Supergeometry of gauge PDE and AKSZ sigma models

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Based on:

M.G., 1606.07532

M.G., A. Verbovetsky, to appear

K. Alkalaev, M.G. 2013

Glenn Barnich, M.G. 2010

M.G. 2010,2012

June 27, 2016,

Bialowieza, Poland

Motivations

- Batalin-Vilkovisky (BV) approach to gauge systems (or its generalizations) is probably the most powerful.
  Batalin, (Fradkin), Vilkovisky, 1981 . . .

- For various topological models their BV formulation can be cast into the form of AKSZ sigma model.
  Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994.

- In so doing the equations of motion, gauge symmetries, etc. are encoded in a homological vector field Q on the target space, which is a Q-manifold (or QP-manifold in Lagrangian case).

- A natural question is weather the same can be done for non-topological systems? How the gauge symmetries, Lagrangians, etc. are encoded in the geometry of the target space?

- For a local gauge theory AKSZ-like formulation has certain advantages over the usual jet-space version of the BV formalism *Henneaux; Barnich, Brandt, Henneaux*

  This has to do with the manifest background independence of AKSZ, which can be employed in studying boundary values, manifest realization symmetries etc.

Batalin-Vilkovisky formalism:

Given equations T_a, gauge symmetries R^i_α, reducibility relations,.... the BRST differential:

$$s = \delta + \gamma + \ldots\,, \qquad s^2 = 0\,, \quad \mathrm{gh}(s) = 1$$

$$\delta = T_a \frac{\partial}{\partial \mathcal{P}_a} + Z^a_A \mathcal{P}_a \frac{\partial}{\partial \pi^A} \ldots\,, \qquad \gamma = c^\alpha R^i_\alpha \frac{\partial}{\partial \phi_i} + \ldots\,.$$

δ – (Koszule-Tate) restriction to the stationary surface
γ – implements gauge invariance condition
ϕ^i – fields, c^α – ghosts,
$\mathcal{P}_a$ – ghost momenta, π^A – reducibility ghost momenta

$$\mathrm{gh}(\phi^i) = 0\,, \quad \mathrm{gh}(c^\alpha) = 1\,, \quad \mathrm{gh}(\mathcal{P}_a) = -1\,, \quad \ldots$$

BRST differential completely defines the system.
Equations of motion and gauge symmetries can be read off from s:

$$s\mathcal{P}_a|_{\mathcal{P}_a=0,\, c^\alpha=0\,,\ldots} = 0\,, \quad \delta_\epsilon \phi^i = (s\phi^i)|_{c^\alpha=\epsilon^\alpha,\, \mathcal{P}_a=0,\, \ldots}$$

If the theory is Lagrangian then: $T_i = \frac{\delta S_0}{\delta \phi^i}$, reducibility relations $R^i_\alpha T_i = 0$ so that $Z^i_\alpha = R^i_\alpha$

Natural bracket structure (antibracket)

$$\left(\phi^i, \mathcal{P}_j\right) = \delta^i_j \qquad \left(c^\alpha, \mathcal{P}_\beta\right) = \delta^\alpha_\beta$$

BV master action

$$s = \left(\cdot, S_{BV}\right), \qquad S_{BV} = S_0 + \mathcal{P}_i R^i_\alpha c^\alpha + \dots$$

Master equation:

$$\left(S_{BV}, S_{BV}\right) = 0 \quad \Longleftrightarrow \quad s^2 = 0$$

Example: YM theory

Fields: A_μ, C (with values in the Lie algebra)

Antifields: $A^{*\mu}$, C^*

Gauge part BRST differential: $\gamma A_\mu = \partial_\mu C + [A_\mu, C]$

Master action:

$$S_{BV} = S_0 + \int d^n x \mathrm{Tr}[A^{*\mu}(\partial_\mu C + [A_\mu, C]) + \frac{1}{2} C^* [C, C]]$$

AKSZ sigma models

M - supermanifold (target space) with coordinates Ψ^A:
Ghost degree – gh()
(odd)symplectic structure σ, gh(σ) $= n - 1$ and hence
(odd)Poisson bracket $\{\cdot\,,\,\cdot\}$, gh($\{\cdot\,,\,\cdot\}$) $= -n + 1$
"BRST potential" $S_M(\Psi)$, gh(S_M) $= n$, master equation
$\{S_M, S_M\} = 0$
(QP structure: $Q = \{\cdot\,, S_M\}$ and $P = \{\cdot\,,\,\cdot\}$)

$\mathcal{X}$ - supermanifold (source space)
Ghost degree gh( )
$\boldsymbol{d}$ – odd vector field, $\boldsymbol{d}^2 = 0$, gh($\boldsymbol{d}$) $= 1$
Tipically, $\mathcal{X} = T[1]X$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$, $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$,
$\mu = 0, \ldots n - 1$

$\Phi : \mathcal{X} \to M$. Fields $\Psi^A(x,\theta) \equiv \Phi^*(\Psi^A)$.

BV master action

$$S_{BV} = \int \left[(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M) \right] , \qquad \text{gh}(S_{BV}) = 0$$

χ is potential for $\sigma = d\chi$. In components:

$$S_{BV} = \int d^n x d^n \theta \left[\chi_A(\Psi(x,\theta)) \boldsymbol{d} \Psi^A(x,\theta) + S_M(\Psi(x,\theta)) \right]$$

BV antibracket

$$\big(F, G\big) = \int d^n x d^n \theta \left(\frac{\delta^R F}{\delta \Psi^A(x,\theta)} \sigma^{AB} \frac{\delta G}{\delta \Psi^B(x,\theta)} \right) , \qquad \text{gh}\big(, \big) = 1$$

$\sigma^{AB}(\Psi)$ – components of the Poisson bivector.

Master equation:

$$\big(S_{BV}, S_{BV}\big) = 0 ,$$

BRST differential:

$$s^{AKSZ}\Psi^A(x,\theta) = \boldsymbol{d}\Psi^A(x,\theta) + Q^A(\Psi(x,\theta))\,, \qquad Q^A = \left\{\Psi^A, S_M\right\}$$

Natural lift of Q and $\boldsymbol{d}$ to the space of maps.

Dynamical fields: those of vanishing ghost degree

$$\Psi^A(x,\theta) = \overset{0}{\Psi}{}^A(x) + \overset{1}{\Psi}{}^A_\mu(x)\theta^\mu + \ldots \qquad \mathrm{gh}(\overset{k}{\Psi}{}^A_{\mu_1\ldots\mu_k}) = \mathrm{gh}(\Psi^A) - k$$

If $\mathrm{gh}(\Psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\Psi}{}^A_{\mu_1\ldots\mu_k}(x)$ is dynamical.

AKSZ equations of motion

$$\sigma_{AB}(\boldsymbol{d}\boldsymbol{\Psi}^A + Q^A) = 0\,, \quad \Rightarrow \quad \boldsymbol{d}\boldsymbol{\Psi}^A(x,\theta) + Q^A(\boldsymbol{\Psi}(x,\theta)) = 0$$

(recall: σ_{AB} is invertible)

AKSZ at the level of equations of motion (nonlagrangian)

$$\{,\}\,,\, S_M \quad \Rightarrow \quad Q = Q^A \frac{\partial}{\partial \boldsymbol{\Psi}^A} \quad Q^2 = 0\,.$$

I.e. target is a generic Q manifold.

target doesn't know dim X! (Recall gh$(S_M) = n =$ dim X)

If gh$(\boldsymbol{\Psi}^A) \geqslant 0 \;\forall\; \boldsymbol{\Psi}^A$ then BV-BRST extended FDA.

Otherwise BV-BRST extended FDA with constraints.

Examples:

Chern-Simons: *AKSZ, 1994*

Target space M:

$M = \mathfrak{g}[1]$, $\mathfrak{g}$ – Lie algebra with invariant inner product.

e_i –basis in $\mathfrak{g}$, C^i – coordinates on $\mathfrak{g}$, gh$(C^i) = 1$, $C = C^i e_i$

$$S_M = \frac{1}{6}\langle C, [C, C]\rangle\,, \qquad \left\{C^i, C^j\right\} = \langle e_i, e_j\rangle^{-1}$$

Source space:

$\mathcal{X} = T[1]X$, X – 3-dim manifold. Fied content

$$C^i(x,\theta) = c^i(x) + \theta^\mu A^i_\mu(x) + \theta^\mu\theta^\nu A^{*i}_{\mu\nu} + (\theta)^3 c^{*i}$$

BV action

$$S_{BV} = \int (\frac{1}{2}\langle C, \boldsymbol{d}C\rangle + \frac{1}{6}\langle C, [C, C]\rangle) = \int \frac{1}{2}\langle A, \boldsymbol{d}A\rangle + \frac{1}{6}\langle A, [A, A]\rangle) + \ldots$$

1d AKSZ systems

Target space M – Extended phase space: $\{,\}$ – Poisson bracket, $S_M = \Omega - \theta H$, Ω – BRST charge, H - BRST invariant Hamiltonian

Source space $\mathcal{X} = T[1](\mathbb{R}^1)$, coordinates t, θ

BV action *M.G., Damgaard, 1999*

$$S_{BV} = \int dt d\theta (\chi_A \boldsymbol{d} \psi^A + \Omega - \theta H)$$

Integration over θ gives BV for the Hamiltoninan action *Fisch, Henneaux, 1989, Batalin, Fradkin 1988*.

Example: coordinates on M: $\widetilde{c}, \widetilde{\mathcal{P}}, \widetilde{x}^\mu, \widetilde{p}_\mu$, BRST charge $\Omega = \widetilde{c}(\widetilde{p}^2 - m^2)$,

$$S_{BV} = \int dt d\theta (\widetilde{p}_\mu \boldsymbol{d} \widetilde{x}^\mu + \widetilde{\mathcal{P}} \boldsymbol{d} \widetilde{c} + \widetilde{c}(p^2 - m^2)) = \int dt (p_\mu \dot{x}^\mu + \lambda(p^2 + m^2))$$

$$\widetilde{x}^\mu(t, \theta) = x^\mu(t) + \theta p_*^\mu(t)\,, \qquad \widetilde{p}_\mu(t, \theta) = p_\mu(t) + \theta x_\mu^*(t)\,,$$

$$\widetilde{c}(t, \theta) = c(t) + \theta \lambda(t)\,, \qquad \ldots$$

– Background-independent

– AKSZ is both Lagrangian and Hamiltonian

AKSZ model: $(M, S_M, \{,\})$ and $(\mathcal{X}, \boldsymbol{d})$.

Let $X = X_S \times \mathbb{R}^1$ *Barnich, M.G, 2003*

$$\Omega_{BFV} = \int_{X_S} [(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M)] \,, \qquad \mathrm{gh}(\Omega_{BFV}) = 1$$

$$\{\cdot\,,\,\cdot\}_{BFV} = \int d^{n-1}x d^{n-1}\theta \, \{\cdot\,,\,\cdot\} \qquad \{\Omega_{BFV}, \Omega_{BFV}\}_{BFV} = 0\,.$$

– Higher BRST charges

Similarly: $X_k \subset X$ – dimension-k submanifold

$$\Omega_{X_k} = \int_{X_k} (\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M))$$

In particular, $\Omega_{BFV} = \Omega_{X_S}$, $S_{BV} = \Omega_X$

– At the level of equations of motion AKSZ is a generalization of so-called unfolded formalism independently developed in the context of HS theories *Vasiliev 1988,...* .

– At the level of equations of motion the same target space gives an AKSZ model for any $X_k \subset X$ or even different X. Useful for "replacing space-time". E.g. *Vasiliev 2002*

(asymptotic) boundary values, e.g. in the context of AdS/CFT

For higher-spin fields *Vasiliev, 2012; Bekaert M.G. 2012*

– Locally in X and M: *Barnich, M.G. 2009*

$$H^g(s^{AKSZ}, \text{local functionals}) \cong H^{g+n}(Q, C^\infty(M))$$

Isomorphism sends $f \in C^\infty(M)$ to functional $F = \int f$. Compatible with the bracket.

– If M finite dimensional and $n > 1$ – the model is topological. What about non-topological?

AKSZ form of PDE

Jet-bundle:

Fiber-bundle $\mathcal{F} \to X$ (for simplicity: direct product of $\mathbb{R}^n \times \mathbb{R}^N$): base space (independent variables or space-time coordinates): x^a, $a = 1, \ldots, n$.

Fiber coordinates (dependent variables or fields) ϕ^i. Jet-bundle:

$$J^0(\mathcal{F}): \quad x^a, \phi^i, \qquad J^1(\mathcal{F}): \quad x^a, \phi^i, \phi^i_a\,, \qquad J^2(\mathcal{F}) \quad x^a, \phi^i, \phi^i_a, \phi^i_{ab}\,, \qquad \ldots$$

Projections:

$$\ldots \to J^N(\mathcal{F}) \to J^{N-1}(\mathcal{F}) \to \ldots \to J^1(\mathcal{F}) \to J^0(\mathcal{F}) = \mathcal{F}$$

Useful to work with $J^\infty(\mathcal{F})$. A local diff. form on $J^\infty(\mathcal{F})$ – a form on $J^N(\mathcal{F})$ pulled back to $J^\infty(\mathcal{F})$.

J^∞ is equipped with the total derivative

$$\partial_a^T = \frac{\partial}{\partial x^a} + \phi_a^i \frac{\partial}{\partial \phi^i} + \phi_{ab}^i \frac{\partial}{\partial \phi_b^i} + \dots$$

For a given section $\phi^i = s^i(x)$ and local function $f[\phi]$

$$(\partial_a^T f)\Big|_{\phi=s,\phi_a=\partial_a s,\dots} = \frac{\partial}{\partial x^a}(f\Big|_{\phi=s,\phi_a=\partial_a s,\phi_{ab}=\partial_a\partial_b s,\dots})$$

Space time differentials dx^a. Horizontal differential

$$d_{\mathsf{h}} \equiv dx^a \partial_a^T \,, \qquad d_{\mathsf{h}}^2 = 0\,.$$

Differential forms:

$$\alpha = \alpha(x, dx, \phi, \phi_a, \ldots)_{I_1 \ldots I_k} d_{\mathsf{v}}\phi^{I_1} \ldots d_{\mathsf{v}}\phi^{I_k}\,, \qquad \phi^I = \phi^i_{a_1 \ldots a_m}$$

Vertical differential:

$$d_{\mathsf{v}} \equiv d - d_{\mathsf{h}} = d_{\mathsf{v}}\phi^I \frac{\partial}{\partial \phi^I}$$

Variational bicomplex (*Vinogradov's* C-spectral sequence):

$$d_{\mathsf{v}}^2 = 0, \qquad d_{\mathsf{v}} d_{\mathsf{h}} + d_{\mathsf{h}} d_{\mathsf{v}} = 0\,, \qquad d_{\mathsf{h}}^2 = 0$$

Bidegree (l, p). On the jet space $H^{>0}(d_{\mathsf{v}}) = 0 = H^{<n}(d_{\mathsf{h}})$ (unless global geometry!). $H^n(d_{\mathsf{h}})$ = local functionals

A system of partially differential equations (PDE) is a collection of local functions on $J^\infty(\mathcal{F})$

$$E_\alpha[\phi, x] \,.$$

The equation manifold (stationary surface): $\mathcal{E} \subset J^\infty(\mathcal{F})$ singled out by:

$$\partial^T_{a_1} \dots \partial^T_{a_l} E_\alpha = 0 \,, \qquad l = 0, 1, 2, \dots$$

understood as the algebraic equations in $J^\infty(\mathcal{F})$. It is usually assumed that x^a, ϕ^i are not constrained, e.g. $\mathcal{E}$ is a bundle over the space-time.

∂^T_a are tangent to $\mathcal{E}$ and hence restricts to $\mathcal{E}$. So do the differentials d_{h} and d_{v}. $\partial^T_a|_{\mathcal{E}}$ determine a dim-n integrable distribution (Cartan distribution).

Definition: *[Vinogradov] A PDE is a manifold $\mathcal{E}$ equipped with an integrable distribution.*

In addition one typically assumes regularity, constant rank, and that $\mathcal{E}$ is a bundle over the spacetime. Use notation $(\mathcal{E}, d_{\mathsf{h}})$.

It is clear when PDEs are to be considered equivalent.

Differential forms on $\mathcal{E}$ form the variational bicomplex of $\mathcal{E}$. Note that in general $H^k(d_{\mathsf{h}}) \neq 0$ for $k < n$.

Scalar field Example:

Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)\,, \qquad \partial_a\partial^a\phi + \frac{\partial V}{\partial\phi} = 0\,.$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \ldots$. Already ϕ_{ab} are not independent. One can e.g. take $\phi_{abc\ldots}$ traceless. The d_{h}-differential on $\mathcal{E}$ reads as

$$d_{\mathsf{h}}x^a = dx^a\,, \quad d_{\mathsf{h}}\phi = dx^a\phi_a,\,, \quad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial\phi})\,, \quad \ldots$$

So if the system is nonlinear, i.e. $\frac{\partial V}{\partial\phi}$ nonlinear in ϕ, d_{h} is also nonlinear.

Intrinsic (unfolded) realization

Given PDE $(\mathcal{E}, d_{\mathsf{h}})$ defined intrinsically one can always find a jet space $\mathcal{J}$ such that $(\mathcal{E}, Q)$ can be realized as a stationary surface of some $E_\alpha[u, x]$.

There is an intrinsic way to realize $(\mathcal{E}, d_{\mathsf{h}})$ explicitly. If x^a, ψ^A coordinates on $\mathcal{E}$ (e.g. $\psi^A = \{\phi, \phi_a, \phi_{ab}, \ldots\}$) promote ψ^A to fields $\psi^A(x) =$ of a new theory and subject them to EOM's

$\boldsymbol{d}(\psi^A(x)) = (d_{\mathsf{h}}\psi^A)(x)$ components: $\frac{\partial}{\partial x^a}\psi^A(x) = (\partial^T_a \psi^A)(x)$

Proposition: *The original PDE $(\mathcal{E}, d_{\mathsf{h}})$ is equivalent to $\boldsymbol{d}\psi^A = d_{\mathsf{h}}\psi^A$*

Comments:

- Version of the unfolded formulation (though only zero forms). Unfolded form of gauge systems involves gauge form-fields. *Vasiliev, 1987,. . .*

- Generalized version of the Proposition involving gauge forms and BRST extension was formulated and proved using BRST technique and Koszule-Tate differential. *Barnich, M.G.,Semikhatov, Tipunin 2004, Barnich, M.G 2010*

New jet-space

Because $\mathcal{E}$ is a bundle over spacetime, take $\mathcal{J}^{new} \equiv J^{\infty}(\mathcal{E})$. More precisely, if x^a, dx^a, ψ^A are coordinates on $\mathcal{E}$ then

$$x^a, \quad dx^a, \quad \psi^A, \quad \psi^A_b, \quad \psi^A_{bc}, \quad \psi^A_{bcd}, \quad \ldots$$

are coordinates on $\mathcal{J}^{new}$.

New jet space is equipped with its own horizontal differential:

$$D_{\mathsf{h}} = dx^a \Big(\frac{\partial}{\partial x^a} + \psi^A_a \frac{\partial}{\partial \psi^A} + \psi^A_{ab} \frac{\partial}{\partial \psi^A_b} + \ldots\Big)$$

"Old" differential d_{h} on $\mathcal{E}$ extends to $\mathcal{J}^{new}$ by $[D_H, Q] = 0$. In the new jet space $\mathcal{J}^{new}$ consider the following PDE

$$D_{\mathsf{h}} \psi^A = d_{\mathsf{h}} \psi^A$$

In this form the new PDE is manifestly isomorphic to $(\mathcal{E}, Q)$ (because manifolds are isomorphic and horizontal differentials are equal by construction)

AKSZ form and reparametrization invariance

Consider dx^a as ghosts ξ^a, change notation $x^a \to z^a$ and extend $\mathcal{E}$ into a supermanifold with coordinates $\Psi^A = \{z^a, \xi^a, \phi^i, \phi^i_a, \phi^i_{ab}, \ldots\}$. It is a Q-manifold:

$$Q = -d_{\mathsf{h}} = -\xi^A \partial^T_a$$

Take $\mathcal{X} = T[1]X$ with coordinates x^μ, θ^μ and consider AKSZ model with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}, Q)$.

Note that now z^a is promoted to a field $z^a(x)$ and ξ^a to $e^a_\mu(x) dx^\mu$.

In fact: we are dealing with parametrized version.
$z^a(x)$ – space-time coordinates understood as fields
$e^a_\mu(x)$ – frame field components.

Gauge transf. for z^a: $\delta z^a = \xi^a$. Q is the BRST differential implementing repatametrization invariance.
Gauge condition $z^a = \delta^a_\mu x^\mu$ give un-parametrized version:

$$\boldsymbol{d}\Psi^A + Q^A(\Psi) = 0 \quad \Rightarrow \quad \boldsymbol{d}\Psi^A(x) - \theta^a(\partial^T_a \Psi^A)(x,\theta) = 0$$

Recall: ∂^T_a – total derivative (vector field in the target).

Extension to gauge theories

If $(\mathcal{E}, d_{\mathsf{h}})$ has gauge symmeries there are parameters ϵ^{α} which are arbitary space time functions. Promote them to ghost variables c^{α} and consider the extension $\bar{\mathcal{E}}$ of $\mathcal{E}$ by the jet-space for c^{α}:

$$C^I = \{c^{\alpha}, \quad c^{\alpha}_{a}, \quad c^{\alpha}_{ab}, \quad \ldots\}$$

The gauge symmetry is encoded in vector field γ satisfying

$$[d_{\mathsf{h}}, \gamma] = 0\,, \qquad \gamma^2 = 0\,, \qquad \mathsf{gh}(\gamma) = 1$$

It can be written as

$$\gamma = C^I R^A_I(\psi)\frac{\partial}{\partial\psi^A} - \frac{1}{2}C^I C^J U^K_{IJ}(\psi)\frac{\partial}{\partial C^K}$$

Vector fields R_I determine an integrable distribution on $\mathcal{E}$ (gauge-distribution), compatible with Cartan distribution.

AKSZ form

Consider AKSZ model with source $(\mathcal{X}, \boldsymbol{d})$ and the target $(\bar{\mathcal{E}}, Q)$, where

$$Q = -d_{\mathsf{h}} + \gamma$$

Total differential familiar in the local BRST cohomology *Stora, 1983, Batnich, Brandt, Henneaux 1993,. . .* .

Equivalent to the parametrized version of gauge system. In addition to $e^a_\mu(x)dx^\mu$ new 1-form fields $A^I_\mu(x)dx^\mu$ associated to C^I.

The equivalence was proved using *Barnich, M.G. 2010*

$$\widetilde{s} = -d_{\mathsf{h}} + \delta + \gamma + \ldots$$

where δ is the Koszule–Tate differential of the stationary surface.

New feature: contractibe pairs for Q: if by local invertible change of coordinates:

$$Qw^a = v^a, \qquad Q\psi^\alpha = Q^\alpha(\psi)$$

then w^a, v^a are contractibe pairs. Their elimination results in the reduced Q-manifold $(Q, \widetilde{\mathcal{E}})$. Eliminating all such trivial pairs one arrives at "minimal" Q-manifold associated the gauge system *Brandt, 1996*

The manifold of generalized connections and tensor fields.

For the AKSZ model trivial pairs give rise to generalized auxiliary fields. Lagrangian: *Dresse, Grégoire, Henneaux, 1990*
EOM: *Barnich, M.G., Semikhatov, Tipunin, 2004*
Their elimination is an equivalence of the respective AKSZ models.

Example of Einstein gravity

For diffeomorphism-invariant theory parameterization brings nothing. It follows x^a, ξ^a can be eliminated together with d_{h}, giving $Q = \gamma$.

After elimination the contractible pairs of Q manifold $\tilde{\mathcal{E}}$:

$$e^a, \quad \omega^{ab}, \quad W^{cd}_{ab}, \quad W^{cd}_{ab|e}, \quad W^{cd}_{ab|e\ldots}$$

– ghosts associated to frame field and spin connection and Weyl tensor and its independent covariant derivatives.

$$Qe^a = \omega^a{}_c\, e^c\,, \qquad Q\omega^{ab} = \omega^a{}_c\, \omega^{cb} + \frac{1}{2} e^c e^d W^{ab}_{cd}\,,$$

$$QW = eW + \omega W + \ldots$$

Minimal BRST complex (Q-manifold) for gravity.
Gives minimal AKSZ formulation (unfolded formulation).

Variational (Lagrangian) equations

Let us get back to equations $E_i[\phi, x] = 0$ on the jet space $J^\infty(\mathcal{F})$. These are said variational (Lagrangian) if

$$\mathcal{E}_i = \frac{\delta^{EL} L}{\delta\phi^i}\,, \qquad\qquad \frac{\delta^{EL} F[u,x]}{\delta\phi^i} \equiv \frac{\partial F}{\partial\phi^i} - \partial^T_a \frac{\partial F}{\partial\phi^i_a} + \partial^T_a \partial^T_b \frac{\partial F}{\partial\phi^i_{ab}} - \ldots$$

for some local function $L = L[\phi, x]$. It is convenient to work in terms of Lagrangian density $\mathcal{L} = (dx)^n\, L$.

Here and below

$$(dx)^n = dx^1 \ldots dx^n\,, \qquad\qquad (dx)^{n-1}_a = \frac{1}{(n-1)!}\epsilon_{ab_2\ldots b_n} dx^{b_1} \ldots dx^{b_n}$$

The notion of Lagrangian is explicitly based on the realization of the equation $(\mathcal{E}, d_{\mathsf{h}})$ in terms of a jet space $\mathcal{J}$. For instance it's possible that $\mathcal{E} \subset \mathcal{J}$ is variational while $\mathcal{E} \subset \mathcal{J}'$ is not. Naive invariant object – the restriction of $\mathcal{L}$ to $\mathcal{E}$, does not make much sense.

Presymplectic structure

It is well-known that $\mathcal{L} = (dx)^n L[x, \phi]$ induces an invariant object on *Crnkovic, Witten, 1987, Hydon 2005,...*

$d_{\mathsf{v}}\mathcal{L} = d\phi^i\, E_i\, (dx)^n - d_{\mathsf{h}}\widehat{\chi}\,,$ components: $\frac{\delta^{EL} L}{\delta \phi^i} = \frac{\partial L}{\partial \phi^i} + \partial_a^T (\widehat{\chi}_i^a)$

for some 1 form $\widehat{\chi} = \widehat{\chi}_i d_{\mathsf{v}}\phi^i + \widehat{\chi}_{ia} d_{\mathsf{v}}\phi^i_a + \ldots$ of degree $n-1$, called presymplectic potential. For $\chi = \widehat{\chi}|_{\mathcal{E}}$ we have

$$d_{\mathsf{h}}\sigma = 0\,, \qquad \sigma = d\chi$$

So we have conserved closed 2-form on $\mathcal{E}$. It's called canonical presymplectic structure.

As an example consider $L(\phi, \phi_a, \phi_{ab})$. One finds:

$$\chi = \left((\frac{\partial L}{\partial \phi^a} - \partial_b^T \frac{\partial L}{\partial \phi_{ab}}) d_{\mathsf{V}}\phi + \frac{\partial L}{\partial \phi_{ab}} d_{\mathsf{V}}\phi_b \right) \Big|_{\mathcal{E}} (dx)_a^{n-1}$$

In particular, for a scalar field with $L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$

$$\chi = \phi^a d_{\mathsf{V}}\phi \; (dx)_a^{n-1} \;\; , \qquad \sigma = d_{\mathsf{V}}\phi^a \, d_{\mathsf{V}}\phi \; (dx)_a^{n-1}$$

More generally:

Definition: *A 2-form σ of degree $n-1$ on $(\mathcal{E}, d_{\mathsf{h}})$ is called compatible presymplectic structure if $d_{\mathsf{h}}\sigma = 0\,, d\sigma = 0$.*

. . . ,Khavkin 2012, Alkalaev, M.G. 2013

Such form in general can be considered irrespective of any realization in terms of jet-space and/or Lagrangian.

Symmetries and conservation laws

A well-known fact: both symmetries and conservation laws can be defined in terms of the equation manifold $(\mathcal{E}, d_{\mathsf{h}})$.

Recall: a vector field $\widehat{V}$ on $\mathcal{J}$ is a symmetry if it is evolutionary i.e. $[d_{\mathsf{h}}, \widehat{V}] = 0$ and tangent to $\mathcal{E} \subset \mathcal{J}$.

Intrinsic terms: a vector field V on $(\mathcal{E}, d_{\mathsf{h}})$ satisfying is called symmetry if $[d_{\mathsf{h}}, V] = 0$ (typically one also requires $V x^a = 0$).

If $\mathcal{E} \subset \mathcal{J}$ is variational then variational symmetries restricted to $\mathcal{E}$ satisfy in addition

$$L_V \sigma = d_{\mathsf{h}} d_{\mathsf{v}} \alpha$$

for some $(n-2, 1)$-form α.

Conservation law (conserved curent) is a degree $n-1$ 0-form K on $\mathcal{E}$ such that $d_{\mathsf{h}}K = 0$. K of the form $K = d_{\mathsf{h}}M$ is trivial.

Any compatible presymplectic structure determines a map from symmetries to conserved currents according to

$$dK = i_V\sigma - d_{\mathsf{h}}\alpha\,,$$

Note: $di_V\sigma = L_V\sigma = 0$. Trivial symmetries are mapped to trivial conserved currents. In the Lagrangian case this is usual Noether theorem. General case was also discussed recently *Sharapov 2016*.

It is different from the Poisson (BV antibracket) bracket map from conservation laws to symmetries. The degenerate version of the bracket is known as Hamilton/Lagrange structure *Kersten, Krasilshchik, Verbovetsky*
Kaparulin, Lyakhovich, Sharapov

Suppose that $(\mathcal{E}, d_{\mathsf{h}}, \sigma)$ is realized as $\mathcal{E} \subset J^\infty(\mathcal{F})$. Then σ determines a Lagrangian form $\mathcal{L}$ on $J^\infty(\mathcal{F})$ such that the EL equations derived from $\mathcal{L}$ are in general consequences of those defining $\mathcal{E}$. *Khavkine 2012, based on earlier: Bridges, Hydon, Lawson 2009, Hydon 2005*

More precisely, if $\mathcal{E}'$ is an equation manifold defined by $\mathcal{L}$ then $\mathcal{E} \subset \mathcal{E}'$. Even if σ is canonical (derived from a Lagrangian) there is no guarantee that constructed $\mathcal{L}$ is equivalent to the starting point Lagrangian.

Intrinsic Lagrangian

Given an equation manifold $(\mathcal{E}, d_{\mathsf{h}}, \sigma)$ equipped with the compatible presymplectic structure one can construct a natural Lagrangian in terms of the $\mathcal{E}$-valued fields.

First: define covariant Hamiltonian (better BRST charge) which is a conserved current associated to d_{h} seen as a symmetry of $\mathcal{E}$. Degree n function $\mathcal{H}$ on $\mathcal{E}$ defined by

$$d_{\mathsf{h}}\mathcal{H} = i_{d_{\mathsf{h}}}\sigma\,, \qquad \text{components:} \quad \frac{\partial}{\partial\psi^A}\mathcal{H} = \sigma_{AB} d_{\mathsf{h}}\psi^B$$

If σ originates from $\mathcal{L}$: $\mathcal{H} = \chi_A d_{\mathsf{h}}\psi^A - \mathcal{L}|_{\mathcal{E}}$ E.g. in the simple case where $\mathcal{L} = (dx)^n L(\phi, \phi_a)$

$$\chi = \left(\frac{\partial L}{\partial\phi_a} d_{\mathsf{v}}\phi\right)\Big|_{\mathcal{E}} (dx)^{n-1}_a\,, \qquad \mathcal{H} = \left(\frac{\partial L}{\partial\phi_a}\phi_a - L\right)\Big|_{\mathcal{E}} (dx)^n$$

cf. de Donder-Weyl covariant Hamiltonian. The intrinsic Lagrangian

More invariant language:
It follows from $d\sigma = 0$ that

$$\sigma = d(\chi + l) \qquad l \text{ is an } (n, 0)\text{-form} \tag{1}$$

For canonical σ $l = \mathcal{L}|_{\mathcal{E}}$.

New jet-space $J^\infty(\mathcal{E})$. Intrinsic Lagrangian

$$\mathcal{L}^C = \mathbf{H}(\pi^*(\chi + l))\,, \tag{2}$$

where $\pi : J^\infty(M) \to M$. $\mathbf{H}$ denotes horizontalization map:

$$\mathbf{H}(dx^a) = dx^a, \qquad \mathbf{H}(d\psi^A_{ab\ldots}) = D_{\mathsf{h}}\psi^A_{ab\ldots}\,. \tag{3}$$

Switching to the language of Q-manifolds:

$$d\mathcal{H} = i_Q \sigma$$

The respective action can be seen as presymplectic generalization

Alkalaev, M.G. 2013

$$S^C = \int \Big(\chi_A \boldsymbol{d}\psi^A(x) - \mathcal{H}(\psi(x))\Big)$$

of AKSZ action. Its equations of motion read as

$$\sigma_{AB}(\boldsymbol{d}\psi^B(x) - Q^B(x)) = 0\,,$$

and hence are consequences of the original $\boldsymbol{d}\psi^B - Q^B = 0$.

For a local theory $\mathcal{L}^C$ does not depend on most of the fields ψ^A. These can be treated as pure-gauge variables with algebraic (shift) gauge transformations. With this interpretation and under certain assumptions we can prove that starting point $\mathcal{L}$ and $\mathcal{L}^C$ are equivalent for a wide class of systems (but not all).

Examples

Scalar field: Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \ldots$. take $\phi_{abc\ldots}$ traceless.
The d_{h} differential

$$d_{\mathsf{h}} x^a = dx^a\,, \qquad d_{\mathsf{h}}\phi = dx^a\phi_a,\,, \qquad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})$$

The presymplectic potential and 2-form:

$$\chi = \left((\frac{\partial L}{\partial \phi^a} - \partial^T_c \frac{\partial L}{\partial \phi_{ca}}) d_{\mathsf{v}}\phi \right) \Big|_{\mathcal{E}} (dx)^{n-1}_a = (dx)^{n-1}_a \phi^a d_{\mathsf{v}}\phi\,,$$

The Hamiltonian obtained from $d\mathcal{H} - i_Q\sigma = 0$:

$$\mathcal{H} = (dx)^n(\phi_a\phi^a - L|_{\mathcal{E}}) = \frac{1}{2}\phi^a\phi_a + V(\phi)$$

The intrinsic Larangian: *Schwinger*

$$\mathcal{L}^c = (dx)^n \left(\phi^a(\partial_a\phi - \frac{1}{2}\phi_a) - V(\phi) \right)$$

Polywave equation

The simplest genuine higher derivative example is $L = \frac{1}{2}\Box\phi\Box\phi = \frac{1}{2}\phi_{aa}\phi_{bb}$
(here and below $\phi_{aa} = \eta^{ab}\phi_{ab}$). Presymplectic potential:

$$\chi = (-\phi_{acc} d_{\mathsf{V}}\phi + \phi_{cc} d_{\mathsf{V}}\phi_a)(dx)^{n-1}_a$$

Hamiltonian

$$\mathcal{H} = (dx)^n(-\phi_{acc}\phi_a + \frac{1}{2}\phi_{cc}\phi_{aa})\,.$$

The intrinsic action takes the form

$$S^C = \int d^n x(-\phi_{acc}(\partial_a\phi - \phi_a) + \phi_{cc}\partial_a\phi_a - \frac{1}{2}\phi_{aa}\phi_{cc})\,.$$

Note that the action depends on only the following variables $\phi, \phi_a, \phi_{aa}, \phi_{acc}$ but NOT on the traceless component of ϕ_{ab} and ϕ_{abc}.

It is equivalent to $\int \phi_{aa}\phi_{cc}$. Indeed, varying ϕ_a and ϕ_{acc} gives $\phi_a = \partial_a\phi$ and $\phi_{acc} = \partial_a\phi_{cc}$ resulting in

$$\int d^n x(\phi_{cc}\partial_a\partial_a\phi - \frac{1}{2}\phi_{aa}\phi_{cc})$$

YM theory

The YM field is A^a taking values in a Lie algebra $\mathfrak{g}$ equipped with an invariant inner product $\langle , \rangle$. We will use notation $A^a_{b_1 \dots b_l}$ for $\partial^T_{b_1} \dots \partial^T_{b_l} A^a$. The Lagrangian:

$$L = \frac{1}{4}\langle F_{ab}, F_{ab}\rangle \,, \qquad F_{ab} := A^b_a - A^a_b + [A^a, A^b] \,.$$

Coordinates on $\mathcal{E}$:

$$x^a, A^a, F_{ab}, S_{ab} := A^b_a + A^a_b, A^a_{bc}, \dots$$

The one form χ is given by

$$\chi = \frac{\partial L}{\partial A^b_a} dA^b (dx)^{n-1}_a = \langle F_{ab}, dA^b\rangle (dx)^{n-1}_a$$

The Hamiltonian

$$\mathcal{H} = (\frac{\partial L}{\partial A^b_a} A^b_a - \frac{1}{4}\langle F_{ab}, F_{ab}\rangle)(dx) = \frac{1}{2}\langle F_{ab}, \frac{1}{2}F_{ab} - [A^a, A^b]\rangle$$

The intrinsic action

$$\int \frac{1}{2}\langle F_{ab}, \partial_a A^b - \partial_b A^a \rangle - \frac{1}{2}\langle F_{ab}, \frac{1}{2}F_{ab} - [A^a, A^b]\rangle =$$

$$\int \frac{1}{2}\langle F_{ab}, \partial_a A^b - \partial_b A^a + [A^a, A^b] - \frac{1}{2}F_{ab}\rangle$$

equivalent to the starting point action through the elimination of F_{ab} by its own equations of motion.

Well-known first-order action for YM.

Example of gravity: frame like Lagrangian

Recall: reducwd Q-manifold $\tilde{\mathcal{E}}$

$$e^a, \quad \omega^{ab}, \quad W^{cd}_{ab}, \quad W^{cd}_{ab|e}, \quad W^{cd}_{ab|e\ldots}$$

– ghosts to which frame field and spin connection are associated and Weyl tensor and its covariant derivatives.

$$Qe^a = \omega^a{}_c\, e^c\,, \qquad Q\omega^{ab} = \omega^a{}_c\, \omega^{cb} + e^c e^d W^{ab}_{cd}\,, \qquad \ldots\,,$$

Presymplectic potential χ and form *Alkalaev, M.G. 2013*

$$\chi = \frac{1}{2}\epsilon_{abcd} d\omega^{ab} e^c e^d\,, \qquad \sigma = d\omega^{ab} de^c \epsilon_{abcd} e^d$$

Hamiltonian (term with Weyl tensor vanishes)

$$\mathcal{H} = Q^A \chi_A = \frac{1}{2}\omega^a_c \omega^{cb}\epsilon_{abcd} e^c e^d$$

Intrinsic action (frame-like GR action):

$$S^C = \int \chi_A(\boldsymbol{d}\psi^A + Q^A) = \int (\boldsymbol{d}\omega^{ab} + \omega^a{}_c \omega^{cb})\epsilon_{abcd} e^c e^d$$

Conclusions

- A Lagrangian system can be defined in terms of its equation manifold $\mathcal{E}$ without refereeing to any particular realization of $\mathcal{E}$ in one or another set of fields and choice of the Lagrangian. While the structure of the equation is encoded in the differential Q the Lagrangian is encoded in the compatible presymplectic structure σ.

- In particular, when looking for a Lagrangian for an equation $\mathcal{E}$ it is enough to study compatible presymplectic structures on $\mathcal{E}$. No need to study possible explicit realizations of $\mathcal{E}$.

- Easy to see whether Lagrangian systems are equivalent or not.

- BRST extension to manifestly gauge systems. Intrinsic

Lagrangian = Frame-like Lagrangian.

- The presymplectic form can be seen to originate from the odd symplectic form of the Batalin-Vilkovisky formalism.

Parent Lagrangian

One way to understand where do the structure of the intrinsic Lagrangian comes from is to consider "parent" action for $L = L(\phi, \phi_a, \phi_{ab})$:

$$S^P = \int \left(L(\phi, \phi_a, \phi_{ab}) + \pi^a(\partial_a \phi - \phi_a) + \pi^{ac}(\partial_a \phi_c - \phi_{ac}) + \ldots \right) .$$

Its equations of motion read as

$$\frac{\partial L}{\partial \phi} - \partial_a \pi^a = 0 \,,$$

$$\pi^a - \frac{\partial L}{\partial \phi_a} + \partial_c \pi^{ca} = 0 \,, \qquad \pi^{ab} - \frac{\partial L}{\partial \phi_{ab}} = 0 \,, \qquad \pi^{ab\ldots} = 0$$

$$\phi_a = \partial_a \phi \,, \qquad \phi_{ab} = \partial_{(a} \phi_{b)} \,, \qquad \ldots$$

Using the last line the derivatives in the first two lines can be replaced with the total derivatives. Using the second line the first equation becomes EL

$$\frac{\partial L}{\partial \phi} - \partial^T_a \frac{\partial L}{\partial \phi_a} + \partial^T_c \partial^T_a \frac{\partial L}{\partial \phi_{ca}} = 0 \,.$$

Introduce 1-form of degree $n-1$:

$$\bar{\chi} = (dx)_a^{n-1}(\pi^a d\phi + \pi^{ab} d\phi_b + \ldots)$$

"parent" Hamiltonian

$$\bar{\mathcal{H}} = (\pi^a \phi_a + \pi^{ab}\phi_{ab} + \ldots - L(\phi, \phi_a, \phi_{ab}))(dx)^n\,.$$

The parent action can be written as

$$S^P = \int (\bar{\chi}_A \boldsymbol{d}\Psi^A - \bar{\mathcal{H}})\,,$$

where Ψ^A stand for all the coordinates $\phi, \phi_{\ldots}$, $\pi^{\cdots}$.

Consider the following submanifold of the space of $x^a, dx^a, \phi, \pi^{\cdots}, \phi_{\dots}$

$$\pi^a - \frac{\partial L}{\partial \phi^a} + \partial_c^T \frac{\partial L}{\partial \phi_{ca}} = 0\,, \qquad \pi^{ab} - \frac{\partial L}{\partial \phi_{ab}} = 0\,, \quad \pi^{ab\dots} = 0\,,$$
$$\partial_{a_1}^T \dots \partial_{a_k}^T (EL) = 0\,,$$

These are consequences of the parent action equations of motion.

The submanifold they single out is $\mathcal{E}$ (equation manifold of L).

$$\chi = \bar{\chi}|_{\mathcal{E}} \qquad \text{Presymplectic potential for } L$$

One can show

$$i_Q d\sigma = d\mathcal{H}\,, \qquad \mathcal{H} = \bar{\mathcal{H}}|_{\mathcal{E}}\,, \qquad \sigma = d\chi$$

Furthermore, χ and $\mathcal{H}$ determine the intrinsic action

$$S^C[\psi] = \int \Big(\chi_A(x, dx^a, \psi)\boldsymbol{d}\psi^A - \mathcal{H}(x, dx^a, \psi)\Big) ,$$

where x^a, ψ^A are coordinates on $\mathcal{E}$. This can be independently arrived at by eliminating auxiliary fields starting from the parent action.