C^*-ALGEBRAS OF PENROSE HYPERBOLIC TILINGS

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Some definitions

An **euclidian tiling** T of $\mathbb{R}^d$ is a collection of non empty compact sets called **tiles** meeting full edge to full edge such that their union is the full space $\mathbb{R}^d$. Let N be a group action on the tiling T (ex: $\mathbb{R}^d$ action by translations). One call the tiling N**-finite** if there exists a finite sequence of tiles $(p_i)_{1\leq i\leq n}$ of T s.t. $\forall t \in T\ \exists! i \in \{1,\cdots,n\}\ \exists! h \in N$ st $t = h\cdot p_i$. The p_i's are called **prototiles** of the tiling.

One said that a tiling :

(i) is **repetitive** if every pattern (ie local configuration) repeats infinitely many times;

(ii) has **finite local complexity** if for every ball of finite diameter there's a finite number of possible patterns;

(iii) is **aperiodic** if the tiling is unstable for the translation action.

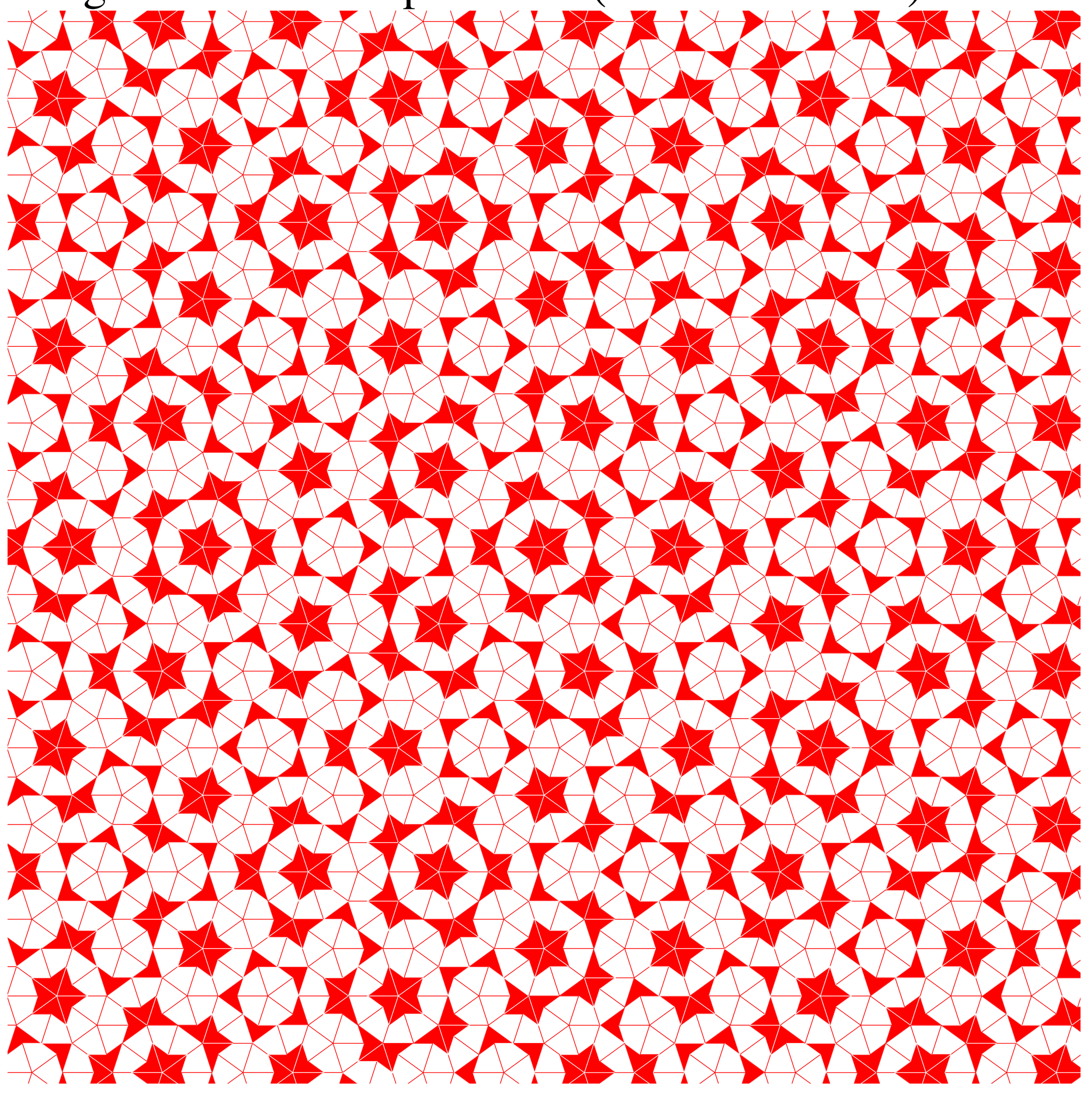
Classical Penrose tiling

A **Penrose tiling** named after Sir Roger PENROSE, is a FLC, aperiodic and repetitive tiling of $\mathbb{R}^d$.

They are linked with quasicristals highlighted by Dan SCHETCHMANN in 1982-1984 which awarded him the Noble Prize in Chemistry in 2011.

Penrose's kites and darts tiling

Also called Penrose P2, the kites and darts is a Penrose tiling build with two prototiles (a kite and a dart).



source:http://tex.stackexchange.com/questions/61437/penrose-tiling-in-tikz

Substitution tilings

One of the main tool to construct tilings and especially Penrose tilings type are substitution.

Let $\{p_i, 1\leq i\leq n\}$ prototiles. A **substitution** is the association of an expansion map Q and a decomposition application σ which tells how $Q(p_k)$ is splitted onto original p_i's for each $1\leq k\leq n$. One can restrict in $\mathbb{R}$ this process to an alphabet $\mathcal{A}$ and an application σ which associate to each letter of $\mathcal{A}$ finite sequence of letters.

A substitution is then the limit of the process expansion-splitting.

Fibonacci substitution

The Fibonacci substitution is given by a 2 letters alphabet $\mathcal{A} = \{a, b\}$ and the application σ defined by $\sigma(a) = ab$ and $\sigma(b) = a$.

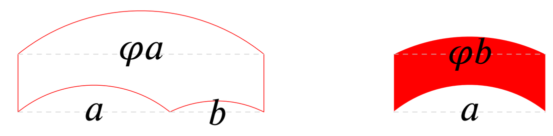
The geometrical representation is the given by two intervals a (white) and b (red) of respective lenghts $\varphi = \frac{1+\sqrt{5}}{2}$ and 1. Below is the first few tiles that compose the tiling.

Hyperbolic tilings and substitutions

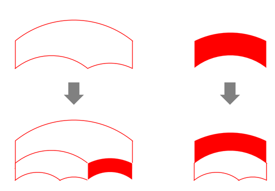
We mean by hyperbolic tilings, tilings of Poincaré half-plane $\mathbb{H}_2$ with the usual metric. One can use substitutions to build such tilings. To obtain the prototiles you simply relate accordingly substitution's prototiles to their images by the substitution (see image on the right).

One just needs to be careful with the definition of the tiling. The easiest way is link with the notion of "forcing borders" and can sum up via n-supertiles eg a tile and its neighbors within n tiles radius.

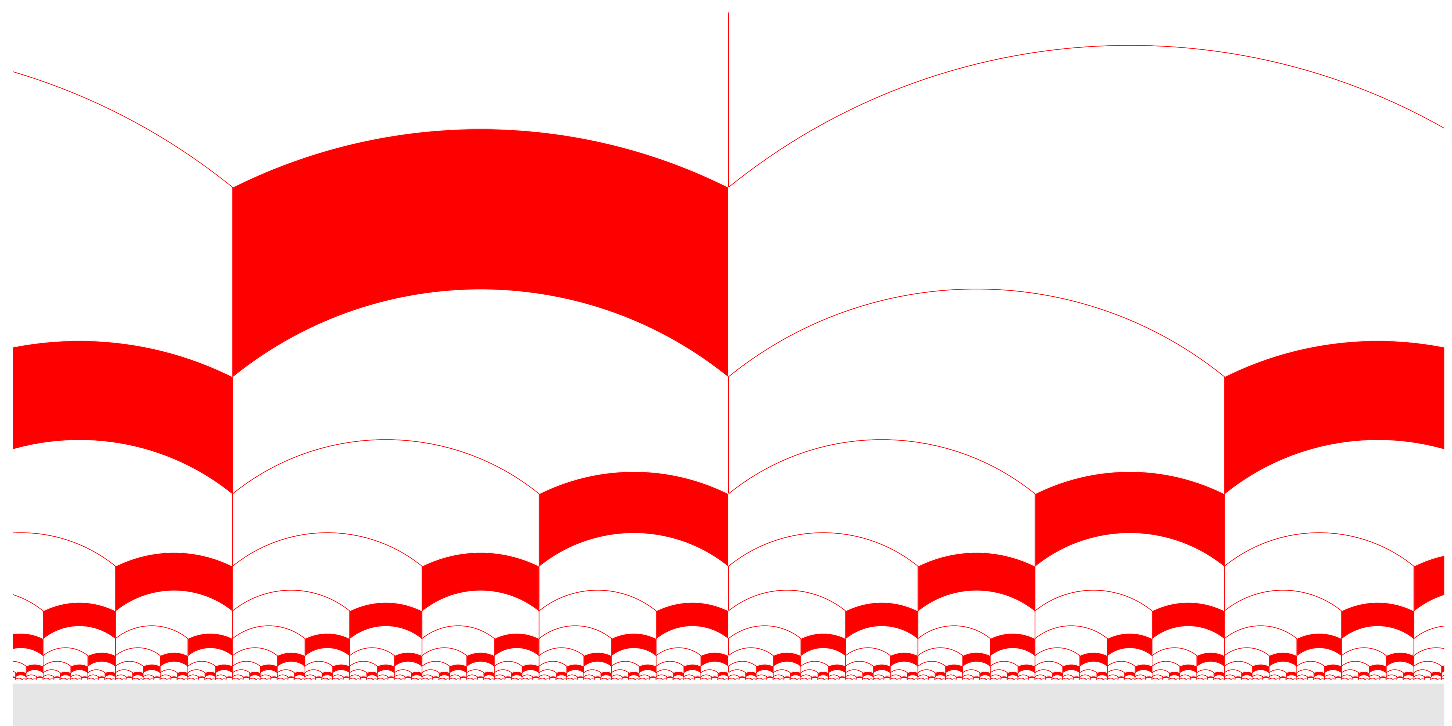
Hyperbolic Fibonacci prototiles



Construction rules



Hyperbolic Fibonacci tiling



Objects associated to the hyperbolic tiling ([2],[1])

We have two groups actions on the tiling P: $G = \{\mathbb{H}_2 \to \mathbb{H}_2, z \mapsto az+b, a>0, b\in\mathbb{R}\}$ et $N = \{\mathbb{H}_2 \to \mathbb{H}_2, z \mapsto z+t, t\in\mathbb{R}\}$. One can define a metric δ sur $G\cdot P$, which leads to two hulls of the tiling P, $X_P^N = \overline{N\cdot P}^{\delta}$ et $X_P^G = \overline{G\cdot P}^{\delta}$.

One can easily remark that $(z \to \varphi^2 z)\cdot P = P$, so to make the action of G free, one needs the use of coloration by r colors. A coloration is a sequence $c \in \{1,\cdots,r\}^{\mathbb{Z}}$, we chose to be aperiodic for the usual shift, then one defines a colorated tiling $P(c)$ where each line is colored by a c_i's. As before we have a colored hull $X_{P(c)}^G$ and the G-action on $P(c)$ is free.

C^*-algebra, K-theory [4]

While G acts $X_{P(c)}$, one can form a cross-product groupoid $X_{P(c)}^G \rtimes G$ with a Haar system λ coming from the Haar measure on G. Using J. Renault's (see [3]), we can construct the reduced C^*-algebra, $C_r^*(X_{P(c)} \rtimes G, \lambda)$ which in our case is nothing but $C(X_{P(c)}^G) \rtimes G$.

One can then try to compute the K-theory of this algebra, namely the two K-groups.

Some facts on the hulls

The hulls are compact metric sets. One can then show that the dynamical system $(X_{P(c)}^G, G)$ is topologically conjugate with $((X_P^N \times \mathcal{Z}_c \times \mathbb{R}_+^*)/\mathcal{R}, G)$ where $\mathcal{Z}_c$ is a Cantor set associated to the coloration and $\mathcal{R}$ only depends on parameters of the substitution.

Some results on associated C^*-algebras

Let $\mathcal{G} = (X_P^N \times \mathcal{Z}_c) \rtimes \mathbb{R}$ be the groupoid associated to the action $t\cdot(P', c') = (P'+t, c'), t\in\mathbb{R}, (P', c') \in X_P^N \times \mathcal{Z}_c$. By carefully choosing an automorphism α_c on $\mathcal{G}$, one can then show the existence of a unique $\mathbb{R}$-equivariant isomorphism of groupoids $\Lambda_{\alpha_c} : \mathcal{G}_{\alpha_c} = (\mathcal{G}\times\mathbb{R})/\alpha_c \to X_{P(c)}^G \rtimes \mathbb{R}$, inducing an isomorphism $\mathbb{R}$-equivariant of C^*-algebras between $C(X_{P(c)}^G) \rtimes \mathbb{R}$ and $C_r^*(\mathcal{G}_{\alpha_c}, \lambda_{\alpha_c})$.

We then deduce a Morita equivalence between $C(X_{P(c)}^G) \rtimes G$ and $C_r^*(\mathcal{G}) \rtimes_{\tilde{\alpha_c}} \mathbb{Z}$, which leads in K-theory to an isomorphism $K_*(C(X_{P(c)}^G) \rtimes G) \xrightarrow{\cong} K_*(C_r^*(\mathcal{G}) \rtimes_{\tilde{\alpha_c}} \mathbb{Z})$.

A further extend to the decoposition leads in fact to

$$K_*\left(\left(C(\mathcal{Z}_c) \otimes \mathcal{K}(\mathrm{L}^2([0,1]) \otimes C(\Xi) \rtimes \mathbb{Z}\right) \rtimes_{\tilde{\alpha_c}} \mathbb{Z}\right),$$

where Ξ is a tranversal of the tiling space.

Further computation

To compute the K-theory of $C(X_P^N) \rtimes \mathbb{R}$, we can use the Anderson-Putnam approximation of the hull X_P^N (see [1]) ie. $X_P^N = \varprojlim(\Gamma_1, \sigma^*)$ where Γ_1 is nothing but a 1-dimensional CW-complex. We can then compute $K_i(C(X_P^N) \rtimes \mathbb{R})$ via integer-valued cohomology of Γ_1.

The next main goal is to link generator of homology to actual generators of K-theory since the link between the two is given by diagramm chasing proof.

Some references

[1] Jared E. Anderson and Ian F. Putnam. Topological invariants for substitution tilings and their associated C^*-algebras. Ergodic Theory Dyn. Syst., 18(3):509–537, 1998.

[2] Hervé Oyono-Oyono and Samuel Petite. C^*-algebras of Penrose hyperbolic tilings. J. Geom. Phys., 61(2):400–424, 2011.

[3] Jean Renault. A groupoid approach to C*-algebras. 1980.

[4] Niels Erik Wegge-Olsen. K-theory and C^*-algebras: a friendly approach. Oxford: Oxford University Press, 1993.

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