nirenberg (*Weinstein*' 82):

$$\|u\|_{L^4}^4 \le C_{GN} \|\nabla u\|_{L^2}^n \|u\|_{L^2}^{4-n}$$

with

$$C_{GN} = \frac{2}{\|Q\|_{L^2}^2}.$$

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-\tau)\Delta}|u|^2u(\tau)d\tau \equiv \mathrm{NLS}(t)u_0$$

$$e^{it\Delta}u_0 \rightsquigarrow \mathrm{NLS}(t)u_0$$

Questions:

When do solutions scatter?

$$u(t) \to e^{it\Delta}v^+ \qquad \text{as} \qquad t \to \infty$$

When Blow-up occurs?

$$\|\nabla u(t)\|_{L^2} \to \infty \qquad \text{as} \qquad t \to T^*$$

Lossless Polariton

$\kappa_x = \kappa_c = 0$

$$i\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \omega_c - \Delta & \gamma \\ \gamma & \omega_x - g|\psi|^2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Conserved quantities

- Number of particles (Mass)

$$M[u](t) = \int \left(|\psi|^2 + |\phi|^2\right) dx$$

- Hamiltonian (Energy)

$$E[u](t) = \int \left(\frac{1}{2}|\nabla\psi|^2 + \omega_c|\phi|^2 + \omega_X|\psi|^2 + \frac{g}{2}|\psi|^4 + 2\gamma Re(\psi\overline{\phi})\right) dx.$$

Lossless Polariton

Invariance

- Spatial translation
- Time translation

Break of Invariance

- Scaling

$$\begin{pmatrix} \phi(\mathbf{x},t) \\ \psi(\mathbf{x},t) \end{pmatrix} \quad \mapsto \quad \lambda \begin{pmatrix} \phi(\lambda\mathbf{x},\lambda^2 t) \\ \psi(\lambda\mathbf{x},\lambda^2 t) \end{pmatrix},$$

$$\begin{pmatrix} \omega_x - g|\psi|^2 & \gamma \\ \gamma & \omega_c - \Delta \end{pmatrix} \mapsto \begin{pmatrix} \lambda^2\omega_c - \Delta & \lambda^2\gamma \\ \lambda^2\gamma & \lambda^2\omega_x - g|\psi|^2 \end{pmatrix}$$

Lossless Polariton

- Galilean transformation

$$\begin{pmatrix} \phi(\mathbf{x},t) \\ \psi(\mathbf{x},t) \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \phi(\mathbf{x}-\xi t,t) \\ \psi(\mathbf{x}-\xi t,t) \end{pmatrix} e^{i(\xi\cdot\mathbf{x}-|\xi|^2 t)},$$

$$\begin{pmatrix} \omega_c - \Delta & \gamma \\ \gamma & \omega_x - g|\psi|^2 \end{pmatrix}$$

$$\downarrow$$

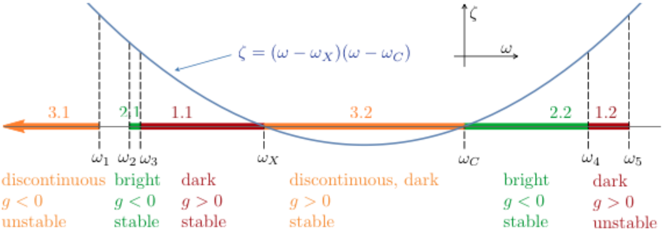
$$\begin{pmatrix} \omega_c - \Delta & \gamma \\ \gamma & \omega_x - |\xi|^2 - g|\psi|^2 - i\xi\cdot\nabla\psi \end{pmatrix}$$

Lossless Polariton

What about the ground state?

$$\begin{aligned}\psi(x,t) &= \psi(x)e^{-i\omega t},\\ \phi(x,t) &= \phi(x)e^{-i\omega t}.\end{aligned}$$

Komineas, Shipman, Venakides



Lossless Polariton with $\omega_c = 0$

Consider

$$i\phi_t = -\Delta\phi + \gamma\psi$$
$$i\psi_t = (\omega_X + g|\psi|^2)\psi + \gamma\phi$$

and

$$\begin{pmatrix} \phi(x,0) \\ \psi(x,0) \end{pmatrix} = \begin{pmatrix} \phi_0(x) \\ 0 \end{pmatrix} \in H^s(\mathbb{R}^n) \qquad \text{with } s > \frac{n}{2}.$$

Existence of solutions to the polariton equations

Given:

$$\|\phi_0\|_{H^s} \leq \alpha N \qquad \text{for} \qquad N > 0, \quad \alpha \in (0,1)$$

There exists a unique solution $\begin{pmatrix} \phi(x,t) \\ \psi(x,t) \end{pmatrix} \in C(I, H^s(\mathbb{R}^n))$ to the polariton system such that

$$\|\phi(t)\|_{H^s} < N \quad \text{and} \quad \|\psi(t)\|_{H^s} < N$$

for

$$0 \leq t \leq \frac{1-\alpha}{2\gamma + |g|N^2}.$$

Short-time behavior

Observing the photon field:

- up to what time is the effect of the exciton on the photon field negligible?

$$\begin{aligned} i\phi_t &= -\Delta\phi \\ i\psi_t &= \omega_X\psi + \gamma\phi\,. \end{aligned} \qquad \text{(Approximation A)}$$

- up to what time thereafter is the effect of the nonlinearity on the photon field negligible?

$$\begin{aligned} i\phi_t &= -\Delta\phi + \gamma\psi \\ i\psi_t &= \omega_X\psi + \gamma\phi\,. \end{aligned} \qquad \text{(Approximation B)}$$

Theorem

Let $\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix}, \begin{pmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{pmatrix} \in C([0,t_3], H^s(\mathbb{R}^n))$ $\epsilon > 0$, $c_1, c_2 \in \mathbb{R}$ s.t. $c_2\epsilon^{1/5} < t_3$.

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} \leftrightarrow \text{ polariton}, \begin{pmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{pmatrix} \leftrightarrow \begin{cases} \text{approx. A} & [0, c_1\epsilon^{1/2}] \\ \text{approx. B} & [c_1\epsilon^{1/2}, c_2\epsilon^{1/5}] \\ \text{polariton} & [c_2\epsilon^{1/5}, t_3] \end{cases}$$

IC

$$\begin{pmatrix} \phi(0) \\ \psi(0) \end{pmatrix} = \begin{pmatrix} \tilde{\phi}(0) \\ \tilde{\psi}(0) \end{pmatrix} = \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \|\phi_0\|_s = M \neq 0$$

Then

$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_s}{\|\phi(t)\|_s} \leq K_1\epsilon + O(\epsilon^2) \qquad 0 \leq t \leq c_1\epsilon^{1/2} \qquad (\epsilon \to 0),$$

$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_s}{\|\phi(t)\|_s} \leq K_2\epsilon + O(\epsilon^{7/5}) \qquad c_1\epsilon^{1/2} \leq t \leq c_2\epsilon^{1/5} \qquad (\epsilon \to 0).$$

Tools

- H^s Estimates
- Triangle inequality

Decay estimate

We are NOT using

$$\left|e^{it\triangle}\phi_0\right| \leq C\frac{1}{t^{n/2}}\left\|\phi_0\right\|_{L^1}.$$

$\mathrm{NLS}_3(\mathbb{R}^3)$ Theorem

$u(t), \tilde{u}(t) \in C([0, t_2], H^s(\mathbb{R}^n))$

$$\epsilon > 0, \;\; c_4 \in \mathbb{R} \text{ s.t. } c_4 \epsilon^{1/2} < t_2.$$

$$u(t) \mapsto \mathrm{NLS}_3(\mathbb{R}^n) \qquad \tilde{u}(t) \mapsto \begin{cases} iu_t = -\Delta u & 0 < t \leq c_4 \epsilon^{1/2} \\ \mathrm{NLS}_3(\mathbb{R}^n) & c_4 \epsilon^{1/2} < t < t_2 \end{cases}$$

IC:

$$u(x,0) = u(x,0) = u_0(x) \quad \text{and} \quad \|u_0\|_s = N$$

Then

$$\frac{\|\tilde{u}(t) - u(t)\|_s}{\|u(t)\|_s} \leq K_4 \epsilon + O(\epsilon^{1/4}) \qquad 0 \leq t \leq c_4 \epsilon^{1/2} \qquad (\epsilon \to 0).$$

$\mathrm{NLS}_3(\mathbb{R}^3)$ Theorem -small data

$u(t), \tilde{u}(t) \in C([0, t_2], H^s(\mathbb{R}^n))$

$$0 \le \alpha \le \tfrac{1}{3}, \quad \epsilon > 0, \; c_3 \in \mathbb{R} \text{ s.t. } c_3\epsilon^{1-2\alpha} < t_2.$$

$$u(t) \mapsto \mathrm{NLS}_3(\mathbb{R}^n) \qquad \tilde{u}(t) \mapsto \begin{cases} iu_t = -\Delta u & 0 < t \le c_3\epsilon^{1-2\alpha} \\ \mathrm{NLS}_3(\mathbb{R}^n) & c_3\epsilon^{1-2\alpha} < t < t_2 \end{cases}$$

IC:

$$u(x,0) = u(x,0) = u_0(x) \quad \text{and} \quad \|u_0\|_s = \epsilon^\alpha$$

Then

$$\frac{\|\tilde{u}(t) - u(t)\|_s}{\|u(t)\|_s} \le K_3\epsilon + O(\epsilon^{2-3\alpha}) \qquad 0 \le t \le c_3\epsilon^{1-2\alpha} \qquad (\epsilon \to 0).$$

Comments

- Exciton bounds
  - Approximation A: $0 < t \leq c_1 \epsilon^{1/2}$

$$\frac{\|\hat{\psi}\|_{L_t^\infty H_x^s}}{\|\psi\|_{L_t^\infty H_x^s}} \approx O(\epsilon)$$

  - Approximation B:$c_1 \epsilon^{1/2} \leq t \leq c_2 \epsilon^{1/5}$

$$\frac{\|\hat{\psi}\|_{L_t^\infty H_x^s}}{\|\psi\|_{L_t^\infty H_x^s}} \approx O(\epsilon^{2/5})$$

- Optimal bounds?

Thank you - Dziękuję Ci - Gracias